ON SOME STOCHASTIC PROPERTIES OF RESIDUAL LIFE AND INACTIVITY TIME AT RANDOM TIME WITH APPLICATIONS IN RELIABILITY

Thesis submitted in partial fulfillment

for the Award of Degree of

Doctor of Philosophy

by

ARIJIT PATRA

RAJIV GANDHI INSTITUTE OF PETROLEUM TECHNOLOGY

JAIS – 229304

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RAJIV GANDHI INSTITUTE OF PETROLEUM TECHNOLOGY
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Dr. Chanchal Kundu , M.Sc., Ph.D.
Associate Professor

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Dr. Chanchal Kundu
Supervisor

Jais, Amethi- 229304; U. P., India. Mob. +91 9454909776; E-mail: ckundu@rgipt.ac.in/chanchal_kundu@yahoo.com

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RGIPT, Jais **Arijit Patra**

November, 2020

DEDICATED TO MY BELOVED PARENTS

Contents

Notations and Abbreviations

$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_+$	the set of positive natural numbers
$\in$	belongs to
X	a continuous random variable
cdf	cumulative distribution function
sf	survival function or reliability function
pdf	probability density function
$F(\cdot)$	*cdf* of X: sometime written as F_X
$f(\cdot)$	*pdf* of X; sometime written as f_X
$\overline{F}(\cdot)$	sf of X; sometime written as $\overline{F}_X$
r	failure rate /hazard rate (hr) of X; sometime written as r_X
$\tilde{r}$	reversed hazard rate (rh) of X; sometime written as $\tilde{r}_X$
$E(X)$	expected value of X
$m(\cdot)$	mean residual life (mrl) of X; sometime written as $m_X(\cdot)$
$\overline{m}(\cdot)$	mean inactivity time (mit) of X; sometime written as $\overline{m}_X(\cdot)$
σ^2	variance residual life (vrl) of X; sometime written as σ_X^2
$\overline{\sigma}^2$	variance inactivity time (vit) of X
$\leq_{st}$	usual stochastic order
$\leq_{hr}$	hazard rate order
$\leq_{rh}$	reversed hazard order
$\leq_{lr}$	likelihood ratio order

$\leq_{mrl}$	mean residual life order
$\leq_{mit}$	mean inactivity time order
$\leq_{vrl}$	variance residual life order
$\leq_{vit}$	variance inactivity time order
$\leq_{cx}$	convex order
$\leq_{icv}$	increasing concave order
(X_1, X_2)	a bivariate random vector
$f(t_1, t_2)$	the joint *pdf* of (X_1, X_2)
$RLRT$	residual life at random time
$ITRT$	inactivity time at random time
$RLMM$	residual lifetime mixture model
$RLRA$	residual life at random age
ILR (DLR)	increasing (decreasing) likelihood ratio
IFR (DFR)	increasing (decreasing) failure rate
IMRL (DMRL)	increasing (decreasing) mean residual life
IVRL (DVRL)	increasing (decreasing) variance residual life
DRHR	decreasing reversed hazard rate
IMIT	increasing mean inactivity time
IVIT	increasing variance inactivity time
NBU (NWU)	new better (worse) than used
NBUCX (NWUCX)	new better (worse) than used in convex order
TP_2	totally positive of order 2
RR_2	reverse regular of order 2
X_t	residual life of X at time $t > 0$
X_Y	residual life of X at a random time Y
$X_{(t)}$	inactivity time of X at time $t > 0$
$X_{(Y)}$	inactivity time of X at a random time Y
X^Y	mixture of residual life of X at a random time Y

F_Y	distribution function of X_Y
$\overline{F}_Y$	survival function of X_Y
F^Y	distribution function of X^Y
$\overline{F}^Y$	survival function of X^Y
$F_{(Y)}$	distribution function of $X_{(Y)}$
$\overline{F}_{(Y)}$	survival function of $X_{(Y)}$
$\overline{T}_s(X, x)$	survival function of equilibrium distribution
$r_s(X, x)$	hazard rate function of equilibrium distribution
$\overset{\mathrm{d}}{=}$	same in distribution

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Abstract

One of the main objectives of statistics is the comparison of random quantities in some stochastic sense. The need for providing a more detailed comparison of two random quantities has been the origin of the theory of stochastic orders that has been used during the last fifty years, at an accelerated rate, in many diverse areas of probability and statistics. Such areas include reliability theory, queuing theory, survival analysis, biology, economics, finance, insurance, actuarial science, operations research, and management science, and other fields in engineering, natural, and social sciences. These comparisons are mainly based on the comparison of some measures associated with these random quantities. As a result, several stochastic orders have been comprehensively discussed in the literature most of which are based on some reliability concepts for residual life and inactivity time at a fixed time. Furthermore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions by means of these stochastic orders. In reliability and life testing, a number of nonparametric classes of life distributions are considered to model the lifetimes of individuals as well as physical, biological, mechanical systems or components. Most of the reliability classes of life distributions are defined in terms of the reliability measures based on residual life and inactivity time and provide probabilistic information on them. These classes by no means are important ones in terms of their applications.

In the past few years, a lot of interest has been evoked on the study of stochastic properties of residual life and inactivity time at a fixed time. However, in real life applications, sometimes the time is not fixed and could be viewed as some nonnegative random variable (dependent or independent with generic lifetime). For example, the idle time of a server in a classical GI/G/1 queuing system is expressed as a residual life at random time (RLRT), the incubation period of a disease, i.e., the time between infection and beginning of a disease identifies with the notion of inactivity time at random time (ITRT). The concepts of RLRT and ITRT have evolved into a deep field of enormous breadth with ample structures of its own, establishing strong ties with numerous striking applications in reliability theory, survival analysis, queuing theory, forensic sciences and many other applied areas.

There exist plenty of scenarios where a statistical comparison of RLRT/ITRT and the study of their associated classes of life distributions are also required. But, only few works seem to have been done in this direction. In this thesis, we shall mainly be concerned about the stochastic properties of RLRT and ITRT. To this aim, first we assume that the generic lifetime and the random time are statistically independent and carry out stochastic comparisons of RLRT and ITRT in two sample problem having same as well as different random ages or observed to fail at same/different random times. We also investigate various properties of RLRT and ITRT based on different (ageing) classes of life distributions. We enhance the study and provide some new preservation properties of generalized ageing classes and generalized stochastic orderings for RLRT and ITRT. To enhance the study further, we consider the case when the generic lifetime and random time are not necessarily independent and obtain stochastic comparison results for RLRT and ITRT including their ageing notions. Finally, we consider a mixture model based on residual lifetime. We perform stochastic comparisons of two mixture models having different base line distributions as well as two different mixing distributions. Preservation properties of some ageing classes have also been discussed for this model.

Some applications of the results derived in this thesis are also illustrated in the context of statistics and reliability theory. The results strengthen some results available in the literature and are expected to be useful in reliability theory, forensic science, econometrics, queueing theory and actuarial science. This thesis is also intended to stimulate further research on stochastic orders and (ageing) classes of life distributions for residual life and inactivity time at random time with their applications.

Key Words and Phrases: Classes of life distributions, residual life (inactivity time) at random time, residual lifetime mixture model, stochastic orders.

Chapter 1

Introduction and a Brief Review of Literature

1.1 Introduction

During the last few decades, global competition in the market place has become more complicated. From customer's point of view, a product that has higher reliability is more acceptable than the product which has less reliability. Recall that, reliability is the ability of a system or component to perform its required functions under stated conditions for a specified period of time. An AC remote control quit functioning, a mobile battery goes dead, a CD disk drive goes bad, a mobile speaker quit, a laptop malfunctioning and a house roof leaks badly. In order to achieve good performance of a product, one has to understand the reliability of the product. Reliability plays a key role in monitoring the quality of various products/systems, which is primarily due to the complexity, sophistication, and automation inherent in modern technology.

Today's consumers are more conscious about product reliability while purchase. Evidently, the question of selecting the best product in terms of multifarious characteristics, viz., reliability, length of lifetime, etc., arises. Before purchase, customers are concerned about the reliability of the product after the warranty period. So it is interesting to compare the remaining lifetimes after the warranty period of different brands electronics items, such as mobile, laptop, hairdryer, trimmer, etc, in order to decide which brand is to be preferred. The necessity of comparing random lifetimes is not restricted to market products but also has a vital significance in several other fields. This can be better understood by putting across an example of COVID-19 which has proliferated around the globe and become the greatest threat to global public health of the century. The comparison of remaining lifetimes of a particular patient at different times of his/her life span after infection helps in determining whether the patient will take to recover or die. This enables the hospital management to decide the order in which patients need to be treated. In information theory, we require to compare two different systems in the sense of less uncertainty of residual lifetime to select the better system. In biological sciences, remaining lifetimes of control group of living organisms need to be compared with group receiving drug in order to conclude about the effectiveness of a particular drug. If one wishes to model the length of hospital stay of a surgical patient or, to setting rates and benefits for life insurance, then we need to take into account their residual lifetimes. It is also of interest to compare the lifetimes of different breeds of animals in order to decide which breed is to be preferred. In econometrics, different income distributions are to be compared in terms of the corresponding income inequalities and the various random prospects need comparison so that the better one can be chosen. Along with the residual life, the comparison of past lives (inactivity times) of components which were observed to fail at a certain time has received considerable attention in diverse areas. For a product, the manufacturer expects a high reliability (no failure) during warranty period to have less claim. In the case of COVID-19 infection, it is also important to take into account the time that has elapsed since a patient had been infected and to compare the inactivity times of virus in the body of patients. The comparison data helps in determining the extent to which the disease has proliferated in the body of patients and the order in which the patients need to be treated (who all need to be treated at earliest). In forensic science, the exact time of failure (e.g., death in case of human beings) is often not known. Here also, one is interested in the time elapsed from the failure. The comparison could also be used in insurance where the length of the period of the first payment to the death of the policy holder is of utmost importance. Residual life or inactivity time of mechanical devices/equipment or of biological organisms are, respectively, the focus of survival analysis and reliability. In manufacturing engineering, items of different brands need to be compared with respect to their residual life/inactivity time for remanufacturing.

Customarily, the comparison of residual lives and inactivity times often concern the scenarios in which the systems have fixed age or failed at fixed time. However, in reliability engineering, survival analysis, and many other applied areas, researchers often encounter the scenario in which a system or a component has survived (or failed) one unknown time point, which could be viewed as the realization of some nonnegative random variable. Therefore, it is highly worthwhile to study residual life and inactivity time at some random time. In real life, technical items can be incepted into operation having already some random age. Assume that we do not know when an operating item has been incepted into operation. Consider the case of buying a second-hand car from a company that sells used cars having some initial random age before being put on sale. Now, if a car is directly purchased from a potential seller in the locality, then the remaining lifetime of that car would be defined by residual life at random time. But, when the car is purchased (picked up at random) from a lot/mixture of used cars available with the car selling company, then the remaining lifetime of the car would be defined by residual life at random age (also called residual lifetime mixture model). In both the cases, it may be of interest to study the concept of residual life not at fixed time but at random time through introducing randomness to the reference time point. The idle time of a server in a classical GI/G/1 queuing system is also expressed as residual life at random time. In the literature, one can find another notion of random time, i.e., inactivity time at random time which is defined for an item failed at an unknown time. In medical science, study of incubation period of a disease, i.e., the time between infection and beginning of a disease, is of great importance. Suppose a person has undergone a medical test to check whether he has been infected by dengue or not, and the test reveals positive. Now the time when he had been infected by the disease is random and the beginning of the disease is also random. So it is interesting to study the incubation period of the disease, represented as inactivity time at random time.

During the last 50 to 60 years, a comparison of random quantities in terms of so-called stochastic orders and the classification of life distributions have received much attention due to their important role in different fields like actuarial science, survival analysis, risks, econometrics, finance and epidemics and so on. Apart from all these, stochastic orders and their associated classes have also an intense application in reliability theory. In stochastic order, we compare two random variables or compare some measures associated with these random quantities. Stochastic orders are very well-established tools to compare random quantities and it helps to design and optimize in order to increase the reliability of complex stochastic systems. On the other hand, in reliability and life testing, survival analysis for modeling situations, inventory theory, biometry, and maintenance, a number of nonparametric classes of life distributions have been defined to describe the reliability characteristics of random lifetimes. Some of these classes represent the notion of ageing which describes how a system or component improves or deteriorates with age. Under the various circumstance, one can characterize the life distribution classes using stochastic comparisons as well. In the literature, many stochastic orders and classes of life distributions are categorized or defined. These are based on some reliability measures for continuous life distributions. We confine ourselves to those of direct association with the residual life and inactivity time. In the same vein of residual life and inactivity time at fixed time, the concept of residual life at random time (RLRT), inactivity time at random time (ITRT) and residual lifetime mixture model (RLMM) have also made rapid stride in recent years. The present thesis deals with the study of stochastic properties of RLRT (including RLMM) and ITRT. Here we extend, generalize and unify the stochastic orders and the (ageing) classes of life distributions for RLRT, ITRT and RLMM in connection with residual life and inactivity time.

For convenience, throughout the thesis the words increasing (decreasing) and non-decreasing (non-increasing) are used interchangeably. All expectations and integrals are implicitly assumed to exist. By '$X \stackrel{\mathrm{d}}{=} Y$' we mean that the random variables X and Y are same in distribution. All the random variables considered here are nonnegative and $a/0$ is assumed to be ∞ where $a > 0$.

1.2 Review of Literature

We divide this section into nine subsections. In the first subsection, we discuss about residual life. A short presentation on inactivity time is made in subsection two. The third subsection deals with the study of some measures defined on residual life and inactivity time. In the fourth subsection, we give some definitions of stochastic orders and their interrelations. In the fifth subsection, we reproduce some definitions of life distribution classes and their interrelations. A brief review on generalized stochastic orders and ageing classes is given in subsection six. In the seventh subsection, we discuss on residual life at random time. A brief review on inactivity time at random time is made in subsection eight. Subsection nine deals with the study of residual lifetime mixture model.

1.2.1 Residual Life

For most products, customers see reliability as one of the important quality characteristic. When a producer discharges a product in the market, a data-sheet is given along with the product, which expresses the life of the component(s). The producer has to bear the repairing charges or replacement cost if the product fails within the warranty period. So customers are keen to know the remaining life of the product after the warranty period. Thus, it is of importance to know the residual life or additional life after warranty period of the product. For any random variable X, let $X_t = (X - t|X > t)$ denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. If the random variable X denotes the lifetime of a unit, then X_t is known as the residual life of that unit after surviving time t. For an element, the residual life represents the remaining period of that element until it will next require restoration, renewal, rehabilitation, reconstruction or replacement. Residual life of a system means the system, all sub-parts and all components thereof will continue to perform their function satisfactorily without refurbishment, replacement or significant maintenance. A nice interpretation of residual life is given by Watson and Wells (1961) in reference to the possibility of improving the useful life of items by eliminating those with short lives. Suppose that every item of a product with lifetime X is put in operation and run until either the item fails or a time t elapses, whichever comes first. Then a fraction of the product, in the long run, that does not fail is defined by X_t. To illustrate the importance of the random variable X_t we give two examples. First, consider a cancer patient undergoing chemotherapy. Then the variable of interest would be how long he/she would survive given that his/her treatment has begun a fixed time, say t, ago which is denoted by X_t. In the second example we assume that, at time t a computer is attacked by a malware. Now it is great to know how much time the computer will be operated after the attack by malware. Residual life is one of the important notions for studying the reliability characteristics of a system and it plays a vital role while choosing an electronic item. Sometimes a multi-component system fails but all the components of the system do not fail, so if we know the remaining life of the components which are not fail can be re-used.

Actuaries apply residual life to setting rates and benefits for life insurance. If t is the deductible for a particular policy and X representing the loss then X_t represents the amount of claim (cf. Gupta and Kirmani, 1998). In economics, for investing landholding residual life is used. It has been found useful in the social sciences for the life length of wars and strikes mentioned by Morrison (1978). It occurs in biomedical sciences to analyze survivorship studies (cf. Gross and Clock, 1975). It is used to model the length of hospital stay of surgical patients. The residual life is an important notion of many other areas such as renewal theory, burn-in, branching process, and dynamic programming. For an extensive discussion based on residual life, one may refer to Nagaraja (1975), Hall and Wellner (1981), Gupta and Gupta (1983), Gupta (1987), Guess and Proschan (1988), Galambos and Hagwood (1992), Embrechts et al. (1997), Navarro et al. (1998), Gupta and Kirmani (2000), Lin (2003), Li and Lu (2003), Navarro et al. (2008), Banjevic, D. (2009), Huang and Su (2012), Gurler (2012), Eryilmaz (2013), Gupta (2013), Tavangar (2014), Gupta et al. (2015), Bairamov and Tavangar (2015), Samadi et al. (2017), Chahkandi et al. (2017), Navarro (2018) and Su and Hung (2018), among others.

1.2.2 Inactivity Time

It is reasonable to infer that in many realistic situations the random variable is related to the past not to the future. For instance, suppose that, at time t, a person underwent a medical test to check whether he/she has been infected by COVID-19 or not, and the test reveals positive. Let X denote the time that he/she has been infected by the virus. Hence, it is known that t is greater than or equal to X. Now the question arises, how much time has elapsed since he/she had been infected by COVID-19. Again, suppose that a chemist runs a reaction for which the end point is unknown. Considering the fact that it is practically impossible to monitor the reaction continuously. Assume that when the chemist checks the reaction at time t, he found that the reaction has already attained its end point. The chemist might be interested to know the exact time at which the reaction was completed. In this case also the same question as above arises. Therefore, the random variable of interest is $X_{(t)} = (t - X|X \leqslant t)$ which is known as inactivity time. The random variable $X_{(t)}$ has received considerable attention in the literature and is also known as reversed residual life or time since failure. It can be interpreted in a situation of replacement policy. To be more specific, in a periodic replacement policy a system is observed after a certain period of time, say, T. So, at times $T, 2T, 3T, \dots$, the system is observed, and it is possible that at time nT the system is found to be down but at time $(n-1)T$ it was functioning, where n is a positive integer. Now, if the random variable X is the failure time of the system, the variable of interest is how much before time nT the system failed, and it is represented as $[nT - X|X \leqslant nT]$. By writing $nT = t$, we have the conditional random variable $X_{(t)}$. Thus, $X_{(t)}$ is the time interval between the observed failure time t and the exact failure time X given that failure has occurred at or before time t. Inactivity time has mainly been used in reliability theory, but it is also useful to describe the behavior of lifetime random variables in survival analysis (cf. Andersen et al., 1993). Some applications in risk theory and econometrics have been studied by Eeckhoudt and Gollier (1995), Kijima and Ohnishi (1999) and Mi (1999), to name a few. The inactivity time could be used in forensic science, where it is of importance to estimate $X_{(t)}$ in order to determine the exact time of death of a human being. It also plays an important role in insurance. Suppose by time t an individual policyholder must pay all the premium for an insurance policy. Unfortunately, suppose at time X the policyholder dies and assume all the premiums are cleared till X. So, $X_{(t)}$ represents the period remaining unpaid by the policyholder due to his/her death (cf. Maiti and Nanda, 2009).

Ruiz and Navarro (1996) were the first to define the inactivity time. In last two decades various types of stochastic orders and associated classes of life distributions have been developed and studied based on the inactivity time function. Li and Lu (2003) studied the inactivity time for series and parallel systems. For a detailed account of various properties, applications and higher order moments of inactivity time, one may refer to Nagaraja (1975), Nanda et al. (2003), Sunoj and Maya (2008), Kundu and Nanda (2010), Gupta (2013), Gupta et al. (2015), Tavangar and Asadi (2015), Bayramoglu and Ozkut (2016), Navarro et al. (2017), Kundu and Sarkar (2017), Kundu and Ghosh (2017), Abouelmagd et al. (2018), Navarro and Cali (2019) and Mahdy (2019), also.

1.2.3 Some Reliability Concepts for Continuous Lifetime Distributions

Let X be a nonnegative random variable representing the random lifetime of a unit or a system having absolutely continuous distribution function

$$F(t) = P(X \leqslant t),\ t > 0.$$

This unit could be the lifetime of a human being, an animal or a plant, or a non-living object. For non-living objects, the total amount of time for which the unit carries out its function satisfactorily is defined as the lifetime of those objects. Then the survival function of X is defined by

$$\begin{aligned} \overline{F}(t) &= P(X > t) \\ &= 1 - F(t),\ t > 0. \end{aligned}$$

In reliability and survival analysis, where a random variable represents the lifetime of a unit, the survival function is the probability to survive of this unit beyond t units of time. If $f(\cdot)$ is the probability density function of X, then the survival function is also represented as

$$\overline{F}(t) = \int_t^\infty f(u)du.$$

There are several measures, defined in reliability theory, based on residual life and inactivity time. These measures are found to be great tools to evaluate the stochastic behaviour of X. Some of the concepts for studying the reliability characteristics of a system are the failure rate or hazard rate, the mean residual life, the variance residual life, the reversed hazard rate, the expected inactivity time and the variance inactivity time. These are the most important notions in the area of engineering, biomedical science, actuarial science, forensic science, reliability theory, survival studies, business, social sciences and many other fields. Note that, hazard rate, mean residual life and variance residual life functions are important to study a system which has survived to an age t (residual life). On the contrary, the other three functions deal with the study of a system which is known to fail at or before some time t (inactivity time).

1. Hazard Rate or Failure Rate Function

In reliability theory, survival analysis, medical research, industrial life testing and other studies, the lifetime distributions are often specified by choosing a relevant failure rate/hazard rate function. The hazard rate function is also known as force of mortality in demography, the inverse of the Mill's ratio in economics, intensity function in the extreme value theory and in epidemiology, it is known as age-specific failure rate. It is time dependent and provides an instantaneous rate of failure. If a system has survived up to time t, the conditional probability of failure in the time interval $(t, t+\Delta t)$, where Δt (>0) is very small, is given by

$$P(t < X \leqslant t + \Delta t | X > t) = \frac{\overline{F}(t) - \overline{F}(t+\Delta t)}{\overline{F}(t)}.$$

Then the hazard rate function $r(t)$ is defined as

$$r(t) = \lim_{\Delta t \to 0} \frac{P(t < X \leqslant t + \Delta t | X > t)}{\Delta t}.$$

The hazard rate function $r(t)$ can be thought of as the intensity of failure of a device in the next small interval of time Δt, given that the device has survived up to time t. Thus, $\Delta t \cdot r(t)$ is the approximate probability of failure before time $t + \Delta t$, given that it has survived up to time (age) t. When X is an absolutely continuous random variable, then

$$r(t) = \frac{f(t)}{\overline{F}(t)} = -\frac{d}{dt}\ln(\overline{F}(t)). \tag{1.2.1}$$

Integrating (1.2.1) with respect to t, we get

$$\overline{F}(t) = \exp\left(-\int_0^t r(u)du\right).$$

There are many general shapes for the hazard rate. Some generic types of hazard rates are monotone hazard rates, either increasing or decreasing, non-monotone hazard rates, either bathtub-shaped or hump-shaped. Increasing hazard rates arise when the unit is wearing out with age. Models with decreasing hazard rates arise when the unit is improving with age. If the hazard rate is decreasing early and eventually begins increasing, then the hazard rate is bathtub-shaped. An example of a bathtub-shaped hazard rate is the case with the age-specific death rate in human life tables. In hump-shaped hazard rate, the hazard rate is increasing early and eventually begins declining. After successful surgery of a human being, where there is an initial risk of death due to infection or other complications just after the surgery and then risk of death is decreasing as the patient recovers, is an example of hump-shaped hazard rate. The concept of hazard rate has a long and exciting usefulness in the literature. It is not possible to nail down the review of this subject in a few pages.

2. Reversed Hazard Rate Function

The reversed hazard rate function was first introduced as a dual function of the hazard rate function by Barlow et al. (1963). The name reversed hazard rate was first mentioned by Lagakos et al. (1988). Sometimes it is called reversed failure rate, retro-hazard rate, or backward hazard rate function. The reversed hazard rate has mainly been applied in reliability engineering, even though initially introduced in actuarial research. It is useful for the analysis of left-censored and right-truncated data. Suppose that a system has failed at or before time t. Then the conditional probability that it survived at least up to time $t-\Delta t$, where Δt (>0) is very small, is given by

$$P(t-\Delta t < X \leqslant t | X \leqslant t) = \frac{F(t)-F(t-\Delta t)}{F(t)}.$$

The reversed hazard rate function $\tilde{r}(t)$ is defined as

$$\tilde{r}(t) = \lim_{\Delta t \to 0} \frac{P(t-\Delta t < X \leqslant t | X \leqslant t)}{\Delta t}.$$

Here $\Delta t \cdot \tilde{r}(t)$ is the approximate probability of failure of a unit in the interval $(t-\Delta t, t]$, given that it has failed at or before time t. When X is an absolutely continuous random variable, then

$$\tilde{r}(t) = \frac{f(t)}{F(t)} = \frac{d}{dt}\ln(F(t)). \tag{1.2.2}$$

Likewise the hazard rate function, the reversed hazard rate function also uniquely determines the distribution function $F(t)$ through the relation

$$F(t) = \exp\left(-\int_t^{\infty} \tilde{r}(u)du\right).$$

In the literature, a number of different applications of reversed hazard rate function have been investigated in the study of lifetime random variables. In forensic sciences, the reversed hazard rate function is quite useful to find out the exact time of failure of a unit or a human being. Ware and DeMets (1976) and Andersen et al. (1993) used the reversed hazard rate function to estimate the survival function of left-censored data. Keilson and Sumita (1982) studied it in the context of stochastic ordering. For a retrospective analysis of epidemiological data on individuals in a group, Lagakos et al. (1988) used the reversed hazard rate function. Same kind of data were studied by Kalbfleisch and Lawless (1989) using reversed hazard rate function. Some applications of reversed hazard rate in the study of continuous-time Markov chains is given in Kijima (1998). Block et al. (1998) have shown that unlike the monotonicity of the hazard rate function, the monotonicity of the reversed hazard rate function is not related to the ageing property of the unit. Different characterizations of the reversed hazard rate have been studied by Block et al. (1998), Sengupta et al. (1999) and Chandra and Roy (2001). Finkelstein (2002a)

expressed the relation between hazard rate and reversed rate functions as

$$\tilde{r}(t) = \frac{r(t)}{\exp\left(\int_0^t r(u)du\right) - 1}.$$

Lawless (2003) developed nonparametric estimator of the survival function for the right truncated data using reversed hazard rate function. Chandra and Roy (2005) defined and studied classes of distributions based on reversed hazard rate and made their implicative relationships. For more details information of reversed hazard rate function, one can refer to Nanda et al. (1998), Kijima and Ohinishi (1999), Gupta and Nanda (2001), Nanda and Shaked (2001), Gupta and Wu (2001), Nanda et al. (2003), Chen et al. (2004), Gupta et al. (2004), Nair and Asha (2004), Ahmad and Kayid (2005), Chandra and Roy (2005), Sunoj and Maya (2006), Gupta and Gupta (2007), Sankaran and Gleeja (2007, 2008), Kayid et al. (2011), Misra and Misra (2013), Veres-Ferrer and Pavia (2014), Burkschat and Torrado (2014), Oliveira and Torrado (2015), Gupta (2015), Abouelmaged et al. (2018) and Kayid et al. (2019), to mention a few.

3. Mean Residual Life Function

The hazard rate is closely related to another significant concept of reliability known as mean residual life (*mrl*), which is defined as

$$\begin{aligned} m(t) \equiv E(X_t) &= E(X - t|X > t) \\ &= \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(u)du. \end{aligned}$$

It is also known as biometric function (cf. Chiang, 1960), life expectancy or expectance of life function (cf. Barlow and Proschan, 1965) and expected remaining life function or the mean excess function (cf. Abdous and Berred, 2005). The *mrl* function is finite for all finite t, and is usually of interest when X is a nonnegative random variable. However, it is possible that $m(\infty) = \lim_{t\to\infty} m(t) = \infty$. Though $m(t)$ is positive but not every nonnegative function is a *mrl* function corresponding to some random variable. For some set of conditions for a function to be a *mrl* function and properties one may refer to Bhattacharjee (1982) and Shaked and Shanthikumar (1991, 2007).

If we have some idea about the expected remaining life for which the component under consideration will continue to work, then it becomes easy to replace that component. Mean residual life is more useful than the failure rate for constructing the maintenance policies. Sometimes, the *mrl* function may be more relevant than the hazard rate function, for example, in industrial reliability studies of repair and replacement strategies. The *mrl* function summarizes the entire residual life distribution, whereas the hazard rate function involves only the risk of immediate failure. The hazard rate and *mrl* functions of X are linked through the relation (cf. Muth, 1977)

$$r(t) = \frac{1 + \frac{d}{dt}m(t)}{m(t)}.$$

Calabria and Pulcini (1987) established that

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{1}{r(t)},$$

provided the latter limit exits, finite and strictly positive. Zahedi (1991) claims that the *mrl* function has more intuitive appeal than the concept of hazard rate function for modeling and analysis of failure data. Moreover, the existence of the probability density function does not required for the existence of the *mrl* function. For a comparison and discussion of the advantages of the use of *mrl* function over the hazard rate function in some applications one may refer to Muth (1977), Gupta (1981), Bhattacharjee (1982), Calabria and Pulcini (1987), Ghai and Mi (1999), Lillo (2000) and Hu et al. (2002), among others.

Mean residual life function is an eminent characteristic in reliability, survival analysis, actuarial studies and various other areas. In studies of human populations, demographers are interested in life expectancy or expectation of life which is simply *mrl* concept in disguise. In binary systems, where the system has two possible states as either working or failed, the *mrl* has been extensively studied in the literature. In economics, for investing landholding it is also used. In maintenance and product quality control, for analyzing burn-in, *mrl* function is used to model the ageing process of a device. In product technology, the *mrl* has been applied to a cutting tool monitoring problem by Chinnam and Baruah (2004). The *mrl* function also arises in many other areas such as biomedical science, actuarial science, economics, renewal theory, optimal disposal of an asset, dynamic programming, and branching process. In the literature, there have been many sources where one can find more applications of the *mrl* function in modeling and analysis of failure data in a number of different areas. For example, see, Chiang (1960), Bjerkedal (1960), Watson and Walls (1961), Bryson and Siddiqui (1969), Weiss and Dishon (1971), Jardine and Kirkham (1973), Hollander and Proschan (1975, 1980), Elandt-Jhonson and Jhonson (1980), Schoenfeld (1980), Kuo (1984), Park (1985), Bhattacharjee (1986), Berger et al. (1988), Guess and Proschan (1988), Guess and Park (1991), Siddiqui and Caglar (1994), Kulkarni and Rattihalli (2002), Gupta and Bradley (2003), Gupta and Kirmani (2004a), Lai and Xie (2006) and Banjevic (2009), Sun and Zhang (2009), Sun and Zhao (2010), Sun et al. (2012), Huynh et al. (2012), Le Son et al. (2013), Yang and Zhou (2014), Gupta (2016), Lin et al. (2018), Pourjafar and Zardasht (2020) and Ma et al. (2020), just to name a few.

In many statistical studies, the *mrl* function is of prime importance and has extensively been studied in the literature. Some tests for alternatives representing decreasing *mrl* function has been developed by Hollander and Proschan (1975). Hall and Wellner (1981) have characterized the *mrl* function and express survival function in terms of it and studied some residual moments and inequalities for *mrl*. Gupta (1981) has established a method of obtaining the moments in terms of the *mrl* function. The class of the *mrl* function has been characterized by Bhattacharjee (1982). Hollander and Proschan (1975) have derived tests that the underlying failure distribution is exponential, versus it has a monotone *mrl* function. But their tests were based on a complete sample. However, data are always not complete, it could be incomplete. Chen et al. (1983) generalized the Hollander and Proschan (1975) tests for monotone *mrl* function using randomly censored data. They have also investigated the efficiency loss due to the presence of censoring. Calabria and Pulcini (1987) presented a useful relationship between the asymptotic values of the *mrl* function and the hazard rate of a generic continuous distribution function. It is also shown that at infinity the derivative of the *mrl* function of a positive random variable always tends to zero. The *mrl* function has attracted many researchers including Meilijson (1972), Hamdan (1972), Swartz (1973), Balkema and De Hann (1974), Gupta (1975), Morrison (1978), Yang (1978), Alzaid (1988), Shaked and Shanthikumar (1991), Tsang and Jardine (1993), Gupta and Akman (1995), Li (1997), Aly (1997), Müller and Scarsini (2006), Raqab and Asadi (2008), Ghebremichael (2009), Shen et al. (2010), Nanda et al. (2010), Eryilmaz (2010), Bayramoglu (2013), Gupta (2015), Chen et al. (2016), Viswakala and Sathar (2019) and Nanda and Kayal (2019), among others.

4. Mean Inactivity Time Function

Due to the important role in reliability theory, survival analysis, forensic science, actuarial science, maintenance policies, and many other areas of applied probability, the mean inactivity time (*mit*) function has received much attention in the last two decades. The *mit* function represents the average past life of a component which was observed to fail at time t. It is also known as expected inactivity time or reversed mean residual life or expected stopped time or mean past lifetime function. The *mit* function of X is defined by

$$\overline{m}(t) = \begin{cases} E(t - X|X \leqslant t), & t > 0, \\ 0, & otherwise. \end{cases}$$

For an absolutely continuous nonnegative random variable we have

$$\overline{m}(t) = \frac{\int_0^t F(u)du}{F(t)}, \; t > 0.$$

Finkelstein (2002a) called the *mit* as mean waiting time and they used it in defining reversed hazard rate function. The reversed hazard rate and mean inactivity time are closely related to each other because both are associated with right truncated random variables and determine a distribution uniquely. They are linked through the relation (cf. Finkelstein, 2002a)

$$\tilde{r}(t) = \frac{1 - \frac{d}{dt}\overline{m}(t)}{\overline{m}(t)}.$$

In real life situation, *mit* is a quantity of interest to assess the lifetime of a system. In actuarial sciences, with the help of *mit* one can determine the expected time elapsed since failure in order to predict the exact time of failure. In insurance, the *mit* could be used, where the expected length of the period of the first payment to the death of the policy holder is of importance. In forensic sciences, it could also be applied to determine the expected time elapsed since death of a human being in order to predict the exact time of death. While describing different maintenance strategies, the *mit* might be of interest. In cases of diseases that can recur, efficiency of a treatment is determined by analyzing the remission period, *i.e.,* disease free survival time. Often the true remission period is unknown due to an inability to continuously monitor patients because of the high cost and effort involved. In such circumstances, the true remission period can be estimated using the *mit* function.

The *mit* function has received much attention in recent years. Navarro et al. (1997) have defined and studied a new stochastic order based on *mit*, which they called reversed mean residual life order. Chandra and Roy (2001) established some properties of *mit* with respect to reversed hazard rate. Finkelstein (2002*a*) focused on the importance of *mit* in defining the reversed hazard rate and also studied its properties. Nanda et al. (2003) have defined and studied some new classes of distributions based on *mit*, and obtained some stochastic ordering results. Kayid and Ahmad (2004) perform some stochastic comparisons based on *mit* order (definition follows) under the convolution and mixture. Ahmad et al. (2005) provided some other preservation and characterization results for the *mit* ordering and increasing *mit* function under convolution, mixture, and shock models. They also studied a closure property under shock models and series systems. Asadi (2006) introduced and studied various properties of *mit* function for components of a parallel system. For more research related to *mit* one may refer to Ahmad and Kayid (2005), Goliforushani and Asadi (2008), Kundu and Nanda (2010), Eryilmaz (2010), Tavangar and Asadi (2010), Gandotra et al. (2011), Izadkhah and Kayid (2013), Gupta (2015), Kayid and Izadkhah (2014, 2018) and Kayid et al. (2018), to mention a few.

5. Variance Residual Life Function

The concept of residual life is of special interest in reliability theory and survival analysis as it measures the remaining life of a mechanical equipments or of a biological organism after it has attained a specific age. Various characteristics of residual life such as its mean, variance, and higher order moments have been studied in the literature. The variance residual lifetime (*vrl*) function is useful in many areas including biometry, actuarial sciences and reliability. It is denoted by $\sigma^2(t)$ and defined as

$$\begin{aligned}
\sigma^2(t) &= \mathrm{Var}(X_t) \\
&= E[(X-t)^2|X>t] - m^2(t) \\
&= \frac{1}{\overline{F}(t)}\int_t^\infty (u-t)^2 f(u)du - m^2(t) \\
&= \frac{2}{\overline{F}(t)}\int_t^\infty (u-t)\overline{F}(u)du - m^2(t) \\
&= \frac{2}{\overline{F}(t)}\int_t^\infty \int_v^\infty \overline{F}(u)dudv - m^2(t).
\end{aligned}$$

It appears in the expression of weights assigned for censored observations (cf. Schmee and Hahn, 1979). It also appears in the estimation of *mrl* function (cf. Hall and Wellner, 1981).

The *vrl* function has been studied in reliability literature in connection with life length problems and a lot of interest has been evoked on the study of *vrl* order and the associated ageing classes (definition follows). Fagiuoli and Pellerey (1993) and Al-Zahrani and Stoyanov (2008) provided some implications and characterizations of *vrl* order. Karlin (1982) studied its monotonic behavior when the density function is log-convex or log-concave. Launer (1984) and Gupta et al. (1987) studied the classes of life distributions having decreasing (increasing) *vrl* function. Gupta (1987, 2006) discussed its monotonicity and study the relationship of decreasing/increasing *vrl* classes with some other known classes of life distributions. Stoyanov and Al-sadi (2004) have studied the closer properties of these classes under the reliability operations such as mixing, convolution and formation of coherent systems. For a comprehensive account of several properties of *vrl* function, see Dallas (1981), Chen et al. (1983), Abouammoh et al. (1990), Kanwar and Madhu (1991), Gupta and Kirmani (1987, 2000, 2004*b*), Block et al. (2002), Stoyanov and Al-Sadi (2004), El-Arishi (2005), Lai and Xie (2006), Abu-Youssef (2004, 2007, 2009), Banjevic (2009), Nair and Sudheesh (2010), Khorashadizadeh et al. (2010, 2013*a*), Huang and Su (2012), Gupta (2015) and Nair er al. (2017), among others.

6. Variance Inactivity Time Function

Another quantity which has also generated interest in recent years is the variance inactivity time (*vit*) function $\overline{\sigma}^2(t)$. It is related to the random variable $X_{(t)}$ and defined as

$$\begin{aligned}
\overline{\sigma}^2(t) &= \mathrm{Var}(X_{(t)}) \\
&= E[(t-X)^2|X \leqslant t] - \overline{m}^2(t) \\
&= \frac{1}{F(t)}\int_0^t (t-u)^2 f(u)du - \overline{m}^2(t) \\
&= \frac{2}{F(t)}\int_0^t (t-u)F(u)du - \overline{m}^2(t) \\
&= \frac{2}{F(t)}\int_0^t \int_0^v F(u)dudv - \overline{m}^2(t).
\end{aligned}$$

The concept of *vit* plays an important role in reliability and life testing. It provides the variance time elapsed since the failure of a device under the assumption that the device has already failed at time t, say. In the analysis of right truncated data, the *vit* function plays the same role as that of the *vrl* function in the analysis of left truncated data. Nanda et al. (2003) have shown that increasing mean inactivity time property is stronger than the increasing variance inactivity time (IVIT) property (definition follows). Mahdy (2012) studied the closure properties of the IVIT class under some reliability operations such as mixing, convolution and formation of coherent systems. Al-Zahrani and Al-Sobhi (2015) have studied some properties of *vit* function and also use it for entropy measure. Al-Zahrani and Stoyanov (2008), Mahdy (2012, 2016), Al-Zahrani and Al-Sobhi (2015) and Kayid and Izadkhah (2016) introduced the variance inactivity time order (definition follows) for continuous random variables and gave some implications and characterizations concerning this order. They also developed some preservation properties of this order under mixture and convolution. Some new applications in the context of economic theory, reliability, statistics and risk theory are also provided. Several properties of *vit* function for discrete random variable have been studied by Khorashadizadeh et al. (2013*b*), Mahdy (2013) and Gupta (2015).

1.2.4 Stochastic Orders based on Above Reliability Concepts

How do people compare two products for choosing a better one than another. For example, what do we mean when we say that an Apple mobile is better than an Oppo? Let X and Y represent the lifetimes of Apple mobile and Oppo mobile, respectively. Then, Apple mobile should be preferred to the Oppo mobile if X is larger than Y in some sense. Thus, we need to check whether X is greater (or smaller) than Y in some sense. Since X and Y are random variables, it is not possible to compare them in traditional way, i.e., the way we compare real numbers. The simplest way of comparing the two products is by the comparison of their means. However, such a comparison is based on only two numbers measuring the centers of the distributions, and, therefore, is often not very informative. In addition, the means sometimes do not exist (if any random variable is considered and not just lifetime random variable). In many practical instances, one may have more detailed information about the random variables than just their means for the purpose of comparison of two random variables. When one is interested in comparing random variables having same means (or that are centered about the same value), one may usually think of comparing in terms of various measures of dispersion, e.g., standard deviation or variance. Again, such a comparison is based on only two numbers having same type of limitations as that of means. In order to overcome these limitations, various stochastic orderings have been studied in the literature for comparison of two random variables. These orderings have many applications in reliability theory, survival analysis, queuing theory, insurance, biology, economics, operations research, actuarial science, management science and many other areas. Following Lehmann (1955), the study on stochastic orders has gained the interest of researchers and it has grown significantly since 1994. There are several sources where the reader can find a deep historical review of stochastic orders, e.g., Barlow and Proschan (1981), Müller and Stoyan (2002), Shaked and Shanthikumar (2007), Belzunce et al. (2015) and the references therein.

In the sequel, we quickly review some well-known stochastic orders which are useful in the present investigation. Of course, one can find some other stochastic orderings which have extensive range of applications in reliability, survival analysis and many other areas.

Let X and Y be two nonnegative random variables with distribution functions $F(t) = P(X \leqslant t)$, $G(t) = P(Y \leqslant t)$ and density functions $f(\cdot)$, $g(\cdot)$, respectively. Write $\overline{F}(t) = 1 - F(t)$ and $\overline{G}(t) = 1 - G(t)$, as survival functions of X and Y, respectively.

Definition 1.2.1. *X is said to be larger than Y in usual stochastic order (denoted by $X \geqslant_{st} Y$) if and only if $\overline{F}(t) \geqslant \overline{G}(t)$ for all $t \geqslant 0$.*

This means, at each point t, the graph of $\overline{F}(t)$ lies above the graph of $\overline{G}(t)$. If we recall the comparison of Apple mobile and Oppo mobile, then by 'Apple is better than Oppo in usual stochastic order', we mean that for any specified time t, the Apple is more likely to have a lifetime exceed t, than the oppo is. But, customers would like to collect more information in terms of the comparison. They may be interested to know, given that both the mobiles have survived up to a fixed warranty period (time t_0), whether Apple is still superior and hence less likely to fail in the near future than Oppo? Then the role of the following stochastic orders come into picture.

Definition 1.2.2. *X is said to be larger than Y in hazard rate order (denoted by $X \geqslant_{hr} Y$) if and only if $r_X(t) \leqslant r_Y(t)$ for all $t \geqslant 0$, where $r_X(t)$ and $r_Y(t)$ are the hazard rate functions of X and Y, respectively.*

This is equivalent to,

$$\frac{\overline{F}(t)}{\overline{G}(t)} \text{ is increasing in } t \geqslant 0.$$

Note that $X \geqslant_{hr} Y$ if and only if $X_t \geqslant_{st} Y_t$, for all $t \geqslant 0$.

Definition 1.2.3. *X is said to be larger than Y in mean residual life order (denoted as $X \geqslant_{mrl} Y$) if and only if $m_X(t) \geqslant m_Y(t)$ for all $t \geqslant 0$, where $m_X(t)$ and $m_Y(t)$ are the mean residual life functions of X and Y, respectively.*

The above can also be written as (cf. Fagiuoli and Pellerey, 1993),

$$\frac{\int_t^\infty \overline{F}(u)du}{\int_t^\infty \overline{G}(u)du} \text{ is increasing in } t \geqslant 0.$$

Suppose an Apple mobile and an Oppo mobile are known to have survived up to time t, then by 'Apple mobile is better than Oppo mobile in hazard rate order' we mean that the probability that Apple mobile fails in the interval $(t, t+\Delta t)$ is less than the probability that the Oppo mobile fails in $(t, t+\Delta t)$, where Δt (>0) very small. In that case the expected residual life (mean life expectancy at age t) for the Apple mobile is more likely than that for the Oppo mobile. It is worthwhile to remark that the usual stochastic order makes a statement about the distribution of two 'new' units with lifetimes X and Y, whereas the hazard rate and mean residual life orderings are making statement about X and Y at an age t. Moreover, in *mrl* order we deal with the comparison of the means of their residual lives. If one wishes to compare the dispersion/variability of the residual lives then the following stochastic order will be considered.

Definition 1.2.4. *X is said to be larger than Y in variance residual life order (denoted by $X \geqslant_{vrl} Y$) if and only if $\sigma_X^2(t) \geqslant \sigma_Y^2(t)$ for all $t \geqslant 0$, where $\sigma_X^2(t)$ and $\sigma_Y^2(t)$ are the variance residual life functions of X and Y, respectively.*

The above relation can also be expressed in the following form,

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \geqslant \frac{\int_t^\infty \int_x^\infty \overline{G}(u)dudx}{\overline{G}(t)}.$$

Or equivalently (cf. Fagiuoli and Pellerey, 1993),

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \int_x^\infty \overline{G}(u)dudx} \text{ is increasing in } t \geqslant 0, \text{ for any } x > 0.$$

But, what if both the mobiles are known to have already failed at time t? How do we compare them? In this case, the following order relations are of importance.

Definition 1.2.5. *X is said to be larger than Y in*

(i) reversed hazard rate order (denoted by $X \geqslant_{rh} Y$) if and only if $\tilde{r}_X(t) \geqslant \tilde{r}_Y(t)$ for all $t \geqslant 0$, where $\tilde{r}_X(t)$ and $\tilde{r}_Y(t)$ are the reversed hazard rate functions of X and Y, respectively. Or equivalently,

$$\frac{F(t)}{G(t)} \textit{ is increasing in } t \geqslant 0;$$

(ii) mean inactivity time order (denoted by $X \geqslant_{mit} Y$) if and only if $\overline{m}_X(t) \leqslant \overline{m}_Y(t)$ for all $t \geqslant 0$, where $\overline{m}_X(t)$ and $\overline{m}_Y(t)$ are the mean inactivity time functions of X and

Y, respectively. This is equivalent to

$$\frac{\int_0^t F(x)dx}{\int_0^t G(x)dx} \text{ is increasing in } t \geqslant 0.$$

If an Apple mobile and an Oppo mobile are known to have failed at time t, then by 'Apple mobile is better than Oppo mobile in reversed hazard rate order' we mean that the probability that Apple mobile has survived up to time $t - \Delta t$ is greater than the probability that the Oppo mobile has survived up to time $t - \Delta t$ (for a small $\Delta t > 0$). Thus, the expected length of the period since failure for Apple mobile is smaller than that for Oppo mobile. Again, in *mit* order the location of two inactivity time random variables are compared. But, when one wishes to compare two inactivity time random variables that have non-ordered means, one is usually interested in the comparison of the dispersion of these quantities. It is of interest to estimate the times that have elapsed since the failure of the Apple and Oppo mobiles and to study the dispersion/variability of these elapsed interval of times. As a result, the stochastic order which is defined on the basis of the *vit* function has been considered by some authors in recent years.

Definition 1.2.6. *X is said to be larger than Y in variance inactivity time order (denoted by $X \geqslant_{vit} Y$) if and only if*

$$\frac{\int_0^t \int_0^x F(u)dudx}{F(t)} \leqslant \frac{\int_0^t \int_0^x G(u)dudx}{G(t)},$$

or equivalently,

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t \int_0^x G(u)dudx} \text{ is increasing in } t \geqslant 0, \text{ for any } x > 0.$$

But, Sometimes customers require more clarifications for the comparison purpose. The following is a stronger stochastic order than the orders discussed so far.

Definition 1.2.7. *X is said to be larger than Y in likelihood ratio order (denoted by $X \geqslant_{lr} Y$) if and only if $f(t)/g(t)$ is increasing in $t \geqslant 0$.*

The likelihood ratio order is a very important stochastic order as in many situations it is easy to verify. It is the most strongest stochastic order which can be considered as a sufficient condition for the hazard rate and the reversed hazard rate orders to hold.

The interrelationships among the orderings discussed earlier are as follows:

$$\begin{array}{ccccccc}
X \leqslant_{lr} Y & \Longrightarrow & X \leqslant_{hr} Y & \Longrightarrow & X \leqslant_{mrl} Y & \Longrightarrow & X \leqslant_{vrl} Y \\
\Downarrow & & \Downarrow & & & & \\
X \leqslant_{rh} Y & \Longrightarrow & X \leqslant_{st} Y & & & & \\
\Downarrow & & & & & & \\
X \leqslant_{mit} Y & \Longrightarrow & X \leqslant_{vit} Y. & & & &
\end{array}$$

1.2.5 Classes of Life Distributions based on the Above Reliability Concepts

The classes of life distributions provide a knowledge of the intrinsic structure of the stochastic properties of random variables and have become an important tool in applied probability. In reliability and life testing, several nonparametric classes of life distributions are considered to model the lifetimes of a biological/mechanical systems or components. Most of these classes which are defined based on the reliability concepts of residual life characterize the ageing properties of the underlying phenomena. Here we recall the basic concepts of ageing and the classes of positive life distributions in context with residual life and inactivity time. It is a well-known fact that, every manufactured or naturally exist item has certain life length and after performing its functions satisfactorily it starts decaying. Thus, in reliability connection, the age of a working unit is the time for which it is already working satisfactorily without failure. By the term ageing, we mean a mathematical specification of degradation/upgradation of an item over time. To be more specific, by ageing we mean a phenomenon whereby an older system has a shorter residual lifetime, in some statistical sense, than a newer or younger one. The notion of ageing describes how a component or system elevate or deteriorates with age. In the literature, this description covers the states, positive ageing and negative ageing. Positive ageing describes the phenomenon that age affects the residual lifetime in some adverse manner, whereas, negative ageing means beneficial effect of age on the random residual lifetime of the unit. No ageing means that the age of a component has no effect on the distribution of residual lifetime of the component. It is considered to be axiomatic that any effect of age on a unit, which contributes to the reduction of its residual lifetime (in some probabilistic sense) is to be taken as an adverse effect and the phenomenon is called positive ageing (cf. Deshpande et al., 1986). Generally, by ageing we mean positive ageing which is common in reliability engineering as components tend to become worse with time due to increased wear and tear. There are many classes of life distributions which have positive as well as negative ageing criteria. Social scientists use some of these classes for studies on job mobility, length of wars, duration of strike, etc. (cf. Morrison, 1978). Ageing classes are used to describe a wear process and arise naturally in medicine, where testing for, and estimation of, residual measure is of paramount importance. Bryson and Siddiqui (1969), Launer (1984), Averous and Meste (1989), Barlow and Proschan (1981), Bondesson (1983), Deshpande et al. (1986) and Fagiuoli and Pellerey (1993), among others, gave a complete classification of ageing classes. There is another set of classes of life distributions that refer the reliability measures based on inactivity time and do not reflect any notion of ageing. Some interesting characterizations of life distributions can be obtained based on the monotonicity of these measures. These classes have many applications, for example, in life insurance, maintenance, economics, product quality control and social studies. These classes are useful for analyzing the monotonic behaviour of left-censored and right-truncated data. Some important classes of life distributions based on the monotonicity of the reversed hazard rate, mean inactivity time and variance inactivity time functions have been studied by Block et al. (1998), Chandra and Roy (2001), Nanda et al. (2003), Kayid and Ahmad (2004), Li and Zuo (2004), Ahmad and Kayid (2005), Ahmad et al. (2005), Li and Xu (2006), Misra et al. (2008), Kundu and Nanda (2010), Mahdy (2012) and Misra and Naqvi (2017), among others. The following are some commonly used classes that are closely related to our main theme.

Let X be a nonnegative random variable with survival function $\overline{F}$, distribution function F and density function f.

Definition 1.2.8. *The distribution of X is said to be increasing (resp. decreasing) failure rate (IFR (resp. DFR)) if and only if $r(t)$ is increasing (resp. decreasing) in $t \geqslant 0$. Or equivalently, X_t is stochastically decreasing (resp. increasing) in $t \geqslant 0$, which can be* *expressed as,*

$$\frac{\overline{F}(x+t)}{\overline{F}(t)} \text{ decreasing (resp. increasing) in } t \geqslant 0, \ for \ any \ x > 0.$$

This is again similar as the following (cf. Shaked and Shanthikumar, 2007)

$$X_t \leqslant_{hr} (\geqslant_{hr}) X, \ for \ all \ t \geqslant 0.$$

Definition 1.2.9. *The distribution of X is said to be increasing (resp. decreasing) mean residual life (IMRL (resp. DMRL)) if and only if $m(t)$ is increasing (resp. decreasing) in $t \geqslant 0$. Or equivalently,*

$$\frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0.$$

Definition 1.2.10. *The distribution of X is said to be increasing (resp. decreasing) variance residual life (IVRL (resp. DVRL)) if and only if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \overline{F}(x)dx} \text{ is increasing (resp. decreasing) in } t \geqslant 0, \ for \ any \ x > 0.$$

This can also be expressed as,

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0.$$

Definition 1.2.11. *The distribution of X is said to be*

(i) decreasing reversed hazard rate (DRHR) if and only if $\tilde{r}(t)$ is decreasing in $t > 0$, for any $x > 0$. This is equivalent to $X_{(t)}$ is stochastically increasing in $t > 0$, which implies,

$$\frac{F(x-t)}{F(t)} \text{ is increasing in } t \geqslant 0, \ for \ any \ x > 0;$$

(ii) increasing mean inactivity time (IMIT) if and only if $\overline{m}(t)$ is increasing in $t \geqslant 0$. Or equivalently,

$$\frac{\int_0^t F(u)du}{F(t)} \text{ is increasing in } t \geqslant 0;$$

(iii) increasing variance inactivity time (IVIT) if and only if

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t F(x)dx} \text{ is increasing in } t \geqslant 0, \text{ for any } x > 0.$$

This can be written as,

$$\frac{\int_0^t \int_0^x F(u)dudx}{F(t)} \text{ is increasing in } t \geqslant 0.$$

Block et al. (1998) have shown that there exists no nonnegative random variable which has increasing reversed hazard rate function. Likewise, there exists no nonnegative random variable for which the dual class of IMIT exists (cf. Chandra and Roy, 2001). Similar to these classes, the decreasing variance inactivity time class also does not exist for a nonnegative random variable. Below we define a stronger ageing class which was first studied by Barlow and Proschan (1965).

Definition 1.2.12. *The distribution of X is said to be increasing (resp. decreasing) likelihood ratio (ILR (resp. DLR)), if and only if $f(t)$ is log-concave (log-convex). This is similar as, $f(x+t)/f(t)$ decreasing (resp. increasing) in $t \geqslant 0$, for any $x > 0$.*

The interrelationships among the classes of life distributions discussed earlier are as follows:

$$\begin{array}{ccccccc} \text{ILR(DLR)} & \Longrightarrow & \text{IFR(DFR)} & \Longrightarrow & \text{DMRL(IMRL)} & \Longrightarrow & \text{DVRL(IVRL)} \\ \Downarrow & & & & & & \\ \text{DRHR} & \Longrightarrow & \text{IMIT} & \Longrightarrow & \text{IVIT.} & & \end{array}$$

1.2.6 Generalized Stochastic Orders and Ageing Classes

Over the recent years, there is a rapid increase in the study of generalized stochastic orderings and generalized ageing classes in the literature. They are used in reliability, queueing theory, econometrics, inventory, actuarial science, stochastic process and many other applied areas. Generalized ordering were discussed by Fishburn (1980), Ekern (1980), O'Brien (1984) and others. Several preservation results on generalized orderings under Poisson shock models have been established by Fagiuoli and Pellerey (1993). Kass et al. (1994) studied the generalized orderings in actuarial sciences. Nanda et al. (1996*a*,*b*) provided some stochastic properties on generalized orderings. Nanda (1997) used generalized orderings in minimal repair policy. Denuit et al. (1998) used some generalized stochastic orderings in the context of insurance and queues. Different mixture properties of generalized ordering was studied in Hesselanger et al. (1998). Along with generalized stochastic orderings, generalized ageing classes also play a critical role in reliability. Fagiuoli and Pellerey (1993) provided a different type of classification of generalized ageing classes in a unified way. Hu et al. (2001) obtained a number of characterizations for generalized orderings and generalized ageing properties of random variables. Some generalized ageing properties of the underlying distribution of renewal process has been discussed by Hu et al. (2004). Cai and Zheng (2009) obtained some characterizations of generalized ageing classes of inter-arrival times by the excess lifetime of a renewal process. For more information related to generalized stochastic orders and generalized ageing classes one may refer to Mukherjee and Chatterjee (1992), Navarro and Hernandez (2004), Belzunce et al. (2008), Nanda and Kundu (2009) and Cai and Zheng (2012), to mention a few. For the nonnegative absolutely continuous random variable X, let $\overline{T}_0(X,x) = \overline{\Phi}_0(X,x) = f(x)$, $\overline{\Phi}_1(X,x) = \overline{F}(x)$, $\forall x \geqslant 0$ and for $s \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ where $\mathbb{N} = \{0,1,2,3,\ldots\}$,

$$\overline{\Phi}_s(X,x) = \int_x^\infty \overline{\Phi}_{s-1}(X,u)du,\ \overline{T}_s(X,x) = \frac{\overline{\Phi}_s(X,x)}{\overline{\Phi}_s(X,0)}.$$

Then

$$\begin{aligned} r_s(X,x) &= \frac{\overline{T}_{s-1}(X,x)}{\int_x^\infty \overline{T}_{s-1}(X,u)du} \\ &= -\frac{\frac{d}{dx}\overline{T}_s(X,x)}{T_s(X,x)}. \end{aligned}$$

Clearly,

$$\overline{T}_1(X,x) = \overline{F}(x)$$

and

$$\overline{T}_2(X,x) = \frac{\int_x^\infty \overline{F}(u)du}{\int_0^\infty \overline{F}(u)du}.$$

Here $r_s(X,x)$ and $\overline{T}_s(X,x)$ are respectively the hazard rate function and survival function of the equilibrium distribution of the distribution with survival function $\overline{T}_{s-1}(X,x)$, $s = 2,3,4\ldots$. Clearly, for $s = 1,2$, $r_1(X,x) = r(x)$ and $r_2^{-1}(X,x) = E(X_x)$ are the failure rate function and the mean residual life function of X, respectively. Singh (1989) and Fagiuoli and Pellerey (1993, 1994) have defined the generalized stochastic orders in a general way so that most of the partial orderings discussed earlier become the particular cases of their general orderings.

Definition 1.2.13. *Let $s \in \mathbb{N}$. For two nonnegative random variables X and Y, X is said to be larger than Y in*

(i) s-ST order (denoted by $X \geqslant_{s-ST} Y$) if and only if $\frac{\overline{T}_s(X,x)}{\overline{T}_s(Y,x)} \geqslant \frac{\overline{T}_s(X,0)}{\overline{T}_s(Y,0)}$, for all $x \geqslant 0$ and $s \in \mathbb{N}$;

(ii) s-FR order (denoted by $X \geqslant_{s-FR} Y$) if and only if $r_s(X,x) \leqslant r_s(Y,x)$ for all $x \geqslant 0$ and $s \in \mathbb{N}$.
Or equivalently, if $\overline{T}_s(X,x)/\overline{T}_s(Y,x)$ is increasing in $x \geqslant 0$ for all $s \in \mathbb{N}$.

Fagiuoli and Pellerey (1993) have defined some ageing classes in a general way depending on the generalized orderings so that most of the ageing classes discussed earlier become the particular cases of their general ageing classes. Following is the definition of s-IFR (s-DFR) ageing class.

Definition 1.2.14. *Suppose that $s \in \mathbb{N}$. Then the distribution of X is said to be s-IFR (s-DFR) if and only if $r_s(X,x)$ is increasing (decreasing) in $x \geqslant 0$.*
Or equivalently, if $\overline{\Phi}_s(X,x+t)/\overline{\Phi}_s(X,x)$ is decreasing (increasing) in x for all $x,t \geqslant 0$.

It is easy to see that:

- $X \leqslant_{0-FR} Y \Leftrightarrow X \leqslant_{lr} Y$;
- $X \leqslant_{1-FR} Y \Leftrightarrow X \leqslant_{hr} Y$;
- $X \leqslant_{2-FR} Y \Leftrightarrow X \leqslant_{mrl} Y$;
- $X \leqslant_{3-FR} Y \Leftrightarrow X \leqslant_{vrl} Y$;
- X is 0-IFR (0-DFR) $\Leftrightarrow$ X is ILR (DLR);
- X is 1-IFR (1-DFR) $\Leftrightarrow$ X is IFR (DFR);
- X is 2-IFR (2-DFR) $\Leftrightarrow$ X is DMRL (IMRL);
- X is 3-IFR (3-DFR) $\Leftrightarrow$ X is DVRL (IVRL).

1.2.7 Residual Life at Random Time (RLRT)

Let X be the lifetime of a system/component survived up to a time t, then the residual life (remaining life) of X at a fixed time t is defined as $X_t = (X - t|X > t)$. In literature, a lot of interest has been evoked on the study of residual life at a fixed time. But, in real life applications, sometimes the time is not fixed and it could be random. If t is replaced by a random variable Y then $X_Y = (X - Y|X > Y)$ represents the residual life of X at a random time Y (RLRT). In order to justify the practical importance and usefulness of RLRT, consider the following examples:

(i) Assuming a women underwent a medical test to check whether she has been infected by HIV or not, and the test reveals positive. Now it is unknown when she was infected by HIV. Let Y represent the time of infection. Let us denote the total lifetime of the woman by a random variable X. Hence, it is clear that X is greater than Y. Then X_Y denotes the remaining life of the woman after being infected by HIV to death.

(ii) Consider a two component series system, where the system is made up of components C_1 and C_2 with random lifetimes X and Y, respectively. If C_2 fails before C_1, then the system will fail to work but C_1 may still be in working condition. In this situation, X_Y represents the residual life of C_1 at a time when C_2 fails.

(iii) RLRT is one of the important notions in reliability and queuing theory. In theory of reliability, it represents the actual working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age Y. In a classical GI/G/1 queue, let

T_n: time between nth and $(n+1)$th arrival, $T_n \sim T$

S_n: service time of the nth customer, $S_n \sim S$

W_n: waiting time in the queue of the nth customer, $W_n \sim W$

I: length of the ideal period between busy periods.

The sign $\sim$ is used to signify 'with distribution function'. It is well-known that $W_{n+1} = max\{0, W_n + S_n - T_n\}$ and $I \stackrel{d}{=} ((W + S) - T | W + S > T)$ (cf. Marshall, 1968). Thus, the idle time of a server in a classical GI/G/1 queuing system can be expressed as RLRT.

See, Stoyan (1983), for more details on, and applications of residual life at random time.

Let X and Y be two nonnegative random variables with a common support Θ, and Y have distribution function G. Further, let the distribution and survival functions of X_Y be represented as F_Y and $\overline{F}_Y$, respectively. If X and Y are mutually independent then the distribution and survival functions of X_Y are respectively defined as,

$$F_Y(x) = \frac{\int_\Theta [F(y+x) - F(y)]dG(y)}{\int_\Theta \overline{F}(y)dG(y)}$$

and

$$\overline{F}_Y(x) = \frac{\int_\Theta \overline{F}(x+y)dG(y)}{\int_\Theta \overline{F}(y)dG(y)}.$$

Further, assume that X and Y are not statistically independent. Let $X^\theta \stackrel{d}{=} (X|Y = \theta)$ and $\overline{F}_\theta$ be the survival function of X^θ, $\theta > 0$. In this case, the survival function of X_Y is given by

$$\overline{F}_Y(x) = \frac{\int_\Theta \overline{F}_\theta(x+\theta)dG(\theta)}{\int_\Theta \overline{F}_\theta(\theta)dG(\theta)}.$$

Stochastic comparisons and ageing properties of RLRT have been investigated in the present two decades. Yue and Cao (2000) established a number of stochastic comparison results for X_Y under the assumption that X, Y_1 and Y_2 are independent. They showed that, if X is DFR (IFR) then $X_{Y_1} \leqslant_{st} (\geqslant_{st}) X_{Y_2}$ and if X is IMRL (DMRL) then $E(X_{Y_1}) \leqslant (\geqslant) E(X_{Y_2})$ with the help of the condition $Y_1 \leqslant_{rh} Y_2$. Under the assumption that Y has DRHR, they further showed that if X is in any one of the classes IFR, DFR, DMRL or IMRL then X_Y also belongs to the same class of X. They also obtained some useful bounds for the distribution and the moment of X_Y. Li and Zuo (2004) established increasing convex order (icx) between two RLRT and illustrated that, if $Y_1 \leqslant_{rh} Y_2$ and X is IMRL (DMRL) then $X_{Y_1} \leqslant_{icx} (\geqslant_{icx}) X_{Y_2}$. Li and Xu (2006) obtained a stochastic ordering in two sample problem. It is shown that if Z (independent with X and Y) is IMIT, then $X_Z \leqslant_{mrl} Y_Z$ provided $X \leqslant_{hr} Y$ and further X_Z is DMRL when X is IFR. Misra et al.

(2008) strengthen some of the results of Yue and Cao (2000) and Li and Zuo (2004). They provided conditions on X and Y under which X_Y has log-concave (log-convex) probability density function. They showed that, if $Y_1 \leqslant_{rh} Y_2$ and X has DFR (IFR), then $X_{Y_1} \leqslant_{hr} (\geqslant_{hr}) X_{Y_2}$. They also obtained if $Y_1 \leqslant_{rh} Y_2$ and X is IMRL (DMRL) then $X_{Y_1} \leqslant_{mrl} (\geqslant_{mrl}) X_{Y_2}$. Since *mrl* order implies *icx* order (cf. Shaked and Shanthikumar, 2007), so Misra et al. (2008) have a stronger version of the result of Li and Zuo (2004) discussed earlier. Nanda and Kundu (2009) and Cai and Zheng (2012) consider generalized stochastic orders and ageing classes for RLRT, where they generalized some of the results of Yue and Cao (2000) and Misra et al. (2008). They studied the s-IFR (s-DFR) ageing classes for X_Y under various assumptions on X and Y. Nanda and Kundu (2009) proved that, if $Y_1 \leqslant_{rh} Y_2$ and X has s-DFR (s-IFR), then $X_{Y_1} \leqslant_{s-FR} (\geqslant_{s-FR}) X_{Y_2}$. Cai and Zheng (2012) obtained bounds of the residual life at exponential random time. Dewan and Khaledi (2014) investigated some new stochastic orderings results among RLRTs in one sample as well as two sample problems based on *lr, hr, rh* and *mrl* orders and also provided simpler proofs of some of the results of Misra et al. (2008). Unfortunately, the results concerning *rh* order is erroneous. The concept of RLRT has also been used by Kayid and Izadkhah (2015*a*) to characterize exponential distribution. On the contrary, the similar investigation in the presence of dependent structure between X and Y is relatively new. Misra and Naqvi (2018*a*) have investigated some stochastic order results for RLRT with respect to *lr, hr* and *mrl* orders and also studied the ageing notion of X_Y in terms of *hr* and *mrl* functions, assuming dependency between the system life and random time. Following the same spirit, Li and Fang (2018) also consider the comparisons of RLRTs. They prove some of the stochastic comparison results of Misra and Naqvi (2018*a*) in a different way. Recently, several stochastic orderings and ageing properties of RLRT for a coherent system have been obtained by Amini-Seresht et al. (2020).

1.2.8 Inactivity Time at Random Time (ITRT)

In survival analysis and reliability theory, it is of interest to study the stochastic properties of the inactivity time of either a system or a component or a human organism. Customary study of the inactivity time often concerns the scenario in which the system has failed at a fixed time. However, in reliability engineering, survival analysis, forensic sciences and many other applied areas, researchers often encounter the scenario in which a system or a component has failed at one unknown time point, which could be viewed as some nonnegative random variable. Therefore, it is highly worthwhile to look upon the concept of the inactivity time at a random time (ITRT). In the following we provide certain instances to illustrate the interpretation of ITRT:

(i) Suppose a person has died at an unknown time due to a car accident, and it is also not known when he met with the accident. In such a case the times of accident and death both are random, say X and Y, respectively. Hence, it is clear that Y is greater than or equal to X. So it is important to compute the duration he was alive after committing the accident.

(ii) Suppose a system collapsed at an unknown time. It has been found that the system failed due to attack by a malware, but it is not known when the system was attacked. So if we represent the failure time of the system by a random variable Y and X is the time when the system was attacked by the malware, then, it is known that $X \leqslant Y$. Now the question is- how much time the system had been running after the attack?

(iii) Consider a two component parallel system, where the system runs with the help of any of the components, failed at an unknown time due to failure of both the components. Assume that, the first component failed before the second at a random time X while the second component failed at a random time Y. So the same question arises as before, how much time the system had been running after the failure of the first component?

Let X be the lifetime of a system/component observed to fail at a random time Y, then the inactivity time of X at a random time Y is defined as $X_{(Y)} = (Y - X | X \leqslant Y)$. ITRT was first introduced by Li and Zuo (2004). It is important in forensic science, software reliability testing, and many other applied areas. It is also used in medical science to describe the incubation period of a disease. For details, a person suffering from a disease might not have got it instantly, rather he may have its agent a long time back. Every individual has some immunity power to fight back with disease agents, but the degree to which they have this immunity decides whether the person will have the disease or not. Thus, if the person finally gets the disease then there is a difference between the times of being infected by the disease agent and the time finally having the disease. Discovering these times is a challenging task due to randomness. Let X be the time of being infected by the disease agent and Y be the time of having the disease. Then $X_{(Y)}$ represents the time between infection by the disease agent and the beginning of the disease, which is also known as incubation period or dormant season. Each individual will have a different incubation period based on his immunity power. For example, it is estimated that the incubation period of COVID-19 is between 2 to 14 days, which is random.

Let $F_{(Y)}$ and $\overline{F}_{(Y)}$ be the distribution and survival functions of $X_{(Y)}$, respectively. If X and Y are mutually independent and have a common support Θ, then

$$F_{(Y)}(x) = \frac{\int_\Theta [F(y) - F(y-x)]dG(y)}{\int_\Theta F(y)dG(y)}$$

and

$$\overline{F}_{(Y)}(x) = \frac{\int_\Theta F(y-x)dG(y)}{\int_\Theta F(y)dG(y)}.$$

When X and Y are mutually non-independent then the survival function of $X_{(Y)}$ is given by the following

$$\overline{F}_{(Y)}(x) = \frac{\int_\Theta F_\theta(\theta - x)dG(\theta)}{\int_\Theta F_\theta(\theta)dG(\theta)},$$

where F_θ is the distribution function of $X^\theta \stackrel{d}{=} (X|Y=\theta),\ \theta > 0$.

In a parallel line with RLRT, the stochastic properties of ITRT have also been investigated in the literature by many researchers. Li and Zuo (2004) established a number of stochastic comparisons and ageing properties for $X_{(Y)}$ taking X and Y independent. They showed that, if X is DRHR then $X_{(Y_1)} \leqslant_{st} X_{(Y_2)}$ and, if X is IMIT then $X_{(Y_1)} \leqslant_{icx} X_{(Y_2)}$ under the assumption $Y_1 \leqslant_{hr} Y_2$. They also proved that, if X is IMIT and Y is IFR then $X_{(Y)}$ is DMRL. Misra et al. (2008) strengthen some of the results of Li and Zuo (2004). They provided conditions on X and Y under which $X_{(Y)}$ has log-concave (log-convex) probability density function. It is shown that, if $Y_1 \leqslant_{hr} Y_2$ and X has DRHR (IRHR), then $X_{(Y_1)} \leqslant_{hr} (\geqslant_{hr}) X_{(Y_2)}$. They also showed that, if $Y_1 \leqslant_{hr} Y_2$ and X has IMIT (DMIT), then $X_{(Y_1)} \leqslant_{mrl} (\geqslant_{mrl}) X_{(Y_2)}$. Misra and Naqvi (2017) have performed some stochastic comparisons between two ITRTs with respect to *lr, hr* and *mrl* orders assuming statistical dependence between the system lifetime and random time. They also investigated the DFR, IFR, IMRL and DMRL ageing properties of ITRT. Recently, Amini-Seresht et al. (2020) have investigated several stochastic orderings and ageing properties of ITRT for a coherent system.

It is noteworthy that $X_Y = Y_{(X)}$ with probability one for continuous distributions. In view of this fact, for all independent random variables X and Y, each result for either RLRT or ITRT can be translated into a result for the other by exchanging the roles of X and Y.

1.2.9 Residual Lifetime Mixture Model (RLMM)

In statistics, mixture models have a great role in the analysis of data due to their flexibility for modeling a wide variety of random phenomena. For analyzing a data set when each observation comes from one specific distribution, we often make some modeling assumptions. However, in many cases, it is restrictive and may not make intuitive sense to assume each sample comes from the same distribution. For example, suppose a data set has been collected on the death of patients, who had cardiovascular disease, due to COVID-19 for everyday in April 2020 without regard to age or country. If the ignored variables (age or country) have a bearing on the characteristic being measured, then the data are said to come from a mixture. Consider another example, suppose one is interested in simulating the price of a randomly chosen book from a book store. It might make sense to model the price of paperback books separately from hardback books since paperback books are typically cheaper than hardbacks. In this example, the price of a book will be modelled as a mixture model. Here we have two mixture components: one is for hardback books, and another is for paperback books. In practical situations, it is hard to find data that are not some kind of a mixture, because there is almost always some relevant covariate that is not observed. Let $\mathcal{F} = \{F(\cdot|\theta) : \theta \in \chi\}$ be a family of distributions indexed by a parameter θ. When θ is considered as a random variable Y with a distribution function G, then the distribution function

$$F^Y(x) = \int_{\chi} F(x|\theta)dG(\theta),$$

is the mixture of $\mathcal{F}$ with respect to G, and G is called the mixing distribution. Formally, a mixture model corresponds to the mixture distribution that represents the probability distribution of observations in the overall population. Now, the corresponding survival function is given by

$$\overline{F}^Y(x) = \int_{\chi} \overline{F}(x|\theta)dG(\theta).$$

A mixture distribution is applied when a population contains two or more subpopulations and the population is found to be heterogeneous. In general, a mixture distribution is a mixture of more than one probability distributions when random variables are drawn from more than one populations. However, the distributions can be made up from the same family of distribution with different parameters. The population can be univariate or multivariate but the distributions must be of same dimensions and all must either be discrete or continuous. The mixture distributions have various applications in different fields such as agriculture, medicine, finance, epidemiology, electrophoresis, life-testing, communication, fisheries research and so on. Some interesting examples of mixture distributions are given below:

(i) Suppose that in a certain part of the world, different kinds of epidemics such as dengue, cholera and COVID-19, have been spread out simultaneously. Each of these three epidemics has its own characteristics and therefore each epidemics have a different distribution (Poisson). Thus, the population attacked by these epidemics is a mixture distribution (mixture of Poisson distributions).

(ii) Consider a herd in a forest where elephants of different ages have different characteristics and particular age group of elephants have its own distribution. It is not possible to ascertain the age of each individual elephants in this case, therefore the population is considered as a whole. Thus, the natural population of elephant in the forest is a mixture of the populations.

Recall that, the residual lifetime of X at age t (> 0) is defined as $X_t = (X - t|X > t)$. Then the corresponding conditional survival function of X_t is given by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}.$$

It is known that in many practical circumstances the age parameter t may not be constant due to various reasons, and the occurrence of heterogeneity is sometimes unpredictable and unexplained. The heterogeneity sometimes may not be possible to be neglected. In reliability engineering applications, one encounters situations where used components of some random past age are incepted into operation as spares. Here random past age of a spare is described by a random variable. Consider a population of used devices, which are still in working conditions, with different ages t_1, t_2 and t_3, say. So the population is represented as mixture where these three types of used devices with different ages are combined. Here the age parameters exhibit random behaviour and are considered as random variable as it varies from one used device to another one. Suppose that the random behaviour of the age is described by a random variable Y with distribution function G. For simplicity, assume that the support of Y is also $[0, \infty)$. To account the influence of the random ages on the residual lifetime distribution and to handle the heterogeneity of the age parameter t in residual lifetime family of distributions, Kayid and Izadkhah (2015b) introduced the concept of residual lifetime mixture model (RLMM) (also called extended mixture model) with survival function

$$\overline{F}^Y(x) = \int_0^\infty \frac{\overline{F}(x+y)}{\overline{F}(y)} dG(y), \tag{1.2.3}$$

which can be interpreted as the average survival probability of X_t with respect to the random age Y. Denote by X^Y, the random variable that has the survival function (1.2.3) with baseline random variable X and mixing random age Y. Then, the distribution function of X^Y is given by

$$F^Y(x) = \int_0^\infty \frac{F(x+y) - F(y)}{\overline{F}(y)} dG(y).$$

The random variable X^Y can be used to study the residual lifetimes of the devices in the total population. Consider a company that sells second hand aircraft of different models. Suppose that used aircrafts of a model (say, Model 1) that are put on sale has a random past life, the time for which this model aircrafts have been used before being put on sale. Let the generic life of this model aircrafts are also random. Now, for a buyer, who wants to buy a second hand aircraft in this model is interested to know the remaining life/residual life of the aircrafts of this model.

In the literature, the notion of residual lifetime mixture model (1.2.3) has been considered as residual life at random age (RLRA), see for example, Finkelstein (2002*b*), Finkelstein and Vaupel (2015), Hazra et al. (2017), Cha and Finkelstein (2018) and Li and Fang (2018). It is crucial for life tables, and widely used in demography and actuarial applications.

Recently, a great attention has been paid on stochastic comparisons and ageing notions of mixture model due to its significant applications in risk theory, reliability and various areas of applied probability and engineering. If mixture models have the same/different kind of mixing distributions and same as well as different generic distributions, then it might be of interest to compare the residual lifetimes of these models. Under the egis of RLRA, stochastic comparisons were investigated by Finkelstein and Vaupel (2015) and Cha and Finkelstein (2018). Hazra et al. (2017) studied stochastic comparisons for the random age and the remaining lifetime based on *st, hr, lr* and *mrl* orders, by choosing same/different generic distributions and same/different random ages. They also studied DFR, DLR and IMRL ageing classes. Li and Fang (2018) discussed stochastic comparisons for the residual lifetime at random ages in the context of statistical dependence between the system lifetime and the random ages based on *st, hr* and *lr* orders, by choosing same/different generic distributions and same/different random ages. Kayid and Izadkhah (2015*b*) discussed stochastic comparisons for the mixtures of residual life distributions based on *st, hr, lr, rh* and *mrl* orders. In addition to this, DFR, DLR and IMRL ageing classes for this model has also been studied by them. More recently, following the same spirit, Misra and Naqvi (2018*b*) further provided stochastic comparisons of RLMM based on *lr, hr, rh* and *mrl* orders, by choosing different baseline distributions and different mixing distributions.

The study on RLMM was inspired by the concept of RLRT, where the random time is considered as random age because it comes from a mixture of a population. One natural question may therefore arise, what is the difference between RLRT and RLMM/RLRA? Typically, RLRT and RLMM are based on different mechanisms, and this point is elucidated in the following example. Consider the case of kidney recipient. Suppose the random variable Y denotes the time for which the kidney have been used (initial age) before being transplanted and X is the total life of the kidney. Now, if a kidney is directly taken from a known source, then the remaining lifetime of that kidney would be defined by RLRT X_Y. But, when the kidney is purchased (picked up at random) from a lot/mixture of a selling foundation, then the remaining lifetime of the kidney would be defined by RLMM/RLRA X^Y. RLRT and RLMM are mainly different in the probability mechanism of the unknown random time. In the former, researchers take the following viewpoint: a system and one associated instrument (either observable or not) start to operate at the same time, and the system obtains the remaining lifetime, coined RLRT, if the instrument fails before the system. In the later, the probability mechanism is addressed in the following way: from a large number of statistically identical objects undergoing an ageing process and thus having different ages (usually unobservable), one object is randomly selected and attains the remaining lifetime, which is called RLMM. For a detailed discussion on the connection between RLRT and RLMM one may refer to Li and Fang (2018).

1.3 The Aim of the Thesis Work

The notions of stochastic orders and the classes of life distributions have a long, exciting and stormy history. Having played important roles in reliability theory, survival analysis, maintenance policies, and many other areas of applied probability, both stochastic orders and the classes of life distributions received much attention during the last few decades. In the literature, several stochastic orders and the classes of life distributions are defined. Here we consider only those stochastic orders and life distribution classes which are based on residual life and inactivity time. These stochastic orders are mainly based on the comparison of some measures associated with the residual life and inactivity time random variables. In the past few decades, the literature on stochastic orders and the (ageing) classes of life distributions for residual life and inactivity time at a fixed time has grown quite voluminous. But, less works seem to have been done for RLRT (including RLMM) and ITRT. The literature survey done so far reveals that there is a good scope to carry out further research on several problems of orderings and ageing properties of RLRT, ITRT and RLMM. Stochastic comparisons and ageing notions of RLRT, RLMM and ITRT have received much attention in the last two decades. For RLRT and ITRT, the stochastic comparisons are conducted under certain conditions on the concerned total life and the random time. Most of these comparisons have been made with respect to *st*, *hr*, *lr* and *mrl* orders for one and/or two sample problems. Some stochastic comparison results in terms of s-FR ordering have also been obtained for RLRT. Furthermore, several ageing properties have been investigated for RLRT (viz. ILR, IFR, IMRL, s-IFR and their dual classes). However, for ITRT, the IFR, DFR, IMRL and DMRL properties have only been discussed. In addition, stochastic comparisons of two different RLMMs having different mixing distributions have been performed with respect to *st*, *hr*, *rh*, *lr* and *mrl* orders. Preservation of some ageing classes has also been looked into for this model. So far, to the best of our knowledge, no work has been investigated related to variablity/dispersion measures for RLRT, ITRT and RLMM. Moreover, the results available in the literature are not sufficient for stochastic comparisons of RLRT and/or ITRT in two sample problem (under independence/dependence structure) based on *rh*, *mit* and s-FR orderings, and their associated classes of life distributions. In this thesis, we shall mainly be concerned with the study of stochastic properties of RLRT (including RLMM) and ITRT. Here we extend, generalize and unify the stochastic orders and the (ageing) classes of life distributions for RLRT, ITRT and RLMM in connection with residual life, inactivity time and variability measures.

In this thesis, we have two goals. Firstly, we think it best to confine ourselves to perform a detailed treatment of stochastic comparisons and (ageing) classes of life distributions for RLRT and ITRT. We develop a theoretical frame work for stochastic properties of RLRT and ITRT under independent and dependent scenarios. We also apply various (generalized) stochastic orderings for the comparisons of RLRTs and ITRTs, and study the notion of life distribution classes. The second goal is to consider the residual lifetime mixture models having different base line distributions as well as different mixing distributions. We investigate preservation of several stochastic orders (and ageing properties) under this model at the disposal of the highest variability in the components of the model. We also look into the potential applications of the results derived in this thesis. The following section will sketch the investigation of the present thesis.

1.4 A Brief Discussion on the Studied Issues

The present thesis is organized into seven chapters of which **Chapter 1** is introductory. In this chapter a concise survey of the literature concerned with the topic and motivation of the thesis have been provided. In the remaining part, the topic is expanded in the surcharged atmosphere of new arrivals of research articles on residual life and inactivity time at random time. In this section, a brief contribution of the rest of the chapters has been highlighted.

In **Chapter 2**, we enhance the study of residual life at random time and inactivity time at random time taking independency between the generic life and random time. We provide some further results on stochastic comparisons of RLRTs and ITRTs. First, we discuss some stochastic orderings results for ITRT in two sample problems when they are observed to fail at two different random times. In particular, first we focus on stochastic comparisons with respect to *lr, hr* and *mrl* orders, and then *vrl* order. Some preliminary study on VIT order and IVIT class have also been made. Then, with the help of some sufficient conditions, we compare two RLRTs or ITRTs based on *vrl* order. The DVRL ageing class has also been studied for RLRT. The work reported here has been published in *Communications in Statistics- Theory & Methods.*

In the literature, lot of work has been done for RLRT and ITRT, where two RLRTs or ITRTs have been compared based on *st, hr, mrl* and *lr* orders. In **Chapter 3**, first we investigate some more stochastic ordering results for RLRT, where we perform stochastic comparisons of X_{Y_1} and X_{Y_2}, the residual lives of X at the different random times Y_1

and Y_2, respectively. With the help of some sufficient conditions we compare them based on *rh, mit* and *vit* orders. Then, we compare two ITRTs of the same random variable X having different random times Y_1 and Y_2 based on *rh, mit* and *vit* orders under the assumption that X and Y_1 (or Y_2) are statistically independent. Finally, DRHR, IMIT and IVIT classes of life distributions for ITRT have been studied. One paper based on this chapter has been published in *Communications in Statistics- Theory & Methods.*

Stochastic comparisons and ageing properties of RLRT/ITRT taking independency between generic lifetime and random time have been investigated in Chapters 2-3. But, they may not necessarily be independent. In **Chapter 4**, we consider non-independency between these random variables and obtain stochastic properties of RLRT and ITRT based on variance residual life. To this aim, first we deal with some stochastic comparison results on RLRT/ITRT in one sample problem. By assuming *hr* order between Z_1 and Z_2 we compare two ITRTs, $X_{(z_1)}$ and $X_{(Z_2)}$, as well as two RLRTs, X_{Z_1} and X_{Z_2} based on *vrl* order. We also compare these two RLRTs and ITRTs in terms of *vrl* order by assuming *rh* order between Z_1 and Z_2. Then, we provide stochastic comparisons of two systems failed at two different random times or having different random ages based on *vrl* order. We also study various ageing notions of RLRT and ITRT based on IVRL and DVRL ageing classes. Some applications of the results derived in this chapter are also illustrated. One article, containing the work discussed here, has been accepted in *Communications in Statistics- Theory & Methods.* Online First.

In **Chapter 5**, we consider generalized stochastic ordering (s-FR) and preservation of some generalized ageing classes (viz. s-IFR, s-DFR) for RLRT and ITRT, where s is a nonnegative integer. First, we carry out stochastic comparisons of RLRTs and ITRTs under s-FR ordering in two sample problems having different random ages or observed to fail at same/different random times. Later, we discuss some new properties of s-DFR (s-IFR) ageing class on the RLRT. The results are interesting in the sense that they give some existing results with less sufficient conditions. We study the preservation properties of s-DFR class for RLRT, where we show if X is s-DFR then so is X_Y. This theorem strengthen the theorem of Cai and Zheng (2012) for s-DFR class in the sense that the extra DRHR property on Y has been relaxed here. A natural question, therefore, may arise, whether the DRHR property on Y for s-IFR class can be relaxed. We address this question through an example. For the converse, we show that if Y is DRHR and X_Y is s-DFR then X is s-DFR. Further, it is shown that ILR (DLR) ageing class is preserved for RLRT and ITRT. Finally, we provide an application in reliability theory, where we compare the lifetimes of two parallel systems. The results strengthen some results available in the literature. Part of the work done in this chapter has been published in *Metrika*.

In **Chapter 6**, we enrich the study of stochastic comparisons and ageing properties for residual lifetime mixture models. To this aim, first we provide few simple characterization results and then compare two different mixture models having different baseline distributions as well as two different mixing distributions based on *lr, hr, mrl* and *vrl* orders. Later, we develop some sufficient conditions which lead to the stochastic comparisons of these mixture models in terms of *rh, mit* and *vit* orders. Furthermore, under the formation of the proposed model we show that ILR, IFR, DMRL, IVRL and DVRL classes are preserved. Some examples to illustrate the applications of the results derived in this chapter to guaranteed lead times and series system are also investigated. A manuscript based on this chapter has been accepted in *Mathematical Methods of Operations Research*.

Finally, in **Chapter 7**, we summarize the major conclusion of the present study along with some possibilities of future research followed by a list of relevant references. It is to be mentioned here that a list of papers published/communicated based on the thesis is also attached at the end.

In order to make each chapter as independent as possible, some useful results may be found to repeated in the upcoming chapters. Also, a few definitions of stochastic orders and (ageing) classes of life distributions have been reproduced with their equivalent form(s).

Chapter 2

Some Stochastic Comparison Results for Residual Life and Inactivity Time at Random Time [1]

In this chapter, we enhance the study of residual life at random time (RLRT) and inactivity time at random time (ITRT). To this aim, first we provide some stochastic ordering results among ITRT in two sample problems when they fail at two different random times. Then, we develop some sufficient conditions which lead to the stochastic comparisons of RLRTs and ITRTs based on variance residual life order. The results are expected to be useful in reliability theory, forensic science, queuing theory and actuarial science.

2.1 Introduction

In recent decades the stochastic comparison of random variables has received much attention due to its important role in reliability theory, life testing, actuarial science, and many other areas of applied probability. The comparison of two random quantities and conceptions of ageing are also one of the main objectives of statistics. For details on

[1] The results discussed in this chapter have been published in *Communications in Statistics- Theory & Methods*, 2018, 47(2), 372-384.

various stochastic orders and ageing classes one may refer to the famous books by Shaked and Shanthikumar (2007), Belzunce et al. (2015), Müller and Stoyan (2002), and Barlow and Proschan (1981), among others.

Let the random variable X denote the lifetime of a unit, having an absolutely continuous distribution function F, survival function $\overline{F} = 1 - F$ and probability density function f. Let $X_t = (X - t|X > t)$ be the residual life of X at time $t > 0$ and $X_{(t)} = (t - X|X \leqslant t)$ be the inactivity time (IT) at time $t > 0$. Their respective reliability functions are given by

$$P(X_t > x) = \frac{\overline{F}(x+t)}{\overline{F}(t)} \quad \text{and} \quad P(X_{(t)} > x) = \frac{F(t-x)}{F(t)}, \quad x, t \geqslant 0.$$

So, the mean residual life (MRL) $m_X(t)$ and variance residual life (VRL) $\sigma_X^2(t)$ of X can be defined as

$$m_X(t) = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)}, \; 0 \leqslant t \leqslant x \;\text{ and }\; \sigma_X^2(t) = E[(X-t)^2|X > t] - [m_X(t)]^2, \; t \geqslant 0.$$

Similarly, the mean inactivity time (MIT) and variance inactivity time (VIT) of X are defined as

$$\overline{m}_X(t) = \frac{\int_0^t F(x)dx}{F(t)}, \; 0 \leqslant x \leqslant t \;\text{ and }\; \overline{\sigma}_X^2(t) = E\left[(t-X)^2|X \leqslant t\right] - [\overline{m}_X(t)]^2, \; t > 0.$$

Let X and Y be two nonnegative independent random variables. The residual life of X at a random time Y (RLRT) is denoted by X_Y and is defined as $X_Y = (X - Y|X > Y)$. The RLRT is one of the important notions in reliability and queuing theory. It represents the actual working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age Y, and the idle time of a server in a classical GI/G/1 queuing system can also be expressed as a RLRT (see Marshall, 1968). The inactivity time at a random time Y (ITRT) is denoted by $X_{(Y)}$ and is defined as $X_{(Y)} = (Y - X|X < Y)$. Stochastic comparison results and ageing properties of residual life and inactivity time at a random time have been investigated by Yue and Cao (2000), Li and Zuo (2004), Misra et al. (2008), Cai and Zheng (2012) and Dewan and Khaledi (2014). For some discussions on VRL and VIT, one may refer to Gupta (2006), Mahdy (2012), Kayid and Izadkhah (2016). In this chapter, we mainly focus our attention to obtain some results with the help of monotonicity, ordering, and the associated ageing classes of life distributions of VRL and VIT. Let F_Y and $F_{(Y)}$ be the distribution functions of X_Y and $X_{(Y)}$, respectively. If X and Y are mutually independent and have a common support Θ, then their survival functions are, respectively,

$$\overline{F}_Y(x) = \frac{\int_\Theta \overline{F}(x+y)dG(y)}{\int_\Theta \overline{F}(y)dG(y)}$$

or equivalently,

$$\overline{F}_Y(x) = \frac{\int_\Theta G(y-x)dF(y)}{\int_\Theta G(y)dF(y)}$$

and

$$\overline{F}_{(Y)}(x) = \frac{\int_\Theta F(y-x)dG(y)}{\int_\Theta F(y)dG(y)}$$

or equivalently,

$$\overline{F}_{(Y)}(x) = \frac{\int_\Theta \overline{G}(y+x)dF(y)}{\int_\Theta \overline{G}(y)dF(y)}.$$

We first recall the definitions of some stochastic orders that will be used in the sequel.

Definition 2.1.1. *For two random variables X and Y, X is said to be smaller than Y in*

(a) usual stochastic order (denoted by $X \leqslant_{st} Y$) if $\overline{F}(x) \leqslant \overline{G}(x)$ for all $x \geqslant 0$;

(b) hazard rate order (denoted by $X \leqslant_{hr} Y$) if $\overline{F}(x)/\overline{G}(x)$ is decreasing in $x \geqslant 0$;

(c) reverse hazard rate order (denoted by $X \leqslant_{rh} Y$) if $F(x)/G(x)$ is decreasing in $x \geqslant 0$;

(d) likelihood ratio order (denoted by $X \leqslant_{lr} Y$) if $f(x)/g(x)$ is decreasing in $x \geqslant 0$;

(e) mean residual life order (denoted by $X \leqslant_{mrl} Y$) if $m_X(t) \leqslant m_Y(t)$ for all $t \geqslant 0$, or equivalently,

$$\frac{\int_t^\infty \overline{F}(u)du}{\int_t^\infty \overline{G}(u)du} \text{ is decreasing in } t \geqslant 0;$$

(f) mean inactivity time order (denoted by $X \leqslant_{mit} Y$) if $\overline{m}_X(t) \geqslant \overline{m}_Y(t)$ for all $t \geqslant 0$, or equivalently,

$$\frac{\int_0^t F(u)du}{\int_0^t G(u)du} \text{ is decreasing in } t \geqslant 0;$$

(g) *variance residual life order (denoted by $X \leqslant_{vrl} Y$) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \leqslant \frac{\int_t^\infty \int_x^\infty \overline{G}(u)dudx}{\overline{G}(t)},$$

or equivalently,

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \int_x^\infty \overline{G}(u)dudx} \text{ is decreasing in } t \geqslant 0;$$

(h) *variance inactivity time order (denoted by $X \leqslant_{vit} Y$) if*

$$\frac{\int_0^t \int_0^x F(u)dudx}{F(t)} \geqslant \frac{\int_0^t \int_0^x G(u)dudx}{G(t)},$$

or equivalently,

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t \int_0^x G(u)dudx} \text{ is decreasing in } t \geqslant 0;$$

(i) *increasing convex order (denoted by $X \leqslant_{icx} Y$) if $\int_x^\infty \overline{F}(u)du \leqslant \int_x^\infty \overline{G}(u)du$.*

The following ageing classes are closely related to our discussion.

Definition 2.1.2. *A random variable X is said to have an*

(a) *increasing (resp. decreasing) failure rate (IFR (resp. DFR)) if X_t is stochastically decreasing (resp. increasing) in $t \geqslant 0$;*

(b) *increasing (resp. decreasing) mean residual life (IMRL (resp. DMRL)) if $m_X(t)$ is increasing (resp. decreasing) in $t \geqslant 0$;*

(c) *increasing mean inactivity time (IMIT) if $\overline{m}_X(t)$ is increasing in $t \geqslant 0$;*

(d) *increasing (resp. decreasing) variance residual life (IVRL (resp. DVRL)) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \overline{F}(x)dx} \text{ is increasing (resp. decreasing) in } t \geqslant 0,$$

or equivalently,

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0;$$

(e) *increasing variance inactivity time (IVIT) if*

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t F(x)dx} \text{ is increasing in } t \geqslant 0,$$

or equivalently,

$$\frac{\int_0^t \int_0^x F(u)dudx}{F(t)} \text{ is increasing in } t \geqslant 0.$$

In the present chapter, we provide some further results on stochastic comparisons of RLRTs and ITRTs. In Section 2.2, we make stochastic comparisons between $(X_1)_{(Y_1)}$ and $(X_2)_{(Y_2)}$, the IT of two systems failed at two different random times. We also show that if Y has DFR and X has IMIT then $X_{(Y)}$ has IMRL. In Section 2.3, we strengthen some of the results of Li and Xu (2006) on *mit* order to *vit* order. We also establish *vrl* order between X_Z and Y_Z and also between $X_{(Z)}$ and $Y_{(Z)}$ under the assumptions that Z possesses a specified ageing property and X, Y are ordered with respect to a certain stochastic order. We also prove that if X is IFR and Z is IVIT, then X_Z is DVRL.

2.2 Stochastic Comparisons and Ageing Properties for ITRT

At first, we briefly review the stochastic comparisons of IT of two systems observed to fail at the same random time. In order to make the presentation self-contained, we restate the following theorem. The proof is a simple consequence of the results available in the literature. It has been pointed out by Li and Zuo (2004) that each result for either RLRT or ITRT, in view of $X_Y = Y_{(X)}$ for continuous distributions, can be translated into a result for the other by exchanging the roles of X and Y.

Theorem 2.2.1. *Let X and Y be two nonnegative random variables representing the lifetimes of two systems failed at random time Z. Let Z be independent of X and Y. If*

i. *$X \leqslant_{rh} Y$ and Z is DFR (IFR), then $X_{(Z)} \leqslant_{hr} (\geqslant_{hr}) Y_{(Z)}$;*

ii. *$X \leqslant_{rh} Y$ and Z is IMRL (DMRL), then $X_{(Z)} \leqslant_{mrl} (\geqslant_{mrl}) Y_{(Z)}$;*

iii. $X \leqslant_{lr} Y$ *and* Z *is ILR (DLR), then* $X_{(Z)} \leqslant_{lr} (\geqslant_{lr}) Y_{(Z)}$.

Now we consider the stochastic comparisons of IT of two systems observed to fail at two different random times. Before discussing the theorem, we provide the following lemma in line with Dewan and Khaledi (2014). First, recall from Karlin (1968) that a nonnegative function $\psi : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, the set of real numbers, is said to be TP$_2$ (*totally positive of order 2*) if $\psi(x,y)\psi(x^*,y^*) \geqslant \psi(x,y^*)\psi(x^*,y)$ for all $x, x^* \in \mathbb{X}$ and $y, y^* \in \mathbb{Y}$ such that $x \leqslant x^*$ and $y \leqslant y^*$, where $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the real line. ψ is said to be RR$_2$ (*reverse regular of order 2*) if the inequality is reversed. For proving the lemma, we use the following result of Shaked and Shanthikumar (2007).

Proposition 2.2.1. *If* $X \leqslant_{st} Y$ *and* ϕ *is any increasing (decreasing) function, then* $E[\phi(X)] \leqslant (\geqslant) E[\phi(Y)]$.

Lemma 2.2.1. *Let* $h_i(x,\theta), i = 1, 2,$ *be a nonnegative real valued function on* $\mathbb{R} \times \mathbb{X}$ *and* $l_i(\theta)$ *be a nonnegative real valued function on* $\mathbb{X}$*, where* $\mathbb{X}$ *is a subset of real line. If*

(i) $\frac{h_2(x,\theta)}{h_1(x,\theta)}$ *is increasing in* x *and* θ,

(ii) $\frac{l_2(\theta)}{l_1(\theta)}$ *is increasing in* θ*, and*

(iii) if either $h_1(x,\theta)$ *or* $h_2(x,\theta)$ *is TP*$_2$ *in* (x,θ),

then

$$s_i(x) = \int_{\mathbb{X}} h_i(x,\theta) l_i(\theta) d\theta$$

is TP$_2$ *in* (i,x)*, where* l_i *is a continuous function with* $\int_{\mathbb{X}} l_i(\theta) d\theta < \infty$.

Proof: First, we prove the required result when $h_1(x,\theta)$ is TP$_2$ in (x,θ). Let $\Theta^*(X)$ denote a random variable with density function given by

$$\frac{h_1(x,\theta) l_1(\theta)}{\int_{\mathbb{X}} h_1(x,\theta) l_1(\theta) d\theta}.$$

Then, the assumption (iii) is equivalent to the fact that $\Theta^*(x_1) \leqslant_{lr} \Theta^*(x_2)$ for $x_1 \leqslant x_2$, which in turn implies that $\Theta^*(x_1) \leqslant_{st} \Theta^*(x_2)$. Let $x_1 \leqslant x_2$, then

$$\begin{aligned}
\frac{s_2(x_2)}{s_1(x_2)} &= \frac{\int_{\mathbb{X}} h_2(x_2,\theta)l_2(\theta)d\theta}{\int_{\mathbb{X}} h_1(x_2,\theta)l_1(\theta)d\theta} \\
&= \int_{\mathbb{X}} \frac{h_2(x_2,\theta)l_2(\theta)}{h_1(x_2,\theta)l_1(\theta)} \frac{h_1(x_2,\theta)l_1(\theta)}{\int_{\mathbb{X}} h_1(x_2,\theta)l_1(\theta)d\theta} d\theta \\
&\geqslant \int_{\mathbb{X}} \frac{h_2(x_2,\theta)l_2(\theta)}{h_1(x_2,\theta)l_1(\theta)} \frac{h_1(x_1,\theta)l_1(\theta)}{\int_{\mathbb{X}} h_1(x_1,\theta)l_1(\theta)d\theta} d\theta \\
&\geqslant \int_{\mathbb{X}} \frac{h_2(x_1,\theta)l_2(\theta)}{h_1(x_1,\theta)l_1(\theta)} \frac{h_1(x_1,\theta)l_1(\theta)}{\int_{\mathbb{X}} h_1(x_1,\theta)l_1(\theta)d\theta} d\theta \\
&= \frac{\int_{\mathbb{X}} h_2(x_1,\theta)l_2(\theta)d\theta}{\int_{\mathbb{X}} h_1(x_1,\theta)l_1(\theta)d\theta} \\
&= \frac{s_2(x_1)}{s_1(x_1)},
\end{aligned}$$

where the first inequality above follows on using assumption (i) that $\frac{h_2(x,\theta)}{h_1(x,\theta)}$ is increasing in θ for each x, assumption (ii) and Proposition 2.2.1. Also, the second inequality follows from the assumption (i) that $\frac{h_2(x,\theta)}{h_1(x,\theta)}$ is increasing in x for each θ. Hence $\frac{s_2(x)}{s_1(x)}$ is increasing in x, which implies that $s_i(x)$ is TP_2 in (i,x). Again, let $h_2(x,\theta)$ be TP_2 in (x,θ). Also, let $\Psi^*(X)$ denote a random variable with density function given by

$$\frac{h_2(x,\theta)l_2(\theta)}{\int_{\mathbb{X}} h_2(x,\theta)l_2(\theta)d\theta}.$$

Then, the assumption (iii) is equivalent to the fact that $\Psi^*(x_1) \leqslant_{lr} \Psi^*(x_2)$ for $x_1 \leqslant x_2$, which implies that $\Psi^*(x_1) \leqslant_{st} \Psi^*(x_2)$. Let $x_1 \leqslant x_2$, then

$$\begin{aligned}
\frac{s_1(x_2)}{s_2(x_2)} &= \frac{\int_{\mathbb{X}} h_1(x_2,\theta)l_1(\theta)d\theta}{\int_{\mathbb{X}} h_2(x_2,\theta)l_2(\theta)d\theta} \\
&= \int_{\mathbb{X}} \frac{h_1(x_2,\theta)l_1(\theta)}{h_2(x_2,\theta)l_2(\theta)} \frac{h_2(x_2,\theta)l_2(\theta)}{\int_{\mathbb{X}} h_2(x_2,\theta)l_2(\theta)d\theta} d\theta \\
&\leqslant \int_{\mathbb{X}} \frac{h_1(x_2,\theta)l_1(\theta)}{h_2(x_2,\theta)l_2(\theta)} \frac{h_2(x_1,\theta)l_2(\theta)}{\int_{\mathbb{X}} h_2(x_1,\theta)l_2(\theta)d\theta} d\theta \\
&\leqslant \int_{\mathbb{X}} \frac{h_1(x_1,\theta)l_1(\theta)}{h_2(x_1,\theta)l_2(\theta)} \frac{h_2(x_1,\theta)l_2(\theta)}{\int_{\mathbb{X}} h_2(x_1,\theta)l_2(\theta)d\theta} d\theta \\
&= \frac{\int_{\mathbb{X}} h_1(x_1,\theta)l_1(\theta)d\theta}{\int_{\mathbb{X}} h_2(x_1,\theta)l_2(\theta)d\theta} \\
&= \frac{s_1(x_1)}{s_2(x_1)},
\end{aligned}$$

where the above first inequality follows on using assumptions (i), (ii) and Proposition 2.2.1.

Also, the second inequality follows from the assumption (i). Hence $\frac{s_1(x)}{s_2(x)}$ is decreasing in x. Or equivalently, $\frac{s_2(x)}{s_1(x)}$ is increasing in x, which implies that $s_i(x)$ is TP_2 in (i, x).

Theorem 2.2.2. *Let X_1, Y_1 and X_2, Y_2 be independent nonnegative random variables. Denote $(X_i)_{(Y_i)}$ the IT of X_i at random time Y_i, $i = 1, 2$. Suppose that $X_1 \leqslant_{lr} X_2$.*

i. If $Y_1 \leqslant_{hr} Y_2$ and either Y_1 or Y_2 is DFR, then $(X_1)_{(Y_1)} \leqslant_{hr} (X_2)_{(Y_2)}$;

ii. If $Y_1 \leqslant_{mrl} Y_2$ and either Y_1 or Y_2 is IMRL, then $(X_1)_{(Y_1)} \leqslant_{mrl} (X_2)_{(Y_2)}$;

iii. If $Y_1 \leqslant_{lr} Y_2$ and either Y_1 or Y_2 is ILR, then $(X_1)_{(Y_1)} \leqslant_{lr} (X_2)_{(Y_2)}$.

Proof: *(i)* Y_i is DFR iff $\overline{G}_i(y + x)$ is TP_2 in $(x, y) \in (0, \infty) \times (0, \infty)$. On the other hand, $Y_1 \leqslant_{hr} Y_2$ iff $\frac{\overline{G}_2(u)}{\overline{G}_1(u)}$ is increasing in $u > 0$ which in turn implies that $\frac{\overline{G}_2(y+x)}{\overline{G}_1(y+x)}$ is increasing in $y > 0$ as well as $x > 0$, and $X_1 \leqslant_{lr} X_2$ iff $\frac{f_2(x)}{f_1(x)}$ is increasing in $x > 0$. Hence the conditions of Lemma 2.2.1 are satisfied by replacing the functions $l_i(\theta)$ with $f_i(x)$ and $h_i(x, \theta)$ with $\overline{G}_i(y + x)$, $i = 1, 2$. Hence from Lemma 2.2.1, we have

$$\int_y^\infty \overline{G}_i(y + x) f_i(x) dx \text{ is } \mathrm{TP}_2 \text{ in } (i, y) \in \{1, 2\} \times (0, \infty).$$

Or equivalently,

$$\frac{\int_y^\infty \overline{G}_2(y + x) dF_2(x)}{\int_y^\infty \overline{G}_1(y + x) dF_1(x)} \text{ is increasing in } y > 0.$$

Hence $(X_1)_{(Y_1)} \leqslant_{hr} (X_2)_{(Y_2)}$.

(ii) Let $\overline{F}^{(i)}_{(Y_i)}$ be the survival function of the random variable $(X_i)_{(Y_i)}$. Y_i is IMRL iff

$$\int_{x+y}^\infty \overline{G}_i(u) du \text{ is } \mathrm{TP}_2 \text{ in } (x, y) \in (0, \infty) \times (0, \infty). \tag{2.2.1}$$

On the other hand $Y_1 \leqslant_{mrl} Y_2$ implies

$$\frac{\int_{x+y}^\infty \overline{G}_2(u) du}{\int_{x+y}^\infty \overline{G}_1(u) du} \text{ is increasing in } x > 0 \text{ and } y > 0. \tag{2.2.2}$$

Also, $X_1 \leqslant_{lr} X_2$ iff

$$\frac{f_2(x)}{f_1(x)} \text{ is increasing in } x > 0. \tag{2.2.3}$$

Therefore, on using (2.2.1), (2.2.2) and (2.2.3) in Lemma 2.2.1, we have

$$\int_0^\infty \left(\int_{x+y}^\infty \overline{G}_i(u)du \right) f_i(x)dx \text{ is TP }_2 \text{ in } (i,y) \in \{1,2\} \times (0,\infty).$$

This gives that

$$\frac{\int_0^\infty \left(\int_y^\infty \overline{G}_i(u+x)du \right) f_i(x)dx}{\int_0^\infty \overline{G}_i(x) f_i(x)dx} \text{ is TP}_2 \text{ in } (i,y) \in \{1,2\} \times (0,\infty),$$

or equivalently,

$$\int_y^\infty \frac{\int_0^\infty \overline{G}_i(u+x) f_i(x)dx}{\int_0^\infty \overline{G}_i(x) f_i(x)dx} du \text{ is TP}_2 \text{ in } (i,y) \in \{1,2\} \times (0,\infty).$$

Hence

$$\int_y^\infty \overline{F}^{(i)}_{(Y_i)}(u)du \text{ is TP}_2 \text{ in } (i,y) \in \{1,2\} \times (0,\infty),$$

which in turn gives that

$$\frac{\int_y^\infty \overline{F}^{(2)}_{(Y_2)}(u)du}{\int_y^\infty \overline{F}^{(1)}_{(Y_1)}(u)du} \text{ is increasing in } y > 0.$$

Therefore, $(X_1)_{(Y_1)} \leqslant_{mrl} (X_2)_{(Y_2)}$ proving (ii).

(iii) The density function of $(X_i)_{(Y_i)}$ is

$$f_{(Y_i)}(y) = -\frac{d}{dy}\overline{F}_{(Y_i)}(y) = \frac{\int_0^\infty g_i(y+x) f_i(x)dx}{P(Y_i > X_i)}, \; i = 1,2.$$

In Lemma 2.2.1, replace $l_i(\theta)$ with $f_i(x)$ and $h_i(x,\theta)$ with $g_i(y+x)$ for $i = 1,2$. The random variable Y_i is ILR iff $g_i(y+x)$ is TP$_2$ in $(x,y) \in (0,\infty) \times (0,\infty)$. On the other hand $Y_1 \leqslant_{lr} Y_2$ iff $\frac{g_2(u)}{g_1(u)}$ is increasing in $u > 0$ which in turn implies that $\frac{g_2(y+x)}{g_1(y+x)}$ is increasing in $y > 0$ as well as $x > 0$, and $X_1 \leqslant_{lr} X_2$ iff $\frac{f_2(x)}{f_1(x)}$ is increasing in $x > 0$. Combining these observations, from Lemma 2.2.1, we have

$$\int_y^\infty g_i(y+x) f_i(x)dx \text{ is TP}_2 \text{ in } (i,y) \in \{1,2\} \times (0,\infty).$$

Or equivalently,

$$\frac{\int_y^\infty g_2(y+x) f_2(x)dx}{\int_y^\infty g_1(y+x) f_1(x)dx} \text{ is increasing in } y > 0.$$

Hence $(X_1)_{(Y_1)} \leqslant_{lr} (X_2)_{(Y_2)}$. □

The following results are due to Li and Zuo (2004) which will be recalled to obtain Theorem 2.2.3.

Proposition 2.2.2. *Suppose that X, Y_1 and Y_2 are mutually independent. If $Y_1 \leqslant_{hr} Y_2$ and X is IMIT then $X_{(Y_1)} \leqslant_{icx} X_{(Y_2)}$.*

Proposition 2.2.3. *Assume that X and Y are independent random variables. If Y has IFR and X has IMIT then $X_{(Y)}$ has DMRL.*

The following theorem is in the same line of Proposition 2.2.3 where Y is assumed to have DFR ageing property.

Theorem 2.2.3. *Assume that X and Y are independent random variables. If Y has DFR and X has IMIT then $X_{(Y)}$ has IMRL.*

Proof: Denote by $(X_{(Y)})_t = (X_{(Y)} - t|t \leqslant X_{(Y)})$. Then its survival function is given by

$$\begin{aligned} \overline{F}_{(Y)}(s|t) &= \frac{\overline{F}_{(Y)}(t+s)}{\overline{F}_{(Y)}(t)} \\ &= \frac{\int_0^\infty F(y-t-s)dG(y)}{\int_0^\infty F(y-t)dG(y)} \\ &= \frac{\int_0^\infty F(u-s)dG(u+t)}{\int_0^\infty F(u)dG(u+t)}. \end{aligned}$$

Let $Y_i = U_i - t_i$ for $0 < t_1 < t_2$ and $i = 1, 2$ where U_1 and U_2 are independent and identically distributed with a common distribution function G. Denote by G_i the distribution function of Y_i, $i = 1, 2$. Then, we have $G_i(u) = G(u + t_i)$ and

$$\overline{F}_{(Y)}(s|t_i) = \frac{\int_0^\infty F(u-s)dG_i(u)}{\int_0^\infty F(u)dG_i(u)} = \overline{F}_{(Y_i)}(s),\ i = 1, 2.$$

Since Y is DFR, it holds that $Y_1 \leqslant_{hr} Y_2$. Also, X is IMIT which on using Proposition 2.2.2 gives $X_{(Y_1)} \leqslant_{icx} X_{(Y_2)}$. Or equivalently, $(X_{(Y)})_{t_1} \leqslant_{icx} (X_{(Y)})_{t_2}$ for all $t_1 < t_2$. According to Theorem 4.A.51 of Shaked and Shanthikumar (2007), X is IMRL if and only if $X_{t_1} \leqslant_{icx} X_{t_2}$ for all $t_1 \leqslant t_2$. Thus, $X_{(Y)}$ is IMRL. $\square$

The following result is reproduced from a theorem of Yue and Cao (2000).

Proposition 2.2.4. *Suppose that X, Y_1 and Y_2 are independent nonnegative random variables. If $Y_1 \leqslant_{rh} Y_2$ and X has DMRL then $E(X_{Y_1}) \geqslant E(X_{Y_2})$.*

Theorem 2.2.4. *Suppose that X, Y_1 and Y_2 are independent nonnegative random variables. If $Y_1 \leqslant_{hr} Y_2$ and X has IMIT then $E(X_{(Y_1)}) \leqslant E(X_{(Y_2)})$.*

Proof: Let $X^* = -X$, $Y_1^* = -Y_2$ and $Y_2^* = -Y_1$. Then, $Y_1 \leqslant_{hr} Y_2$ implies that $Y_1^* \leqslant_{rh} Y_2^*$. Moreover, X has IMIT implies that X^* has DMRL. On using Proposition 2.2.4, it follows that $E(X^*_{Y_1^*}) \geqslant E(X^*_{Y_2^*})$. Hence, the result follows on observing that $X^*_{Y_1^*}$ (*resp.* $X^*_{Y_2^*}$) has the same distribution as that of $X_{(Y_2)}$ (*resp.* $X_{(Y_1)}$).

2.3 Results for RLRT and ITRT based on *vrl (vit)* Order

In this section, first we carry out a preliminary study on *vit* order and IVIT class. Later we provide some stochastic comparison results on RLRT and ITRT in two sample problems with certain ageing properties. Let us start with a simple result. Before that consider the following result (cf. Theorem 39 of Hu et al., 2001).

Proposition 2.3.1. *If $X_t \leqslant_{vrl} X$ for any $t \geqslant 0$, then X is of DVRL.*

Now, the IVIT class is studied in the following theorem based on *vit* order.

Theorem 2.3.1. *If $X_t \leqslant_{vit} X$ for any $t \geqslant 0$, then X is of IVIT.*

Proof: For any $t \geqslant 0, s > 0$ and $y > 0$, $X_t \leqslant_{vit} X$ is equivalent to

$$\frac{\int_0^s \int_0^y F_t(x)dxdy}{F_t(s)} \geqslant \frac{\int_0^s \int_0^y F(x)dxdy}{F(s)},$$

which in turn gives that

$$\int_0^s \int_0^y \frac{F(t+x) - F(t)}{F(t+s) - F(t)} dxdy \geqslant \int_0^s \int_0^y \frac{F(x)}{F(s)} dxdy.$$

The above reduces to

$$\frac{\int_0^s \int_0^y [F(t+x) - F(t)]dxdy}{\int_0^s \int_0^y F(x)dxdy} \geqslant \frac{F(t+s) - F(t)}{F(s)}.$$

Or equivalently,

$$\frac{\int_0^s \int_0^y F(t+x)dxdy}{\int_0^s \int_0^y F(x)dxdy} - \frac{F(t+s)}{F(s)} \geqslant \frac{\frac{s^2}{2}F(t)}{\int_0^s \int_0^y F(x)dxdy} - \frac{F(t)}{F(s)} \geqslant 0,$$

which in turn implies that

$$\frac{\int_0^s \int_0^y F(t+x)dxdy}{\int_0^s \int_0^y F(x)dxdy} \geqslant \frac{F(t+s)}{F(s)}.$$

Hence, for any $t \geqslant 0$ and $s, y > 0$,

$$\frac{\int_0^{t+s} \int_0^y F(x)dxdy}{F(t+s)} \geqslant \frac{\int_0^s \int_0^y F(t+x)dxdy}{F(t+s)} \geqslant \frac{\int_0^s \int_0^y F(x)dxdy}{F(s)},$$

which asserts that X is of IVIT. □

The following example shows that the converse of the above theorem is not true.

Example 2.3.1. *Let X follow the power distribution*

$$F(x) = \begin{cases} x^{1/2}, & 0 < x < 1 \\ 1, & \textit{otherwise.} \end{cases}$$

Then

$$v_X(s) = \frac{\int_0^s \int_0^x F(u)dudx}{F(s)} = \begin{cases} \frac{4}{15}s^2, & 0 < s < 1 \\ \frac{1}{2}\left(s^2 - \frac{7}{15}\right), & s \geqslant 1, \end{cases}$$

which is increasing in s. Thus, X is of IVIT. For $s = t = 1/2$, it can be shown that $v_X(s) = 1/15$ and

$$v_{X_t}(s) = \frac{\int_0^s \int_0^x [F(t+u) - F(t)]\, dudx}{F(t+s) - F(t)} = \frac{43\sqrt{2} - 64}{120(\sqrt{2} - 2)} < v_X(s),$$

which asserts that $X_t \leqslant_{vit} X$ is not true. □

The next result provides a useful characterization of the *vit* order.

Theorem 2.3.2. *$X_Y \leqslant_{vit} X$ for any Y that is independent of X if and only if $X_t \leqslant_{vit} X$ for all $t \geqslant 0$.*

Proof: To prove the 'if part', let us assume $X_t \leqslant_{vit} X$ for all $t \geqslant 0$, which implies that

$$\frac{\int_0^s \int_0^y [F(t+x) - F(t)]dxdy}{\int_0^s \int_0^y F(x)dxdy} \geqslant \frac{F(t+x) - F(t)}{F(s)}.$$

Or equivalently,

$$\int_0^s \int_0^y [F(t+x) - F(t)]dxdy \geqslant \frac{F(t+x) - F(t)}{F(s)} \int_0^s \int_0^y F(x)dxdy. \qquad (2.3.1)$$

Now, from (2.3.1), we get

$$\begin{aligned} \frac{\int_0^\infty \int_0^s \int_0^y [F(t+x) - F(t)]dxdydG(t)}{\int_0^\infty [F(t+s) - F(t)]dG(t)} &\geqslant \frac{\int_0^\infty \left[\frac{F(t+s)-F(t)}{F(s)} \int_0^s \int_0^y F(x)dxdy\right] dG(t)}{\int_0^\infty [F(t+s) - F(t)]dG(t)} \\ &= \frac{\int_0^s \int_0^y F(x)dxdy}{F(s)}, \end{aligned}$$

for any $s \geqslant 0$. This gives that $X_Y \leqslant_{vit} X$. Conversely, assume that $X_Y \leqslant_{vit} X$ holds for any nonnegative random variable Y independent of X. Then $X_t \leqslant_{vit} X$ for all $t \geqslant 0$ follows by taking Y as a degenerate variable. □

The following lemmas will be used to prove the upcoming theorems.

Lemma 2.3.1. *(Joag-Dev et al., 1995). Let $\psi(x, y)$ be any TP_2 function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each i. Denote*

$$H_i(y) = \int_{\mathbb{X}} \psi(x, y)dF_i(x).$$

If $\overline{F}_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$ and if $\psi(x, y)$ is increasing in x for each y, then $H_i(y)$ is TP_2 in $y \in \mathbb{Y}$ and $i \in \{1, 2\}$.

Lemma 2.3.2. *(Khaledi and Shaked, 2010). Let $\psi(x, y)$ be any RR_2 function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each i. Denote*

$$H_i(y) = \int_{\mathbb{X}} \psi(x, y)dF_i(x).$$

If $F_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$ and if $\psi(x, y)$ is decreasing in x for each y, then $H_i(y)$ is RR_2 in $y \in \mathbb{Y}$ and $i \in \{1, 2\}$.

Now we provide some important theorems on stochastic comparisons of RLRTs and ITRTs based on *vrl* and *vit* orders.

Theorem 2.3.3. *Assume that Z is independent of X and Y. If $X \leqslant_{hr} Y$ and Z is of IVIT, then $X_Z \leqslant_{vrl} Y_Z$.*

Proof: Denote by $F_1,\ F_2$ and H the distribution functions of $X,\ Y$ and Z, respectively. Since Z is of IVIT, we have for all $y \geqslant 0$ and $\triangle > 0$

$$\frac{\int_0^{y+\triangle}\int_0^x H(u)dudx}{\int_0^{y+\triangle} H(u)du} \geqslant \frac{\int_0^{y}\int_0^x H(u)dudx}{\int_0^{y} H(u)du}.$$

Now, on using the above, we have

$$\begin{aligned}\frac{d}{dy}\left(\frac{\int_0^{y+\triangle}\int_0^x H(u)dudx}{\int_0^{y}\int_0^x H(u)dudx}\right) &= \frac{\int_0^{y+\triangle} H(u)du}{\int_0^{y}\int_0^x H(u)dudx} - \frac{\int_0^{y} H(u)du(\int_0^{y+\triangle}\int_0^x H(u)dudx)}{(\int_0^{y}\int_0^x H(u)dudx)^2}\\ &\leqslant 0.\end{aligned}$$

Hence,

$$\left(\frac{\int_0^{y+\triangle}\int_0^x H(u)dudx}{\int_0^{y}\int_0^x H(u)dudx}\right) \text{ is decreasing in } y \geqslant 0.$$

Thus,

$$\frac{\int_0^{y_2-t_2}\int_0^x H(u)dudx}{\int_0^{y_1-t_2}\int_0^x H(u)dudx} \geqslant \frac{\int_0^{y_2-t_1}\int_0^x H(u)dudx}{\int_0^{y_1-t_1}\int_0^x H(u)dudx}, \tag{2.3.2}$$

for all $0 < t_1 \leqslant t_2 < y_1 \leqslant y_2$. Denote

$$\psi(y,t) = \begin{cases} \int_0^{y-t}\int_0^x H(u)dudx, & y \geqslant t\\ 0, & y < t.\end{cases}$$

Then (2.3.2) gives that

$$\psi(y_1,t_1)\psi(y_2,t_2) \geqslant \psi(y_1,t_2)\psi(y_2,t_1), \tag{2.3.3}$$

for all $(t_1,t_2,y_1,y_2) \in S = \{(t_1,t_2,y_1,y_2) : 0 < t_1 \leqslant t_2 < y_1 \leqslant y_2\}$. It can be easily verified that (2.3.3) is also valid for those $(t_1,t_2,y_1,y_2) \in \{(t_1,t_2,y_1,y_2) : 0 < t_1 \leqslant t_2,\ 0 < y_1 \leqslant y_2\} - S$. Thus, $\psi(y,t)$ is TP_2 in $(y,t) \in (0,\infty)\times(0,\infty)$. For $i = 1, 2$, let

$$\mathcal{H}_i(t) = \frac{\int_0^\infty \psi(y,t)dF_i(y)}{\int_0^\infty H(y)dF_i(y)}.$$

Now, $X \leqslant_{hr} Y$ implies that $\overline{F}_i(x)$ is TP$_2$ in $(i,x) \in \{1,2\} \times (0,\infty)$ and $\psi(y,t)$ is increasing in $y > 0$ for each fixed $t > 0$. Thus, from Lemma 2.3.1, it follows that $\mathcal{H}_i(t)$ is TP$_2$ in $(i,t) \in \{1,2\} \times (0,\infty)$. Then

$$
\begin{aligned}
\frac{\mathcal{H}_2(t)}{\mathcal{H}_1(t)} &= \frac{\int_0^\infty \psi(y,t)dF_2(y)}{\int_0^\infty \psi(y,t)dF_1(y)} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_0^\infty \int_0^{y-t} \int_0^x H(u)dudxdF_2(y)}{\int_0^\infty \int_0^{y-t} \int_0^x H(u)dudxdF_1(y)} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_0^{y-t} \int_0^x H(u)dudxdF_2(y)}{\int_t^\infty \int_0^{y-t} \int_0^x H(u)dudxdF_1(y)} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_t^y \int_0^{y-x} H(u)dudxdF_2(y)}{\int_t^\infty \int_t^y \int_0^{y-x} H(u)dudxdF_1(y)} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty \int_0^{y-x} H(u)dudF_2(y)dx}{\int_t^\infty \int_x^\infty \int_0^{y-x} H(u)dudF_1(y)dx} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty \int_x^y H(y-u)dudF_2(y)dx}{\int_t^\infty \int_x^\infty \int_x^y H(y-u)dudF_1(y)dx} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty \int_u^\infty H(y-u)dF_2(y)dudx}{\int_t^\infty \int_x^\infty \int_u^\infty H(y-u)dF_1(y)dudx} \times \frac{\int_0^\infty H(y)dF_1(y)}{\int_0^\infty H(y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty (\overline{F}_2)_Z(u)dudx}{\int_t^\infty \int_x^\infty (\overline{F}_1)_Z(u)dudx},
\end{aligned}
$$

is increasing in $t \geqslant 0$. Hence the result follows. $\square$

The following result is due to Li and Xu (2006).

Proposition 2.3.2. *Assume that Z is independent of X and Y. If $X \leqslant_{hr} Y$ and Z is of IMIT, then $X_Z \leqslant_{mrl} Y_Z$.*

In combination with Theorem 2.3.3 and Proposition 2.3.2 it holds that, if $X \leqslant_{hr} Y$, then

$$
\begin{array}{ccc}
Z \text{ is IMIT} & \Longrightarrow & X_Z \leqslant_{mrl} Y_Z \\
\Downarrow & & \Downarrow \\
Z \text{ is IVIT} & \Longrightarrow & X_Z \leqslant_{vrl} Y_Z.
\end{array}
$$

The following theorem gives the stochastic comparison of ITRT. The proof is a dual version of Theorem 2.3.3.

Theorem 2.3.4. *Assume that Z is independent of X and Y. If $X \leqslant_{hr} Y$ and Z is of IVRL, then $X_{(Z)} \leqslant_{vrl} Y_{(Z)}$.*

Proof: Denote by $F_1,\ F_2$ and H the distribution functions of $X,\ Y$ and Z, respectively. Since Z is of IVRL, we have for all $y \geqslant 0$ and $\triangle > 0$,

$$\frac{\int_{y+\triangle}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_{y+\triangle}^{\infty}\overline{H}(u)du} \geqslant \frac{\int_{y}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_{y}^{\infty}\overline{H}(u)du}.$$

Or equivalently,

$$\frac{\int_y^{\infty}\overline{H}(u)du\left(\int_{y+\triangle}^{\infty}\int_x^{\infty}\overline{H}(u)dudx\right)}{\int_y^{\infty}\int_x^{\infty}\overline{H}(u)dudx} \geqslant \int_{y+\triangle}^{\infty}\overline{H}(u)du.$$

On using this fact, we have

$$\begin{aligned}\frac{d}{dy}\left(\frac{\int_{y+\triangle}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_y^{\infty}\int_x^{\infty}\overline{H}(u)dudx}\right) &= -\frac{\int_{y+\triangle}^{\infty}\overline{H}(u)du}{\int_y^{\infty}\int_x^{\infty}\overline{H}(u)dudx} + \frac{\int_y^{\infty}\overline{H}(u)du\left(\int_{y+\triangle}^{\infty}\int_x^{\infty}\overline{H}(u)dudx\right)}{\left(\int_y^{\infty}\int_x^{\infty}\overline{H}(u)dudx\right)^2}\\ &\geqslant 0.\end{aligned}$$

Hence,

$$\frac{\int_{y+\triangle}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_y^{\infty}\int_x^{\infty}\overline{H}(u)dudx} \text{ is increasing in } y \geqslant 0.$$

Thus,

$$\frac{\int_{y_2+t_2}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_{y_1+t_2}^{\infty}\int_x^{\infty}\overline{H}(u)dudx} \geqslant \frac{\int_{y_2+t_1}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}{\int_{y_1+t_1}^{\infty}\int_x^{\infty}\overline{H}(u)dudx}, \tag{2.3.4}$$

for all $0 < t_1 \leqslant t_2 < y_1 \leqslant y_2$. Denote

$$\psi(y,t) = \begin{cases} \int_{y+t}^{\infty}\int_x^{\infty}\overline{H}(u)dudx, & y > 0\\ 0, & y \leqslant 0.\end{cases}$$

Then (2.3.4) gives that

$$\psi(y_1,t_1)\psi(y_2,t_2) \geqslant \psi(y_1,t_2)\psi(y_2,t_1), \tag{2.3.5}$$

for all $(t_1, t_2, y_1, y_2) \in S = \{(t_1, t_2, y_1, y_2) : 0 < t_1 \leqslant t_2 < y_1 \leqslant y_2\}$. It can be verified that (2.3.5) is also valid for those $(t_1, t_2, y_1, y_2) \in \{(t_1, t_2, y_1, y_2) : 0 < t_1 \leqslant t_2;\ 0 < y_1 \leqslant y_2\} - S$. Thus $\psi(y, t)$ is TP_2 in $(y, t) \in (0, \infty) \times (0, \infty)$. Let

$$\mathcal{H}_i(t) = \frac{\int_0^\infty \psi(y, t) dF_i(y)}{\int_0^\infty \overline{H}(y) dF_i(y)}$$

Now $X \leqslant_{hr} Y$ gives that $\overline{F}_i(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times (0, \infty)$ and $\psi(y, t)$ is increasing in $y > 0$ for each fixed $t > 0$. From Lemma 2.3.1 it follows that $\mathcal{H}_i(t)$ is TP_2 in $(i, t) \in \{1, 2\} \times (0, \infty)$. Then

$$\begin{aligned}
\frac{\mathcal{H}_2(t)}{\mathcal{H}_1(t)} &= \frac{\int_0^\infty \psi(y, t) dF_2(y)}{\int_0^\infty \psi(y, t) dF_1(y)} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_0^\infty \int_{y+t}^\infty \int_x^\infty \overline{H}(u) du dx dF_2(y)}{\int_0^\infty \int_{y+t}^\infty \int_x^\infty \overline{H}(u) du dx dF_1(y)} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_0^\infty \int_t^\infty \int_{x+y}^\infty \overline{H}(u) du dx dF_2(y)}{\int_0^\infty \int_t^\infty \int_{x+y}^\infty \overline{H}(u) du dx dF_1(y)} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_0^\infty \int_t^\infty \int_x^\infty \overline{H}(y+u) du dx dF_2(y)}{\int_0^\infty \int_t^\infty \int_x^\infty \overline{H}(y+u) du dx dF_1(y)} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_t^\infty \int_0^\infty \int_x^\infty \overline{H}(y+u) du dF_2(y) dx}{\int_t^\infty \int_0^\infty \int_x^\infty \overline{H}(y+u) du dF_1(y) dx} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty \int_0^\infty \overline{H}(y+u) dF_2(y) du dx}{\int_t^\infty \int_x^\infty \int_0^\infty \overline{H}(y+u) dF_1(y) du dx} \times \frac{\int_0^\infty \overline{H}(y) dF_1(y)}{\int_0^\infty \overline{H}(y) dF_2(y)} \\
&= \frac{\int_t^\infty \int_x^\infty (\overline{F}_2)_{(Z)}(u) du dx}{\int_t^\infty \int_x^\infty (\overline{F}_1)_{(Z)}(u) du dx},
\end{aligned}$$

is increasing in $t \geqslant 0$. Hence the result follows. $\square$

In the following, we extend the above theorem in two sample problems when they fail at two different random times.

Theorem 2.3.5. *Let X_1, Y_1 and X_2, Y_2 be independent nonnegative random variables. Assume that, $X_1 \leqslant_{lr} X_2$. If $Y_1 \leqslant_{vrl} Y_2$ and either Y_1 or Y_2 is IVRL then $(X_1)_{(Y_1)} \leqslant_{vrl} (X_2)_{(Y_2)}$.*

Proof: Let F_i and G_i be the distribution functions of X_i and Y_i, $i = 1, 2$, respectively.

Y_i is IVRL iff

$$\int_{x+y}^{\infty}\int_t^{\infty}\overline{G}_i(u)dudt \text{ is TP}_2 \text{ in } (x,y)\in(0,\infty)\times(0,\infty).$$

On the other hand, $Y_1 \leqslant_{vrl} Y_2$ implies that

$$\frac{\int_{x+y}^{\infty}\int_t^{\infty}\overline{G}_2(u)dudt}{\int_{x+y}^{\infty}\int_t^{\infty}\overline{G}_1(u)dudt} \text{ is increasing in } x>0 \text{ and } y>0,$$

and $X_1 \leqslant_{lr} X_2$ iff $\frac{f_2(x)}{f_1(x)}$ is increasing in $x>0$. So from Lemma 2.2.1, we have

$$\int_0^{\infty}\left(\int_{x+y}^{\infty}\int_t^{\infty}\overline{G}_i(u)dudt\right)f_i(x)dx \text{ is TP}_2 \text{ in } (i,y)\in\{1,2\}\times(0,\infty).$$

This gives that

$$\frac{\int_0^{\infty}(\int_y^{\infty}\int_t^{\infty}\overline{G}_i(u+x)dudt)f_i(x)dx}{\int_0^{\infty}G_i(x)f_i(x)dx} \text{ is TP}_2 \text{ in } (i,y)\in\{1,2\}\times(0,\infty),$$

or equivalently,

$$\int_y^{\infty}\int_t^{\infty}\frac{\int_0^{\infty}\overline{G}_i(u+x)f_i(x)dx}{\int_0^{\infty}\overline{G}_i(x)f_i(x)dx}dudt \text{ is TP}_2 \text{ in } (i,y)\in\{1,2\}\times(0,\infty).$$

Hence

$$\int_y^{\infty}\int_t^{\infty}\overline{F}^{(i)}_{(Y_i)}(u)dudt \text{ is TP}_2 \text{ in } (i,y)\in\{1,2\}\times(0,\infty),$$

which in turn gives that

$$\frac{\int_y^{\infty}\int_t^{\infty}\overline{F}^{(2)}_{(Y_2)}(u)dudt}{\int_y^{\infty}\int_t^{\infty}\overline{F}^{(1)}_{(Y_1)}(u)dudt} \text{ is increasing in } y>0.$$

Hence, $(X_1)_{(Y_1)} \leqslant_{vrl} (X_2)_{(Y_2)}$. □

In continuation with Theorem 2.3.4 we have the following result.

Theorem 2.3.6. *Assume that Z is independent of X and Y. If $X \leqslant_{rh} Y$ and Z is of DVRL, then $X_{(Z)} \geqslant_{vrl} Y_{(Z)}$.*

Proof: Since Z is DVRL so from Theorem 2.3.4 it follows that $\psi(y,t)$ is RR_2 in $(y,t)\in(0,\infty)\times(0,\infty)$. Again $X \leqslant_{rh} Y$ implies that $F_i(x)$ is TP_2 in $(i,x)\in\{1,2\}\times(0,\infty)$ and

$\psi(y,t)$ is decreasing in $y > 0$ for each fixed $t > 0$. From Lemma 2.3.2, it follows that $\mathcal{H}_i(t)$ as defined in Theorem 2.3.4 is RR_2 in $(i,t) \in \{1,2\} \times (0,\infty)$, that is,

$$\frac{\int_t^\infty \int_x^\infty (\overline{F}_2)_{(Z)}(u)dudx}{\int_t^\infty \int_x^\infty (\overline{F}_1)_{(Z)}(u)dudx} \text{ is decreasing in } t \geqslant 0,$$

which gives the required result. $\square$

The following result is reproduced from Misra et al. (2008).

Proposition 2.3.3. *If $X \leqslant_{rh} Y$ and Z has DMRL, then $Z_Y \leqslant_{mrl} Z_X$.*

On using $X_Y = Y_{(X)}$ in Proposition 2.3.3 it holds in combination with Theorem 2.3.6 that, if $X \leqslant_{rh} Y$, then

$$\begin{array}{ccc} Z \text{ is DMRL} & \Longrightarrow & X_{(Z)} \geqslant_{mrl} Y_{(Z)} \\ \Downarrow & & \Downarrow \\ Z \text{ is DVRL} & \Longrightarrow & X_{(Z)} \geqslant_{vrl} Y_{(Z)}. \end{array}$$

To conclude this section, an application of Theorem 2.3.3 is provided which characterizes the DVRL class based on RLRT.

Theorem 2.3.7. *Let Z be independent of X. If X is IFR and Z has IVIT, then X_Z has DVRL.*

Proof: According to Theorem 1.B.38 of Shaked and Shanthikumar (2007), X is IFR if and only if $X_t \leqslant_{hr} X$ for all $t \geqslant 0$. By Theorem 2.3.3, $(X_t)_Z \leqslant_{vrl} X_Z$ for all $t \geqslant 0$. Note that $(X_Z)_t \stackrel{\text{st}}{=} (X_t)_Z$ for all $t \geqslant 0$, its holds that $(X_Z)_t \leqslant_{vrl} X_Z$ for all $t \geqslant 0$. Now from Theorem 2.3.1, it follows immediately that X_Z is of DVRL.

Chapter 3

Further Results on Residual Life and Inactivity Time at Random Time[1]

Stochastic comparisons of residual life (inactivity time) of a random variable X at random time Y (RLRT/ITRT) with respect to likelihood ratio, (reversed) hazard rate, mean residual life and variance residual life orders have been investigated in the literature. In this chapter, we provide some more stochastic ordering results for RLRT and ITRT based on reversed hazard rate, mean inactivity time and variance inactivity time orders. We also study various properties of ITRT based on DRHR, IMIT and IVIT classes of life distributions.

3.1 Introduction

Let X be an absolutely continuous nonnegative random variable representing the lifetime of a unit and let F, $\overline{F}$ and f be the distribution function, reliability function and density function of X, respectively. Denote by $X_t = (X - t|X > t)$, the residual life of X at time $t > 0$ and $X_{(t)} = (t - X|X \leqslant t)$, the inactivity time of X at time $t > 0$. Their respective

[1]A manuscript containing the work presented here has been published in *Communications in Statistics-Theory & Methods*, 2020, 49(5), 1261-1271.

distribution functions are given by

$$P(X_t < x) = \frac{F(x+t) - F(t)}{\overline{F}(t)} \quad \text{and} \quad P(X_{(t)} < x) = \frac{F(t) - F(t-x)}{F(t)}, \quad x, t \geqslant 0.$$

If t is replaced by a random variable Y, then $X_Y = (X - Y|X > Y)$ represents the residual lifetime of X at random time Y and $X_{(Y)} = (Y - X|X < Y)$ denotes the inactivity time of X at a random time Y. The RLRT is one of the important notions in reliability and queue theory (see Stoyan, 1983, for more details). For example, the idle period of a classical GI/G/1 queuing system is expressed as RLRT (see Marshall, 1968 and Kayid et al., 2017). ITRT is used in medical science to describe the dormant season (or incubation period) of a disease, i.e., the time between infection and the beginning of a disease. The concept of RLRT has also been used by Kayid and Izadkhah (2015*a*) to characterize exponential distribution.

During the past two decades, stochastic comparisons and ageing properties of RLRT and ITRT have been studied and discussed extensively in the literature. See, for example, Yue and Cao (2000), Li and Zuo (2004), Li and Xu (2006), Misra et al. (2008), Cai and Zheng (2012), Dewan and Khaledi (2014) and Misra and Naqvi (2017, 2018*a*). They perform stochastic comparisons of RLRTs and ITRTs based on various stochastic orders e.g., likelihood ratio, (reversed) hazard rate, mean residual life, variance residual life, increasing convex and usual stochastic order. In this chapter we mainly focus our attention to obtain some stochastic comparison results for RLRT and ITRT. These comparisons have been made with respect to reversed hazard rate, mean inactivity time and variance inactivity time orders. Distributional behaviours of ITRT have also been clarified by DRHR, IMIT and IVIT classes of life distributions. For some results on reversed hazard rate and mean inactivity time orders and their associated classes at a fixed time one may refer to Block et al. (1998), Chandra and Roy (2001), Nanda et al. (2003), Kayid and Ahmad (2004), Ahmad and Kayid (2005) and Ahmad et al. (2005), to mention a few. Needless to say that in mean inactivity time order we compare the means of their associated inactivity times. However, the means sometimes do not exist and therefore they are often not very informative. In many instances in applications one has more detailed information, for the purpose of comparison of two inactivity times that have non-ordered means, one is usually interested in the comparison of the dispersion of these random variables. As a result, stochastic comparison based on variance inactivity time order have been investigated in the literature. For some characterizations, preservation properties and applications of the variance inactivity time order one may refer to Mahdy (2012) and Kayid and Izadkhah (2016).

Let X and Y be two mutually independent nonnegative random variables with a common support Θ, and Y have distribution function G. Let the distribution functions of X_Y and $X_{(Y)}$ be represented as F_Y and $F_{(Y)}$, respectively and defined as

$$F_Y(x) = \frac{\int_\Theta [F(y+x) - F(y)]dG(y)}{\int_\Theta \overline{F}(y)dG(y)}$$

and

$$F_{(Y)}(x) = \frac{\int_\Theta [F(y) - F(y-x)]dG(y)}{\int_\Theta F(y)dG(y)}.$$

We first recall definitions of some stochastic orders and classes of life distributions that will be used in this chapter. For details on these topics one may refer to the famous books by Barlow and Proschan (1981), Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Belzunce et al. (2015), among others.

Definition 3.1.1. *For two random variables X and Y, X is said to be smaller than Y in*

(a) hazard rate order (denoted by $X \leqslant_{hr} Y$) if $\overline{F}(x)/\overline{G}(x)$ is decreasing in $x \geqslant 0$;

(b) reversed hazard rate order (denoted by $X \leqslant_{rh} Y$) if $F(x)/G(x)$ is decreasing in $x \geqslant 0$;

(c) mean inactivity time order (denoted by $X \leqslant_{mit} Y$) if

$$\frac{\int_0^t F(u)du}{\int_0^t G(u)du} \text{ is decreasing in } t \geqslant 0;$$

(d) variance inactivity time order (denoted by $X \leqslant_{vit} Y$) if

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t \int_0^x G(u)dudx} \text{ is decreasing in } t \geqslant 0.$$

Definition 3.1.2. *A random variable X is said to have*

(a) decreasing reversed hazard rate (DRHR) if $X_{(t)}$ is stochastically increasing in $t > 0$;

(b) increasing mean inactivity time (IMIT) if

$$\frac{\int_0^t F(u)du}{F(t)} \text{ is increasing in } t \geqslant 0;$$

(c) increasing variance inactivity time (IVIT) if

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t F(x)dx} \text{ is increasing in } t \geqslant 0.$$

The purpose of this chapter is to study the stochastic comparisons of RLRTs and ITRTs based on *rh, mit* and *vit* orders and obtain several properties of ITRT based on their associated classes for life distributions. In Section 3.2, we provide some sufficient conditions under which two RLRTs are stochastically comparable according to *rh, mit* and *vit* orders. Section 3.3 deals with analogous results for ITRT along with its properties based on DRHR, IMIT and IVIT classes. Throughout the chapter, the random variables are assumed to be nonnegative and absolutely continuous.

3.2 Stochastic Comparisons on RLRT

Here we carry out stochastic comparisons of RLRT of the same random variable X having different random ages Y_1 and Y_2 based on reversed hazard rate, mean inactivity time and variance inactivity time orders under the assumption that X and Y_1 (or Y_2) are statistically independent. Before stating our main results, we consider the following lemmas which will be helpful in deriving the upcoming results and may also be of independent interest to researchers. First recall from Karlin (1968) that a nonnegative function $\psi : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, the set of real numbers, is said to be TP_2 (*totally positive of order 2*) if $\psi(x, y)\psi(x^*, y^*) \geqslant \psi(x, y^*)\psi(x^*, y)$ for all $x, x^* \in \mathbb{X}$ and $y, y^* \in \mathbb{Y}$ such that $x \leqslant x^*$ and $y \leqslant y^*$, where $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the real line. ψ is said to be RR_2 (*reverse regular of order 2*) if the inequality is reversed.

Lemma 3.2.1. *(Joag-Dev et al., 1995). Let $\psi(x, y)$ be any TP_2 function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each i.*

Denote

$$H_i(y) = \int_{\mathbb{X}} \psi(x, y) dF_i(x).$$

If $\overline{F}_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$ and if $\psi(x, y)$ is increasing in x for each y, then $H_i(y)$ is TP_2 in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$.

Lemma 3.2.2. *(Nanda and Kundu, 2009). Let $s(x, y)$ be an RR_2 (TP_2) function (not necessarily a survival function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, and $F_i(x)$ be TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$, where $F_i(x)$ is a distribution function in x for each i. Assume that $s(x, y)$ is decreasing in x for every y. Then*

$$\phi_i(y) = \int_{\mathbb{X}} s(x, y) dF_i(x)$$

is RR_2 (TP_2) in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$. Conversely, if $\phi_i(y)$ is RR_2 (TP_2) in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$ whenever $F_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$, then $s(x, y)$ is RR_2 (TP_2) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.

The following theorem provides sufficient conditions for stochastic comparisons between X_{Y_1} and X_{Y_2}, the residual lifetime of X at two different random times Y_1 and Y_2, with respect to reversed hazard rate order.

Theorem 3.2.1. *Let Y_1 and Y_2 represent the times after which an item with lifetime X have survived. Further, let X be independent of Y_1 and Y_2. Suppose that $F(y+x) - F(y)$ is increasing (decreasing) in y and $Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$. If $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$, then $X_{Y_1} \leqslant_{rh} X_{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$, then

$$\frac{F(t_2 + x) - F(t_2)}{F(t_1 + x) - F(t_1)} \text{ is increasing in } x,$$

which implies that $F(y + x) - F(y)$ is TP_2 in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. Again, $\overline{G}_i(y)$ is TP_2 in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$ if $Y_1 \leqslant_{hr} Y_2$. Assume that $F(y + x) - F(y)$ is increasing in y. Therefore, from Lemma 3.2.1, it follows that

$$\int_0^\infty [F(y + x) - F(y)] dG_i(y) \text{ is } \mathrm{TP}_2 \text{ in } i \in \{1, 2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty [F(y+x)-F(y)]dG_2(y)}{\int_0^\infty [F(y+x)-F(y)]dG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{rh} X_{Y_2}$. On the other hand, if $Y_1 \leqslant_{rh} Y_2$ then $G_i(y)$ is TP_2 in $i \in \{1,2\}$ and $y \in \mathbb{Y}$. Suppose $F(y+x)-F(y)$ is decreasing in y. Combining these observations, from Lemma 3.2.2, we have

$$\int_0^\infty [F(y+x)-F(y)]dG_i(y) \text{ is } \text{TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty [F(y+x)-F(y)]dG_2(y)}{\int_0^\infty [F(y+x)-F(y)]dG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{rh} X_{Y_2}$. □

In the following example we provide an application of Theorem 3.2.1.

Example 3.2.1. *(i) Let the random variable X have the distribution function*

$$F(x) = \begin{cases} e^x - 1, & 0 \leqslant x \leqslant a \\ 1, & x > a \end{cases}$$

where $a = 0.69314719$ is the root of the equation $e^x - 2 = 0$. Then, $F(y+x)-F(y) = e^y(e^x-1)$ is increasing in $y > 0$, for any $x \in [0,a]$. Now,

$$\frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)} = \frac{e^{t_2}}{e^{t_1}},$$

so, $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$. Therefore, from Theorem 3.2.1, $X_{Y_1} \leqslant_{rh} X_{Y_2}$ for any Y_1, Y_2 independent of X such that $Y_1 \leqslant_{hr} Y_2$.

(ii) Let the random variable X have the distribution function

$$F(x) = 1 - \frac{1}{(x+1)}, \ x > 0.$$

Here, $F(y+x)-F(y) = \frac{x}{(y+1)(x+y+1)}$ is decreasing in $y > 0$. Since,

$$\begin{aligned} \frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)} &= \frac{(t_1+1)(x+t_1+1)}{(t_2+1)(x+t_2+1)} \\ &= \frac{(t_1+1)}{(t_2+1)}\left(1 - \frac{t_2-t_1}{x+t_2+1}\right) \ \textit{is increasing in } x \textit{ for all } t_1 \leqslant t_2, \end{aligned}$$

so, $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$. Thus, from Theorem 3.2.1, $X_{Y_1} \leqslant_{rh} X_{Y_2}$ for any $Y_1 \leqslant_{rh} Y_2$ independent of X.

It is worthwhile to remark that the reversed inequality between X_{Y_1} and X_{Y_2} in the above theorem is also true in view of some sufficient conditions.

Theorem 3.2.2. *Let X be independent of Y_1 and Y_2. Suppose that $F(y+x)-F(y)$ is decreasing in y and $Y_1 \leqslant_{rh} Y_2$. If $X_{t_1} \geqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$, then $X_{Y_1} \geqslant_{rh} X_{Y_2}$.*

Proof: Let $X_{t_1} \geqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$, then

$$\frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)} \text{ is decreasing in } x,$$

which implies that $F(y+x)-F(y)$ is RR_2 in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. On the other hand, $G_i(y)$ is TP_2 in $i \in \{1,2\}$ and $y \in \mathbb{Y}$ if $Y_1 \leqslant_{rh} Y_2$. Assume that $F(y+x)-F(y)$ is decreasing in y. Therefore, from Lemma 3.2.2, it follows that

$$\int_0^\infty [F(y+x)-F(y)]dG_i(y) \text{ is } \mathrm{RR}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Thus,

$$\frac{\int_0^\infty [F(y+x)-F(y)]dG_2(y)}{\int_0^\infty [F(y+x)-F(y)]dG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{Y_1} \geqslant_{rh} X_{Y_2}$. □

Consider the following example in support of the above result.

Example 3.2.2. *Let X be a random variable having distribution function*

$$F(x) = \begin{cases} \sin x, & 0 \leqslant x \leqslant \pi/2 \\ 1, & x > \pi/2. \end{cases}$$

Then it can be seen that $F(y+x)-F(y) = \sin(y+x) - \sin y = 2\sin(x/2)\cos(x/2+y)$, which is decreasing in $y > 0$. Now,

$$\begin{aligned} \frac{d}{dx}\left[\frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)}\right] &= \frac{d}{dx}\left[\frac{\cos(t_2+x/2)}{\cos(t_1+x/2)}\right] \\ &= \frac{-\frac{1}{2}\cos(t_1+x/2)\sin(t_2+x/2) - \frac{1}{2}\cos(t_2+x/2)\sin(t_1+x/2)}{[\cos(t_1+x/2)]^2} \\ &= -\frac{\frac{1}{2}\sin(x+t_1+t_2)}{[\cos(t_1+x/2)]^2} \\ &\leqslant 0. \end{aligned}$$

Hence,

$$\frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)} \text{ is decreasing in } x.$$

Or equivalently, $X_{t_1} \geqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$. Therefore, from Theorem 3.2.2, $X_{Y_1} \geqslant_{rh} X_{Y_2}$ for any Y_1, Y_2 independent of X so that $Y_1 \leqslant_{rh} Y_2$.

Next, we provide sufficient conditions for stochastic comparison of RLRT with respect to MIT order.

Theorem 3.2.3. *Let X be independent of Y_1 and Y_2. Assume that $\int_0^x [F(y+u)-F(y)]du$ is increasing (decreasing) in y and $Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$. If $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $t_1 \leqslant t_2$, then $X_{Y_1} \leqslant_{mit} X_{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $t_1 \leqslant t_2$, then

$$\frac{\int_0^x [F(t_2+u)-F(t_2)]du}{\int_0^x [F(t_1+u)-F(t_1)]du} \text{ is increasing in } x,$$

which in turn gives that

$$\int_0^x [F(y+u)-F(y)]du \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}. \tag{3.2.1}$$

Assume that $Y_1 \leqslant_{hr} Y_2$. Then

$$\overline{G}_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y}. \tag{3.2.2}$$

Further, assume that

$$\int_0^x [F(y+u)-F(y)]du \text{ is increasing in } y. \tag{3.2.3}$$

Therefore, in view of (3.2.1), (3.2.2) and (3.2.3), we have from Lemma 3.2.1 that

$$\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{mit} X_{Y_2}$. On the other hand,

$$G_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y} \tag{3.2.4}$$

in accordance with $Y_1 \leqslant_{rh} Y_2$. Also, let

$$\int_0^x [F(y+u) - F(y)]du \text{ be decreasing in } y. \tag{3.2.5}$$

Therefore, on using (3.2.1), (3.2.4) and (3.2.5), we have from Lemma 3.2.2 that

$$\int_0^\infty \int_0^x [F(y+u) - F(y)]dudG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X},$$

which implies that

$$\frac{\int_0^\infty \int_0^x [F(y+u) - F(y)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y+u) - F(y)]dudG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{mit} X_{Y_2}$. □

The following example illustrates the above theorem.

Example 3.2.3. *Let the random variable X follow the distribution*

$$F(x) = 1 - \frac{1}{(x+1)^2}, \ x > 0.$$

Then,

$$\int_0^x [F(y+u) - F(y)]du = \frac{x^2}{(y+1)^2(x+y+1)} \textit{ is decreasing in } y > 0.$$

Now,

$$\begin{aligned} \frac{\int_0^x [F(t_2+u) - F(t_2)]du}{\int_0^x [F(t_1+u) - F(t_1)]du} &= \frac{(t_1+1)^2(x+t_1+1)}{(t_2+1)^2(x+t_2+1)} \\ &= \frac{(t_1+1)^2}{(t_2+1)^2}\left(1 - \frac{t_2 - t_1}{x+t_2+1}\right) \textit{ is increasing in } x \textit{ for all } t_1 \leqslant t_2, \end{aligned}$$

which in turn gives that $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $t_1 \leqslant t_2$. Therefore, from Theorem 3.2.3, $X_{Y_1} \leqslant_{mit} X_{Y_2}$ for any Y_1, Y_2 independent of X so that $Y_1 \leqslant_{rh} Y_2$.

In the following we discuss about *vit* order for RLRT.

Theorem 3.2.4. *For three random variables X, Y_1 and Y_2 where X is independent of Y_1 and Y_2, suppose that $\int_0^x \int_0^v [F(y+u) - F(y)]dudv$ is increasing (decreasing) in y and $Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$. If $X_{t_1} \leqslant_{vit} X_{t_2}$ for all $t_1 \leqslant t_2$, then $X_{Y_1} \leqslant_{vit} X_{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{vit} X_{t_2}$ for all $t_1 \leqslant t_2$ then

$$\frac{\int_0^x \int_0^v [F(t_2+u)-F(t_2)]dudv}{\int_0^x \int_0^v [F(t_1+u)-F(t_1)]dudv} \text{ is increasing in } x,$$

which implies that

$$\int_0^x \int_0^v [F(y+u)-F(y)]dudv \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Also,

$$\overline{G}_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y} \text{ if } Y_1 \leqslant_{hr} Y_2.$$

Again, since $\int_0^x \int_0^v [F(y+u)-F(y)]du$ is increasing in y, so from Lemma 3.2.1 it follows that

$$\int_0^\infty \int_0^x \int_0^v [F(y+u)-F(y)]dudvdG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Thus,

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y+u)-F(y)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(t+u)-F(t)]dudvdG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{vit} X_{Y_2}$. Again, if $Y_1 \leqslant_{rh} Y_2$ then $G_i(y)$ is TP$_2$ in $i \in \{1,2\}$ and $y \in \mathbb{Y}$. Assume that

$$\int_0^x \int_0^v [F(y+u)-F(y)]du \text{ is decreasing in } y,$$

Therefore, from Lemma 3.2.2 it follows that

$$\int_0^\infty \int_0^x \int_0^v [F(y+u)-F(y)]dudvdG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y+u)-F(y)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(t+u)-F(t)]dudvdG_1(y)} \text{ is increasing in } x.$$

Hence $X_{Y_1} \leqslant_{vit} X_{Y_2}$. $\square$

To conclude, we give some insight into the sufficient conditions that would be required if the inequalities in Theorems 3.2.3 and 3.2.4 are reversed.

Theorem 3.2.5. *Let X be independent of Y_1 and Y_2 such that $Y_1 \leqslant_{rh} Y_2$.*

(a) If $\int_0^x [F(y+u)-F(y)]du$ *is decreasing in* y *and* $X_{t_1} \geqslant_{mit} X_{t_2}$ *for all* $t_1 \leqslant t_2$*, then* $X_{Y_1} \geqslant_{mit} X_{Y_2}$.

(b) If $\int_0^x \int_0^v [F(y+u)-F(y)]dudv$ *is decreasing in* y *and* $X_{t_1} \geqslant_{vit} X_{t_2}$ *for all* $t_1 \leqslant t_2$*, then* $X_{Y_1} \geqslant_{vit} X_{Y_2}$.

Proof: *(a)* Let $X_{t_1} \geqslant_{mit} X_{t_2}$ for all $t_1 \leqslant t_2$, then

$$\frac{\int_0^x [F(t_2+u)-F(t_2)]du}{\int_0^x [F(t_1+u)-F(t_1)]du} \text{ is decreasing in } x,$$

which in turn gives that

$$\int_0^x [F(y+u)-F(y)]du \text{ is RR}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}. \tag{3.2.6}$$

Again,

$$G_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y} \tag{3.2.7}$$

in accordance with $Y_1 \leqslant_{rh} Y_2$. Also,

$$\int_0^x [F(y+u)-F(y)]du \text{ is decreasing in } y. \tag{3.2.8}$$

Therefore, in view of (3.2.6), (3.2.7) and (3.2.8), we have from Lemma 3.2.2 that

$$\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_i(y) \text{ is RR}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y+u)-F(y)]dudG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{Y_1} \geqslant_{mit} X_{Y_2}$.

(b) Let $X_{t_1} \geqslant_{vit} X_{t_2}$ for all $t_1 \leqslant t_2$ then

$$\frac{\int_0^x \int_0^v [F(t_2+u)-F(t_2)]dudv}{\int_0^x \int_0^v [F(t_1+u)-F(t_1)]dudv} \text{ is decreasing in } x,$$

which implies that

$$\int_0^x \int_0^v [F(y+u)-F(y)]dudv \text{ is RR}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Also, $G_i(y)$ is TP_2 in $i \in \{1,2\}$ and $y \in \mathbb{Y}$ if $Y_1 \leqslant_{rh} Y_2$. Again, since $\int_0^x \int_0^v [F(y+u) - F(y)]du$ is decreasing in y, so from Lemma 3.2.2 it follows that

$$\int_0^\infty \int_0^x \int_0^v [F(y+u) - F(y)]dudvdG_i(y) \text{ is } \mathrm{RR}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Thus,

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y+u) - F(y)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(t+u) - F(t)]dudvdG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{Y_1} \geqslant_{vit} X_{Y_2}$.

3.3 Results on ITRT

In this section, we perform stochastic comparisons between $X_{(Y_1)}$ and $X_{(Y_2)}$, the inactivity time of X at the different random times Y_1 and Y_2. We provide some sufficient conditions under which $X_{(Y_1)}$ is comparable with $X_{(Y_2)}$ according to the reversed hazard rate order, the mean inactivity time order and the variance inactivity time order. We also study some properties of ITRT based on DRHR, IMIT and IVIT classes.

The following result presents sufficient conditions for comparing $X_{(Y_1)}$ and $X_{(Y_2)}$ with respect to rh order.

Theorem 3.3.1. *Let Y_1 and Y_2 be the times of observation of the failure of a system with lifetime X. Further, let Y_1 and Y_2 be independent of X. Suppose that*

- *$F(y) - F(y-x)$ is increasing (decreasing) in y;*
- *$Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$.*

If $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then $X_{(Y_1)} \leqslant_{rh} X_{(Y_2)}$.

Proof: Since $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then

$$\frac{F_{(t_2)}(x)}{F_{(t_1)}(x)} \text{ is increasing in } x,$$

which yields

$$\frac{F(t_2) - F(t_2 - x)}{F(t_1) - F(t_1 - x)} \text{ is increasing in } x.$$

Therefore, $F(y)-F(y-x)$ is TP_2 in $x\in\mathbb{X}$ and $y\in\mathbb{Y}$. Now, $\overline{G}_i(y)$ is TP_2 in $i\in\{1,2\}$ and $y\in\mathbb{Y}$ as $Y_1\leqslant_{hr}Y_2$. Again, $F(y)-F(y-x)$ is increasing in y. Combining these observations, from Lemma 3.2.1, we have

$$\int_0^\infty [F(y)-F(y-x)]dG_i(y) \text{ is } \text{TP}_2 \text{ in } i\in\{1,2\} \text{ and } x\in\mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty [F(y)-F(y-x)]dG_2(y)}{\int_0^\infty [F(y)-F(y-x)]dG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)}\leqslant_{rh}X_{(Y_2)}$. Again, $G_i(y)$ is TP_2 in $i\in\{1,2\}$ and $y\in\mathbb{Y}$ if $Y_1\leqslant_{rh}Y_2$. Suppose that $F(y)-F(y-x)$ is decreasing in y, so from Lemma 3.2.2, we get

$$\int_0^\infty [F(y)-F(y-x)]dG_i(y) \text{ is } \text{TP}_2 \text{ in } i\in\{1,2\} \text{ and } x\in\mathbb{X}.$$

Thus,

$$\frac{\int_0^\infty [F(y)-F(y-x)]dG_2(y)}{\int_0^\infty [F(y)-F(y-x)]dG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)}\leqslant_{rh}X_{(Y_2)}$. □

The following example gives an application of Theorem 3.3.1.

Example 3.3.1. *(i) Let the random variable X have the distribution function*

$$F(x)=\begin{cases} x^2, & 0\leqslant x\leqslant 1\\ 1, & x>1.\end{cases}$$

Then $F(y)-F(y-x)=x(2y-x)$ is increasing in $y>x$ and for any $x\in[0,1]$. Also,

$$\frac{F(t_2)-F(t_2-x)}{F(t_1)-F(t_1-x)}=\frac{2t_2-x}{2t_1-x} \textit{ is increasing in } x\in[0,1],$$

which gives that $X_{(t_1)}\leqslant_{rh}X_{(t_2)}$ for all $t_1\leqslant t_2$. Therefore, from Theorem 3.3.1, we obtain $X_{(Y_1)}\leqslant_{rh}X_{(Y_2)}$ for any two random variables Y_1,Y_2 independent of X so that $Y_1\leqslant_{hr}Y_2$.
(ii) Let the random variable X follow exponential distribution

$$F(x)=1-e^{-x},\ x>0.$$

It can easily be seen that $F(y)-F(y-x)=e^{-y}(e^x-1)$ is decreasing in $y>x$ and

$$\frac{F(t_2)-F(t_2-x)}{F(t_1)-F(t_1-x)}=\frac{e^{-t_2}}{e^{-t_1}},$$

so $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ *for all* $t_1 \leqslant t_2$. *Therefore, from Theorem 3.3.1, we get* $X_{(Y_1)} \leqslant_{rh} X_{(Y_2)}$ *for any* $Y_1 \leqslant_{rh} Y_2$, *independent of* X.

Next, we provide a result concerning comparison of ITRT with respect to *mit* order.

Theorem 3.3.2. *Let* X *be independent of* Y_1 *and* Y_2. *Assume that* $Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$ *and* $\int_0^x [F(y) - F(y-u)]du$ *is increasing (decreasing) in* y. *If* $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ *for all* $t_1 \leqslant t_2$, *then* $X_{(Y_1)} \leqslant_{mit} X_{(Y_2)}$.

Proof: Since $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$, so

$$\frac{\int_0^x F_{(t_2)}(u)du}{\int_0^x F_{(t_1)}(u)du} \text{ is increasing in } x.$$

Or equivalently,

$$\frac{\int_0^x [F(t_2) - F(t_2 - u)]du}{\int_0^x [F(t_1) - F(t_1 - u)]du} \text{ is increasing in } x.$$

Therefore,

$$\int_0^x [F(y) - F(y-u)]du \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}. \tag{3.3.1}$$

On the other hand,

$$\overline{G}_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y} \tag{3.3.2}$$

in accordance with $Y_1 \leqslant_{hr} Y_2$. Assume that

$$\int_0^x [F(y) - F(y-u)]du \text{ is increasing in } y. \tag{3.3.3}$$

Therefore, on using (3.3.1), (3.3.2) and (3.3.3), we obtain from Lemma 3.2.1

$$\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)} \leqslant_{mit} X_{(Y_2)}$. If $Y_1 \leqslant_{rh} Y_2$ then $G_i(y)$ is TP$_2$ in $i \in \{1,2\}$ and $y \in \mathbb{Y}$. Also, let

$$\int_0^x [F(y) - F(y-u)]du \text{ be decreasing in } y.$$

Then, from Lemma 3.2.2, we have

$$\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)} \leqslant_{mit} X_{(Y_2)}$. □

Now, we compare two inactivity time random variables with respect to *vit* order.

Theorem 3.3.3. *For three independent random variables X, Y_1 and Y_2, suppose that $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ is increasing (decreasing) in y and $Y_1 \leqslant_{hr} (\leqslant_{rh}) Y_2$. If $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then $X_{(Y_1)} \leqslant_{vit} X_{(Y_2)}$.*

Proof: On using the fact that $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ we have

$$\frac{\int_0^x \int_0^v F_{(t_2)}(u)dudv}{\int_0^x \int_0^v F_{(t_1)}(u)dudv} \text{ is increasing in } x,$$

which gives that

$$\frac{\int_0^x \int_0^v [F(t_2) - F(t_2-u)]dudv}{\int_0^x \int_0^u [F(t_1) - F(t_1-u)]dudv} \text{ is increasing in } x.$$

Therefore,

$$\int_0^x \int_0^v [F(y) - F(y-u)]dudv \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Now, $Y_1 \leqslant_{hr} Y_2$ implies that $\overline{G}_i(y)$ is TP$_2$ in $i \in \{1,2\}$ and $y \in \mathbb{Y}$. Again,

$$\int_0^x \int_0^v [F(y) - F(y-u)]dudv \text{ is increasing in } y.$$

Therefore, from Lemma 3.2.1, we have

$$\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

This implies that

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)} \leqslant_{vit} X_{(Y_2)}$. Similarly, $G_i(y)$ is TP_2 in $i \in \{1,2\}$ and $y \in \mathbb{Y}$ if $Y_1 \leqslant_{rh} Y_2$. Suppose that

$$\int_0^x \int_0^v [F(y) - F(y-u)]dudv \text{ is decreasing in } y.$$

Therefore, it follows from Lemma 3.2.2 that

$$\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

This implies that

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_1(y)} \text{ is increasing in } x.$$

Hence $X_{(Y_1)} \leqslant_{vit} X_{(Y_2)}$. □

It is to be mentioned here that the converse of Theorems 3.3.1-3.3.3 are also true.

Theorem 3.3.4. *Let X be independent of Y_1 and Y_2. Assume that $Y_1 \leqslant_{rh} Y_2$.*

(a) If $F(y) - F(y-x)$ is decreasing in y and $X_{(t_1)} \geqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then $X_{(Y_1)} \geqslant_{rh} X_{(Y_2)}$.

(b) If $\int_0^x [F(y) - F(y-u)]du$ is decreasing in y and $X_{(t_1)} \geqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then $X_{(Y_1)} \geqslant_{mit} X_{(Y_2)}$.

(c) If $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ is decreasing in y and $X_{(t_1)} \geqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then $X_{(Y_1)} \geqslant_{vit} X_{(Y_2)}$.

Proof: Let $Y_1 \leqslant_{rh} Y_2$, then

$$G_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y}. \tag{3.3.4}$$

(a) If $X_{(t_1)} \geqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$, then

$$\frac{F_{(t_2)}(x)}{F_{(t_1)}(x)} \text{ is decreasing in } x,$$

which yields

$$\frac{F(t_2) - F(t_2 - x)}{F(t_1) - F(t_1 - x)} \text{ is decreasing in } x.$$

Therefore,

$$F(y) - F(y-x) \text{ is } \mathrm{RR}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}. \tag{3.3.5}$$

Again, let

$$F(y) - F(y-x) \text{ be decreasing in } y. \tag{3.3.6}$$

Therefore, on using (3.3.4), (3.3.5) and (3.3.6), we have from Lemma 3.2.2,

$$\int_0^\infty [F(y) - F(y-x)]dG_i(y) \text{ is } \mathrm{RR}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty [F(y) - F(y-x)]dG_2(y)}{\int_0^\infty [F(y) - F(y-x)]dG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{(Y_1)} \geqslant_{rh} X_{(Y_2)}$.

(b) Suppose $X_{(t_1)} \geqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$, so that

$$\frac{\int_0^x F_{(t_2)}(u)du}{\int_0^x F_{(t_1)}(u)du} \text{ is decreasing in } x.$$

Or equivalently,

$$\frac{\int_0^x [F(t_2) - F(t_2-u)]du}{\int_0^x [F(t_1) - F(t_1-u)]du} \text{ is decreasing in } x.$$

Therefore,

$$\int_0^x [F(y) - F(y-u)]du \text{ is } \mathrm{RR}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Again, assume that

$$\int_0^x [F(y) - F(y-u)]du \text{ is decreasing in } y.$$

Combining these observations with (3.3.4), we obtain from Lemma 3.2.2

$$\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_i(y) \text{ is } \mathrm{RR}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_2(y)}{\int_0^\infty \int_0^x [F(y) - F(y-u)]dudG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{(Y_1)} \geqslant_{mit} X_{(Y_2)}$.

(c) Let $X_{(t_1)} \geqslant_{vit} X_{(t_2)}$. Then we have

$$\frac{\int_0^x \int_0^v F_{(t_2)}(u)dudv}{\int_0^x \int_0^v F_{(t_1)}(u)dudv} \text{ is decreasing in } x,$$

which gives that

$$\frac{\int_0^x \int_0^v [F(t_2) - F(t_2 - u)]dudv}{\int_0^x \int_0^u [F(t_1) - F(t_1 - u)]dudv} \text{ is decreasing in } x.$$

Therefore,

$$\int_0^x \int_0^v [F(y) - F(y-u)]dudv \text{ is } \mathrm{RR}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Again, suppose that

$$\int_0^x \int_0^v [F(y) - F(y-u)]dudv \text{ is decreasing in } y.$$

Therefore, on using these facts with (3.3.4) we have

$$\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_i(y) \text{ is } \mathrm{RR}_2 \text{ in } i \in \{1, 2\} \text{ and } x \in \mathbb{X}.$$

This implies that

$$\frac{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_2(y)}{\int_0^\infty \int_0^x \int_0^v [F(y) - F(y-u)]dudvdG_1(y)} \text{ is decreasing in } x.$$

Hence $X_{(Y_1)} \geqslant_{vit} X_{(Y_2)}$. □

Now, we study DRHR, IMIT and IVIT classes of life distributions for ITRT. It is worth mentioning that the DRHR, IMIT and IVIT properties are used to characterize the behaviour of a random variable X representing the lifetime of a system that was found to fail at a fixed time $t > 0$. The stochastic monotonicity of $X_{(t)}$ in relation to ageing properties of X has been the subject of interest by many researchers. Here we consider the situation where the observed failure time and exact time of failure of the system both are random. We provide conditions on X and Y to obtain the distributional behaviors of ITRT $X_{(Y)}$ that are clarified by these nonparametric classes of life distributions. In the following two theorems we discuss DRHR property of $X_{(Y)}$.

Theorem 3.3.5. *Let $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$. Further, let $F(y) - F(y-x)$ be increasing in y. If Y is IFR then $X_{(Y)}$ is DRHR.*

Proof: According to Theorem 1.B.38 of Shaked and Shanthikumar (2007), Y is IFR if and only if $Y_t \leqslant_{hr} Y$ for all $t \geqslant 0$. Then, from Theorem 3.3.1, we have $X_{(Y_t)} \leqslant_{rh} X_{(Y)}$. Note that $(Y_t)_X \stackrel{\text{st}}{=} (Y_X)_t$ for all $t \geqslant 0$, or equivalently, $X_{(Y_t)} \stackrel{\text{st}}{=} \left(X_{(Y)}\right)_t$ for all $t \geqslant 0$. Therefore, $\left(X_{(Y)}\right)_t \leqslant_{rh} X_{(Y)}$ for all $t \geqslant 0$. Hence $X_{(Y)}$ is DRHR.

Theorem 3.3.6. *Assume that $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $F(y) - F(y-x)$ is decreasing in y. If $Y_t \leqslant_{rh} Y$ for all $t \geqslant 0$, then $X_{(Y)}$ is DRHR.*

Proof: Assume that $X_{(t_1)} \leqslant_{rh} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $F(y) - F(y-x)$ is decreasing in y. If $Y_t \leqslant_{rh} Y$ then, from Theorem 3.3.1, we have $X_{(Y_t)} \leqslant_{rh} X_{(Y)}$. Hence $X_{(Y)}$ is DRHR.□

Now we discuss the properties of ITRT based on IMIT class.

Theorem 3.3.7. *Suppose that $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$. If $\int_0^x [F(y) - F(y-u)]du$ is increasing in y and Y is IFR then $X_{(Y)}$ is IMIT.*

Proof: Y is IFR if and only if $Y_t \leqslant_{hr} Y$ for all $t \geqslant 0$. Since $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $\int_0^x [F(y) - F(y-u)]du$ is increasing in y, so from Theorem 3.3.2, we get $X_{(Y_t)} \leqslant_{mit} X_{(Y)}$ for all $t \geqslant 0$. In view of $X_{(Y_t)} \stackrel{\text{st}}{=} \left(X_{(Y)}\right)_t$, we obtain $\left(X_{(Y)}\right)_t \leqslant_{mit} X_{(Y)}$ for all $t \geqslant 0$. According to Proposition 2.4 of Li and Xu (2006), if $X_t \leqslant_{mit} X$ for any $t \geqslant 0$, then X is IMIT. Hence $X_{(Y)}$ is IMIT.

Theorem 3.3.8. *Let $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$. If $\int_0^x [F(y) - F(y-u)]du$ is decreasing in y and $Y_t \leqslant_{rh} Y$ for all $t \geqslant 0$, then $X_{(Y)}$ is IMIT.*

Proof: Let $X_{(t_1)} \leqslant_{mit} X_{(t_2)}$ for all $t_1 \leqslant t_2$. If $\int_0^x [F(y) - F(y-u)]du$ is decreasing in y and $Y_t \leqslant_{rh} Y$ for all $t \geqslant 0$, then from Theorem 3.3.2, we get $X_{(Y_t)} \leqslant_{mit} X_{(Y)}$ for all $t \geqslant 0$. Hence $X_{(Y)}$ is IMIT. □

Here we characterize the IVIT class based on ITRT.

Theorem 3.3.9. *Suppose that $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ is increasing in y. If Y is IFR then $X_{(Y)}$ is IVIT.*

Proof: Let $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ be increasing in y. Also, $Y_t \leqslant_{hr} Y$ for all $t \geqslant 0$, if Y is IFR. So from Theorem 3.3.3 we obtain $X_{(Y_t)} \leqslant_{vit} X_{(Y)}$, or equivalently, $\left(X_{(Y)}\right)_t \leqslant_{vit} X_{(Y)}$ for all $t \geqslant 0$. According to Theorem 2.3.1 of Chapter 2, if $X_t \leqslant_{vit} X$ for any $t \geqslant 0$, then X is IVIT. Hence $X_{(Y)}$ is IVIT.

Theorem 3.3.10. *Assume that $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ is decreasing in y. If $Y_t \leqslant_{rh} Y$ for all $t \geqslant 0$, then $X_{(Y)}$ is IVIT.*

Proof: Assume that $X_{(t_1)} \leqslant_{vit} X_{(t_2)}$ for all $t_1 \leqslant t_2$ and $\int_0^x \int_0^v [F(y) - F(y-u)]dudv$ is decreasing in y. If $Y_t \leqslant_{rh} Y$ for all $t \geqslant 0$, then from Theorem 3.3.3 we obtain $\left(X_{(Y)}\right)_t \leqslant_{vit} X_{(Y)}$ for all $t \geqslant 0$. Hence $X_{(Y)}$ is IVIT.

Chapter 4

Stochastic Properties of RLRT (ITRT) based on VRL [1]

Stochastic comparisons and ageing properties of residual life (inactivity time) of a random variable X at random time Y (RLRT/ITRT) taking X and Y independent have been investigated in the literature. In this chapter, we consider X and Y not necessarily independent and obtain stochastic comparison results for RLRT and ITRT based on variance residual life (*vrl*) order. We also study various ageing properties of RLRT/ITRT based on the associated classes of life distributions of VRL. Some applications of the results derived in this chapter are also illustrated.

4.1 Introduction

Let X and Y be jointly distributed random variables (not necessarily independent). The residual life and the inactivity time of X at a fixed time $t > 0$ are defined as the random variables $X_t = (X - t|X > t)$ and $X_{(t)} = (t - X|X \leqslant t)$. In the same vein, if t is replaced by the random variable Y, then the residual life of X at a random time Y (RLRT) is defined by $X_Y = (X - Y|X > Y)$ and the inactivity time of X at a random time Y (ITRT)

[1] One article, containing the work discussed here has appeared in *Communications in Statistics- Theory & Methods*, 2020, DOI: 10.1080/03610926.2020.1812655 (Online First).

is defined by $X_{(Y)} = (Y - X|X \leqslant Y)$. In industrial engineering, the RLRT represents the original working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age Y, and the idle time of a server in a classical GI/G/1 queuing system can also be expressed as a RLRT (see Marshall, 1968). ITRT is used in medical science to describe the dormant season (or incubation period) of a disease, i.e., the time between infection and the beginning of a disease. In reliability theory, let X and Y represent the lifetimes of two components C_1 and C_2, respectively. If C_1 stops working before C_2, then $X_{(Y)}$ denotes the duration for which C_2 will continue to work after the failure of C_1. Note that the stochastic comparisons of $X_{(Y)}$ and X_Y enlighten us on robustness of components C_1 and C_2 working under the same environment.

Stochastic comparisons and ageing properties of X_Y and $X_{(Y)}$ under the assumption that X and Y are independent, have been studied and discussed extensively by Yue and Cao (2000), Li and Zuo (2004), Li and Xu (2006), Misra et al. (2008), Cai and Zheng (2012), Dewan and Khaledi (2014) and Li and Fang (2018). Most of the comparisons have been made with respect to usual stochastic, hazard rate, reversed hazard rate, likelihood ratio, mean residual life, mean inactivity time and increasing convex orders. But, another context where the stochastic orders arise is in the comparison of random variables in terms of their variability or dispersion. The basic way to decide if one random variable has greater variability than another one, is comparing their variances. For example, in risk theory, the comparison is made in terms of the variances in order to avoid situation of high uncertainty or variability. Therefore, it may be of interest to study the dispersion of RLRT and ITRT. Although, in Chapter 2 and Chapter 3 we perform stochastic comparisons of RLRTs and ITRTs based on *vrl* and *vit* orders, in general, the results are not sufficient for stochastic comparisons of RLRT/ITRT based on variability measures. Moreover, due to technical complexity, all the studies carried out so far have assumed that X and Y are independently distributed. But, in most of the practical situations the assumption of independence is seldom valid and it is necessary to take into account their dependence structure. This shows the relevance and usefulness of studying RLRT/ITRT based on *vrl* order in the presence of dependence structure between X and Y.

It is to be noted that the VRL and VIT functions are useful in many areas of statistics including biometry, actuarial science, reliability theory and a lot of interest has been evoked on the study of *vrl/vit* order and their associated classes at a fixed time. See, for example, Launer (1984), Gupta (1987), Gupta et al. (1987), Kanjo (1996), Nanda et al. (2003), Abu-Youssef (2004), Gupta (2006), Kundu and Nanda (2010), Mahdy (2012) and Kayid and Izadkhah (2016), to mention a few. To this end, we put the conditional random variables RLRT and ITRT in the framework of the VRL and the VIT quantities. To get a flavor on the interpretation of residual life (inactivity time) of RLRT (ITRT), consider the following example. In clinical trials, it often happens that the time at which a person goes to clinic for examination of a disease is different from (actually higher than) the time he got infected. Let X, Y represent the time of infection and onset of a disease and t denote the time when the disease was clinically diagnosed. Then $\left(X_{(Y)}\right)_{(t)}$, the inactivity time of ITRT signifies the time elapsed since beginning of the disease and its clinically observed time. On the contrary, if X is taken as the time of recovery or death then X_Y identifies with the period of illness i.e., the time between beginning of a disease and its recovery. Again, suppose that the person undergoes a lab test and identified with the disease after time t of the inception of the disease i.e., at time $(Y+t)$. Under the assumption that the treatment of the disease commenced immediately after diagnosis, $(X_Y)_t$ describes the time it will take to cure the disease or remaining treatment time. Thus, from a stochastic comparison vantage point, the variability of RLRT/ITRT also needs to be taken into account along with their means.

Recently, Misra and Naqvi (2017, 2018*a*) have investigated some stochastic order related results and ageing properties of RLRT and ITRT with respect to likelihood ratio, hazard rate and mean residual life orders assuming a dependence between X and Y. Even though some works can be found on stochastic comparisons of RLRT/ITRT with respect to various stochastic orders, no works based on *vrl/vit* order, to the best of our knowledge, till now, seem to have been done under dependence condition. Motivated by this, in this chapter, we provide some further results on stochastic comparisons and ageing properties of RLRT/ITRT based on *vrl* order and the associated classes of life distributions, taking X and Y not necessarily independent. We extend the related results of Chapter 2 derived under independence of X and Y to include situations where X and Y may be dependent random variables.

Let (X, Z_1) and (Y, Z_2) be two sets of jointly distributed lifetime random variables (not necessarily independent) with common support $[0,\infty) \times [0,\infty)$. Let F, G and H_i be the cumulative distribution functions (*cdf*s), $\overline{F}$, $\overline{G}$ and $\overline{H}_i$ be the survival functions (*sf*s) and f, g and h_i be the probability density functions (*pdf*s) of X, Y and Z_i, $i = 1, 2$, respectively. Let X^θ (Y^θ) denote the random variable having the same distribution as the conditional distribution of X (Y) given that $Z_1 = \theta$ $(Z_2 = \theta)$. Let f_θ, F_θ and $\overline{F}_\theta$ $(g_\theta$, G_θ and $\overline{G}_\theta)$, respectively, be the *pdf*, the *cdf* and the *sf* of X^θ (Y^θ), $\theta > 0$. The *sf*s of X_{Z_1} and $X_{(Z_1)}$, respectively, are given by the following

$$\overline{F}_{Z_1}(x) = \frac{\int_0^\infty \overline{F}_\theta(x+\theta)dH_1(\theta)}{\int_0^\infty \overline{F}_\theta(\theta)dH_1(\theta)}$$

and

$$\overline{F}_{(Z_1)}(x) = \frac{\int_0^\infty F_\theta(\theta-x)dH_1(\theta)}{\int_0^\infty F_\theta(\theta)dH_1(\theta)}.$$

The organization of this chapter is as follows: In Section 4.2, we include all the definitions and lemmas which will be used in proving the main results of this chapter. In Section 4.3, we provide some further results on stochastic comparisons of RLRTs and ITRTs based on *vrl* order, taking X and Y not necessarily independent. Section 4.4 presents results on preservation of ageing properties of RLRT/ITRT based on IVRL (DVRL) and IVIT classes. Finally, in Section 4.5, we provide several situations where the results of Section 4.3 can be applied.

4.2 Preliminaries

In this section, we provide some definitions and lemmas which will be intensively used in deriving the theorems discussed later and may also be of independent interest to researchers. We first recall definitions of some stochastic orders and classes of life distributions that will be used in the present chapter. For details on these topics one may refer to the famous books by Barlow and Proschan (1981), Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Belzunce et al. (2015), among others.

Definition 4.2.1. *For two random variables X and Y, X is said to be smaller than Y in*

(a) hazard rate order (denoted by $X \leqslant_{hr} Y$) if $\overline{F}(x)/\overline{G}(x)$ is decreasing in $x \geqslant 0$;

(b) reversed hazard rate order (denoted by $X \leqslant_{rh} Y$) if $F(x)/G(x)$ is decreasing in $x \geqslant 0$;

(c) likelihood ratio order (denoted by $X \leqslant_{lr} Y$) if $f(x)/g(x)$ is decreasing in $x \geqslant 0$;

(d) mean residual life order (denoted by $X \leqslant_{mrl} Y$) if

$$\frac{\int_t^\infty \overline{F}(x)dx}{\int_t^\infty \overline{G}(x)dx} \text{ is decreasing in } t \geqslant 0;$$

(e) mean inactivity time order (denoted by $X \leqslant_{mit} Y$) if

$$\frac{\int_0^t F(x)dx}{\int_0^t G(x)dx} \text{ is decreasing in } t \geqslant 0;$$

(f) variance residual life order (denoted by $X \leqslant_{vrl} Y$) if

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \int_x^\infty \overline{G}(u)dudx} \text{ is decreasing in } t \geqslant 0;$$

(g) variance inactivity time order (denoted by $X \leqslant_{vit} Y$) if

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t \int_0^x G(u)dudx} \text{ is decreasing in } t \geqslant 0.$$

Definition 4.2.2. *A random variable X is said to have*

(a) increasing (resp. decreasing) likelihood ratio (ILR (resp. DLR)) if for any $a > 0$, $f(t+a)/f(t)$ is decreasing (resp. increasing) in $t \geqslant 0$;

(b) increasing (resp. decreasing) failure rate (IFR (resp. DFR)) if X_t is stochastically decreasing (resp. increasing) in $t \geqslant 0$;

(c) decreasing reverse hazard rate (DRHR) if $X_{(t)}$ is stochastically increasing in $t > 0$;

(d) increasing (resp. decreasing) mean residual life (IMRL (resp. DMRL)) if

$$\frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0;$$

(e) increasing mean inactivity time (IMIT) if

$$\frac{\int_0^t F(u)du}{F(t)} \textit{ is increasing in } t \geqslant 0;$$

(f) increasing (resp. decreasing) variance residual life (IVRL (resp. DVRL)) if

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \overline{F}(x)dx} \textit{ is increasing (resp. decreasing) in } t \geqslant 0;$$

(g) increasing variance inactivity time (IVIT) if

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t F(x)dx} \textit{ is increasing in } t \geqslant 0.$$

Now we consider the following lemmas which will be helpful in proving the upcoming results of this chapter. First, recall from Karlin (1968) that a nonnegative function $\psi : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, the set of real numbers, is said to be TP_2 (*totally positive of order 2*) if $\psi(x, y)\psi(x^*, y^*) \geqslant \psi(x, y^*)\psi(x^*, y)$ for all $x, x^* \in \mathbb{X}$ and $y, y^* \in \mathbb{Y}$ such that $x \leqslant x^*$ and $y \leqslant y^*$, where $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the real line. ψ is said to be RR_2 (*reverse regular of order 2*) if the inequality is reversed.

Lemma 4.2.1. *(Joag-Dev et al., 1995). Let $\psi(x, y)$ be any TP_2 function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each i. Denote*

$$H_i(y) = \int_{\mathbb{X}} \psi(x, y)dF_i(x).$$

If $\overline{F}_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$ and if $\psi(x, y)$ is increasing in x for each y, then $H_i(y)$ is TP_2 in $y \in \mathbb{Y}$ and $i \in \{1, 2\}$.

Lemma 4.2.2. *(Nanda and Kundu, 2009). Let $s(x, y)$ be an RR_2 (TP_2) function (not necessarily a survival function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, and $F_i(x)$ be TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$, where $F_i(x)$ is a distribution function in x for each i. Assume that $s(x, y)$ is decreasing in x for every y. Then*

$$\phi_i(y) = \int_{\mathbb{X}} s(x, y)dF_i(x)$$

is RR_2 (TP_2) in $i \in \{1,2\}$ and $y \in \mathbb{Y}$. Conversely, if $\phi_i(y)$ is RR_2 (TP_2) in $i \in \{1,2\}$ and $y \in \mathbb{Y}$ whenever $F_i(x)$ is TP_2 in $i \in \{1,2\}$ and $x \in \mathbb{X}$, then $s(x,y)$ is RR_2 (TP_2) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.

Lemma 4.2.3. *(Misra and Naqvi, 2018a). Let $h_i(x,\theta) : [0,\infty) \times [0,\infty) \to \mathbb{R},\ i = 1,2,$ be a function and $l_i(\theta),\ i = 1,2,$ be the Lebesgue pdf of the random variable $T_i,\ i = 1,2$. Let*

$$\psi(x) = \frac{\int_0^\infty h_2(x,\theta) l_2(\theta) d\theta}{\int_0^\infty h_1(x,\theta) l_1(\theta) d\theta},\ x > 0.$$

Now, we define the following two pair of conditions for the positive function $h_i,\ i = 1,2$:

(A) $h_1(x,\theta)$ or $h_2(x,\theta)$ is TP_2 in $(x,\theta) \in (0,\infty) \times (0,\infty)$;

(B) $\frac{h_2(x,\theta)}{h_1(x,\theta)}$ increases in $x \in (0,\infty)$ and $\theta \in (0,\infty)$

and

(C) $h_1(x,\theta)$ or $h_2(x,\theta)$ is RR_2 in $(x,\theta) \in (0,\infty) \times (0,\infty)$;

(D) $\frac{h_2(x,\theta)}{h_1(x,\theta)}$ decreases in $x \in (0,\infty)$ and increases in $\theta \in (0,\infty)$.

Suppose A and B (C and D) hold. Further, if any of the following three conditions hold:

(i) $T_1 \leqslant_{lr} T_2$.

(ii) $T_1 \leqslant_{hr} T_2$ and $h_1(x,\theta)$ or $h_2(x,\theta)$ is increasing in $\theta \in (0,\infty)$.

(iii) $T_1 \leqslant_{rh} T_2$ and $h_1(x,\theta)$ or $h_2(x,\theta)$ is decreasing in $\theta \in (0,\infty)$.

Then, the function $\psi(x)$ increases (decreases) in $x \in [0,\infty)$.

The proof of the next lemma follows on using Theorem 1.B.48 of Shaked and Shanthikumar (2007), and making arguments similar to those used in proving Lemma 1 of Misra and Naqvi (2017).

Lemma 4.2.4. *Let $\psi_i : (0,\infty) \times (0,\infty) \to [0,\infty),\ i = 1,2,$ be nonnegative functions such that* *(i)* $\psi_1(x_1,\theta_1)\psi_2(x_2,\theta_2)-\psi_1(x_2,\theta_1)\psi_2(x_1,\theta_2)$ *decreases in* $\theta_1 \in (0,\theta_2)$, *for all* $0 \leqslant x_1 \leqslant x_2$*;*

(ii) $\psi_1(x_1,\theta_1)\psi_2(x_2,\theta_2)-\psi_1(x_2,\theta_1)\psi_2(x_1,\theta_2)+\psi_1(x_1,\theta_2)\psi_2(x_2,\theta_1)-\psi_1(x_2,\theta_2)\psi_2(x_1,\theta_1) \geqslant 0$, *for all* $0<\theta_1 \leqslant \theta_2$ *and* $0 < x_1 \leqslant x_2$.

Let T_i *be a random variable having probability density function* $l_i(\theta)$, $i=1,2$, *and support* $[0,\infty)$. *Suppose that* $T_1 \leqslant_{rh} T_2$, *then*

$$\psi(x)=\frac{\int_0^\infty \psi_2(x,\theta)l_2(\theta)d\theta}{\int_0^\infty \psi_1(x,\theta)l_1(\theta)d\theta} \text{ increases in } x \in (0,\infty).$$

Lemma 4.2.5. *(Misra and Naqvi, 2017). Let* $\{K_\theta(\cdot):\theta>0\}$ *and* $\{L_\theta(\cdot):\theta>0\}$ *be the families of nonnegative functions defined on* $\mathbb{R}$ *such that, for any* $\theta>0$, $K_\theta(t)=L_\theta(t)=0$, *whenever* $t \leqslant 0$, $K_\theta(t)$, $L_\theta(t)>0$ *for all* $t>0$. *Suppose that* $T_1 \leqslant_{lr} T_2$ *and the following conditions hold:*

(i) $L_\theta(\theta-x)$ *or* $K_\theta(\theta-x)$ *is* TP_2 *in* $(x,\theta)\in(0,\infty)\times(0,\infty)$*;*

(ii) $K_\theta(\theta-x)/L_\theta(\theta-x)$ *is increasing in* $x\in(0,\theta)$ *for every fixed* $\theta>0$*;*

(iii) $K_\theta(\theta-x)/L_\theta(\theta-x)$ *is increasing in* $\theta\in(x,\infty)$ *for every fixed* $x>0$.

Then,

$$\psi_1(x)=\frac{\int_0^\infty K_\theta(\theta-x)l_2(\theta)d\theta}{\int_0^\infty L_\theta(\theta-x)l_1(\theta)d\theta},\ x>0,$$

is increasing in $x\in(0,\infty)$.

Lemma 4.2.6. *(Misra and Naqvi, 2017). Suppose that* $T_1 \leqslant_{hr} T_2$ *and the following conditions hold:*

(i) $K_\theta(\theta-x)$ *is* TP_2 *in* $(x,\theta)\in(0,\infty)\times(0,\infty)$ *and, for every* $\theta>0$, $K_\theta(\theta-x)$ *is increasing in* $\theta\in(x,\infty)$*;*

(ii) $K_\theta(\theta-x)/L_\theta(\theta-x)$ *is increasing in* $x\in(0,\theta)$ *for every fixed* $\theta>0$*;*

(iii) $K_\theta(\theta-x)/L_\theta(\theta-x)$ *is increasing in* $\theta\in(x,\infty)$ *for every fixed* $x>0$.

Then,

$$\psi_1(x) = \frac{\int_0^\infty K_\theta(\theta - x) l_2(\theta) d\theta}{\int_0^\infty L_\theta(\theta - x) l_1(\theta) d\theta}, \ x > 0,$$

is increasing in $x \in (0, \infty)$.

With the help of the Lemma 4.2.4, we establish the following lemma. Here we obtain the same property of the function $\psi_1(x)$ as in Lemma 4.2.5, but with different assumptions.

Lemma 4.2.7. *Suppose that* $T_1 \leqslant_{rh} T_2$ *and the following conditions hold:*

(i) $L_\theta(\theta - x)$ *is* TP_2 *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$ *and, for every* $\theta > 0$, $L_\theta(\theta - x)$ *is decreasing in* $\theta \in (x, \infty)$*;*

(ii) $K_\theta(\theta - x)/L_\theta(\theta - x)$ *is increasing in* $x \in (0, \theta)$ *for every fixed* $\theta > 0$*;*

(iii) $K_\theta(\theta - x)/L_\theta(\theta - x)$ *is increasing in* $\theta \in (x, \infty)$ *for every fixed* $x > 0$.

Then,

$$\psi_1(x) = \frac{\int_0^\infty K_\theta(\theta - x) l_2(\theta) d\theta}{\int_0^\infty L_\theta(\theta - x) l_1(\theta) d\theta}, \ x > 0,$$

is increasing in $x \in (0, \infty)$.

Proof: Since $T_1 \leqslant_{rh} T_2$, it suffices to show that conditions (i) and (ii) of Lemma 4.2.4 are satisfied with $\psi_1(x, \theta) = L_\theta(\theta - x)$ and $\psi_2(x, \theta) = K_\theta(\theta - x)$, $(x, \theta) \in (0, \infty) \times (0, \infty)$. Let $0 < \theta_1 \leqslant \theta_2 < \infty, \ 0 < x_1 \leqslant x_2 < \infty$,

$$\begin{aligned} \Delta_1 &= \psi_1(x_1, \theta_1)\psi_2(x_2, \theta_2) - \psi_1(x_2, \theta_1)\psi_2(x_1, \theta_2) \\ &= L_{\theta_1}(\theta_1 - x_1) K_{\theta_2}(\theta_2 - x_2) - L_{\theta_1}(\theta_1 - x_2) K_{\theta_2}(\theta_2 - x_1) \end{aligned}$$

and

$$\begin{aligned} \Delta_2 &= \psi_1(x_1, \theta_1)\psi_2(x_2, \theta_2) - \psi_1(x_2, \theta_1)\psi_2(x_1, \theta_2) \\ &\quad + \psi_1(x_1, \theta_2)\psi_2(x_2, \theta_1) - \psi_1(x_2, \theta_2)\psi_2(x_1, \theta_1) \\ &= L_{\theta_1}(\theta_1 - x_1) K_{\theta_2}(\theta_2 - x_2) - L_{\theta_1}(\theta_1 - x_2) K_{\theta_2}(\theta_2 - x_1) \\ &\quad + L_{\theta_2}(\theta_2 - x_1) K_{\theta_1}(\theta_1 - x_2) - L_{\theta_2}(\theta_2 - x_2) K_{\theta_1}(\theta_1 - x_1). \end{aligned}$$

We need to show that for every $0 < x_1 \leqslant x_2 < \infty$, Δ_1 is decreasing in $\theta_1 \in (0, \theta_2)$ and $\Delta_2 \geqslant 0$. For $0 < \theta_1 \leqslant \theta_2 < \infty$ and $0 < x_1 \leqslant x_2 < \infty$, consider the following exhaustive cases:

Case 1. $0 < \theta_2 \leqslant x_1$.
Here, $\Delta_1 = 0$, *for all* $\theta_1 \in (0, \theta_2)$, *which is clearly decreasing in* $\theta_1 \in (0, \theta_2)$. *Also,* $\Delta_2 = 0$.

Case 2. $x_1 < \theta_2 \leqslant x_2$.
Under this case, $\Delta_1 = 0$, *which is decreasing in* $\theta_1 \in (0, \theta_2)$, *and, also* $\Delta_2 = 0$.

Case 3. $x_2 < \theta_2$.
In this case

$$\Delta_1 = \begin{cases} 0, & \theta_1 \leqslant x_1 \\ c_1(\theta_1), & x_1 < \theta_1 \leqslant x_2 \\ c_2(\theta_1), & x_2 < \theta_1. \end{cases}$$

Here $c_1(\theta_1) = L_{\theta_1}(\theta_1 - x_1)K_{\theta_2}(\theta_2 - x_2)$ *which, by condition (i), decreases in* $\theta_1 \in (0, \theta_2)$ *and*

$$c_2(\theta_1) = L_{\theta_1}(\theta_1 - x_2)\left[K_{\theta_2}(\theta_2 - x_2)\frac{L_{\theta_1}(\theta_1 - x_1)}{L_{\theta_1}(\theta_1 - x_2)} - K_{\theta_2}(\theta_2 - x_1)\right]$$

is decreasing in $\theta_1 \in (0, \theta_2)$ *by condition (i). Thus, it follows that* Δ_1 *is decreasing in* $\theta_1 \in (0, \theta_2)$. *Now,*

$$\Delta_2 = \begin{cases} 0, & \theta_1 \leqslant x_1 \\ d_1(\theta_1), & x_1 < \theta_1 \leqslant x_2 \\ d_2(\theta_1), & x_2 < \theta_1. \end{cases}$$

Where,

$$\begin{aligned} d_1(\theta_1) &= L_{\theta_1}(\theta_1 - x_1)K_{\theta_2}(\theta_2 - x_2) - L_{\theta_2}(\theta_2 - x_2)K_{\theta_1}(\theta_1 - x_1) \\ &= \frac{K_{\theta_2}(\theta_2 - x_2)}{L_{\theta_2}(\theta_2 - x_2)}L_{\theta_1}(\theta_1 - x_1)L_{\theta_2}(\theta_2 - x_2) - L_{\theta_2}(\theta_2 - x_2)K_{\theta_1}(\theta_1 - x_1) \\ &\geqslant \left[\frac{K_{\theta_2}(\theta_2 - x_1)}{L_{\theta_2}(\theta_2 - x_1)} - \frac{K_{\theta_1}(\theta_1 - x_1)}{L_{\theta_1}(\theta_1 - x_1)}\right] L_{\theta_1}(\theta_1 - x_1)L_{\theta_2}(\theta_2 - x_2) \geqslant 0, \end{aligned}$$

using conditions (ii) and (iii), respectively. Now,

$$\begin{aligned} d_2(\theta_1) &= L_{\theta_1}(\theta_1 - x_1)K_{\theta_2}(\theta_2 - x_2) - L_{\theta_1}(\theta_1 - x_2)K_{\theta_2}(\theta_2 - x_1) \\ &\quad + L_{\theta_2}(\theta_2 - x_1)K_{\theta_1}(\theta_1 - x_2) - L_{\theta_2}(\theta_2 - x_2)K_{\theta_1}(\theta_1 - x_1) \\ &\geqslant \frac{K_{\theta_2}(\theta_2 - x_1)}{L_{\theta_2}(\theta_2 - x_1)}[L_{\theta_1}(\theta_1 - x_1)L_{\theta_2}(\theta_2 - x_2) - L_{\theta_1}(\theta_1 - x_2)L_{\theta_2}(\theta_2 - x_1)] \\ &\quad + \frac{K_{\theta_1}(\theta_1 - x_1)}{L_{\theta_1}(\theta_1 - x_1)}[L_{\theta_1}(\theta_1 - x_2)L_{\theta_2}(\theta_2 - x_1) - L_{\theta_1}(\theta_1 - x_1)L_{\theta_2}(\theta_2 - x_2)] \\ &= \left[\frac{K_{\theta_2}(\theta_2 - x_1)}{L_{\theta_2}(\theta_2 - x_1)} - \frac{K_{\theta_1}(\theta_1 - x_1)}{L_{\theta_1}(\theta_1 - x_1)}\right] \\ &\quad \times [L_{\theta_1}(\theta_1 - x_1)L_{\theta_2}(\theta_2 - x_2) - L_{\theta_1}(\theta_1 - x_2)L_{\theta_2}(\theta_2 - x_1)] \geqslant 0, \end{aligned}$$

using conditions (i)-(iii). Hence the result follows.

4.3 Stochastic Comparisons based on *vrl* Order

Here we enhance the study on stochastic comparisons of RLRTs and ITRTs based on *vrl* order under the assumption that X and Y are not necessarily independent. To this aim, first we deal with some simple stochastic comparison results on RLRT/ITRT in one sample problem. Later, we provide stochastic comparisons of two systems failed at two different random times or having different random ages based on *vrl* order. Before discussing the main results, we briefly review the results available in the literature on stochastic comparisons of RLRTs and ITRTs. In order to make the presentation self-contained, we only restate the results without proof. In the first two propositions, two RLRTs/ITRTs are compared with respect to *lr* order.

Proposition 4.3.1. *(Misra, N. and Naqvi, S., 2018a). Let X, Z_1 and Y, Z_2 be nonnegative random variables not necessarily independent. Suppose $Z_1 \leqslant_{lr} Z_2$ and the following assumptions hold:*

(i) X^θ has ILR (DLR) for every $\theta > 0$ and $X^{\theta_1} \leqslant_{lr} (\geqslant_{lr}) X^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$;

or,

Y^θ has ILR (DLR) for every $\theta > 0$ and $Y^{\theta_1} \leqslant_{lr} (\geqslant_{lr}) Y^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$;

(ii) $X^{\theta} \leqslant_{lr} (\geqslant_{lr}) Y^{\theta}$, *for all* $\theta > 0$;

(iii) $\frac{g_{\theta}(x+\theta)}{f_{\theta}(x+\theta)}$ *is increasing in* $\theta \in (0, \infty)$ *for every fixed* $x > 0$.

Then, $X_{Z_1} \leqslant_{lr} (\geqslant_{lr}) Y_{Z_2}$.

Proposition 4.3.2. *(Misra, N. and Naqvi, S., 2017). Suppose* $Z_1 \leqslant_{lr} Z_2$ *and the following assumptions hold:*

(i) *for every* $\theta \geqslant 0$, X^{θ} *has ILR and* $X^{\theta_2} \leqslant_{lr} X^{\theta_1}$, *for all* $\theta_1 \leqslant \theta_2$;

or,

for every $\theta \geqslant 0$, Y^{θ} *has ILR and* $Y^{\theta_2} \leqslant_{lr} Y^{\theta_1}$, *for all* $\theta_1 \leqslant \theta_2$;

(ii) $X^{\theta} \leqslant_{lr} Y^{\theta}$, *for all* $\theta \geqslant 0$;

(iii) *for every fixed* $x > 0$, $\frac{g_{\theta}(\theta-x)}{f_{\theta}(\theta-x)}$ *is increasing in* $\theta \in (x, \infty)$.

Then, $X_{(Z_1)} \leqslant_{lr} Y_{(Z_2)}$.

The following two results present stochastic comparisons in terms of the hazard rate order.

Proposition 4.3.3. *(Misra, N. and Naqvi, S., 2018a). Let* X, Z_1 *and* Y, Z_2 *be nonnegative random variables not necessarily independent. Suppose* $Z_1 \leqslant_{lr} Z_2$ *and the following assumptions hold:*

(i) X^{θ} *has IFR (DFR) for every* $\theta > 0$ *and* $X^{\theta_1} \leqslant_{hr} (\geqslant_{hr}) X^{\theta_2}$, *for all* $\theta_1 \leqslant \theta_2$;

or,

Y^{θ} *has IFR (DFR) for every* $\theta > 0$ *and* $Y^{\theta_1} \leqslant_{hr} (\geqslant_{hr}) Y^{\theta_2}$, *for all* $\theta_1 \leqslant \theta_2$;

(ii) $X^{\theta} \leqslant_{hr} (\geqslant_{hr}) Y^{\theta}$, *for all* $\theta > 0$;

(iii) $\frac{\overline{G}_{\theta}(x+\theta)}{\overline{F}_{\theta}(x+\theta)}$ *is increasing in* $\theta \in (0, \infty)$ *for every fixed* $x > 0$.

Then, $X_{Z_1} \leqslant_{hr} (\geqslant_{hr}) Y_{Z_2}$.

Proposition 4.3.4. *(Misra, N. and Naqvi, S., 2017). Suppose* $Z_1 \leqslant_{lr} Z_2$ *and the following assumptions hold:*

(i) for every $\theta \geqslant 0$, X^{θ} has DRHR and $X^{\theta_2} \leqslant_{rh} X^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;

or,

for every $\theta \geqslant 0$, Y^{θ} has DRHR and $Y^{\theta_2} \leqslant_{rh} Y^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;

(ii) $X^{\theta} \leqslant_{rh} Y^{\theta}$, for all $\theta \geqslant 0$;

(iii) for every fixed $x > 0$, $\frac{G_\theta(\theta-x)}{F_\theta(\theta-x)}$ is increasing in $\theta \in (x, \infty)$.

Then, $X_{(Z_1)} \leqslant_{hr} Y_{(Z_2)}$.

Further, it may be of interest to examine the trade-off required in conditions of Proposition 4.3.4 if the *lr* ordering between Z_1 and Z_2 is replaced by *hr* ordering.

Proposition 4.3.5. *(Misra, N. and Naqvi, S., 2017). Suppose conditions (i)-(iii) of Proposition 4.3.4 hold. If $Z_1 \leqslant_{hr} Z_2$ and $G_\theta(\theta - x)$ is increasing in $\theta \in (x, \infty)$, then $X_{(Z_1)} \leqslant_{hr} Y_{(Z_2)}$.*

The following proposition provides sufficient conditions under which X_{Z_1} and Y_{Z_2} are ordered under *mrl* order.

Proposition 4.3.6. *(Misra, N. and Naqvi, S., 2018a). Let X, Z_1 and Y, Z_2 be nonnegative random variables not necessarily independent. Suppose $Z_1 \leqslant_{lr} Z_2$ and the following assumptions hold:*

(i) X^{θ} has DMRL (IMRL) for every $\theta > 0$ and $X^{\theta_1} \leqslant_{mrl} (\geqslant_{mrl}) X^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$;

or,

Y^{θ} has DMRL (IMRL) for every $\theta > 0$ and $Y^{\theta_1} \leqslant_{mrl} (\geqslant_{mrl}) Y^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$;

(ii) $X^{\theta} \leqslant_{mrl} (\geqslant_{mrl}) Y^{\theta}$, for all $\theta > 0$;

(iii) $\frac{\int_{x+\theta}^{\infty} \overline{G}_\theta(u)du}{\int_{x+\theta}^{\infty} \overline{F}_\theta(u)du}$ is increasing in $\theta \in (0, \infty)$ for every fixed $x > 0$.

Then, $X_{Z_1} \leqslant_{mrl} (\geqslant_{mrl}) Y_{Z_2}$.

Now, two inactivity time random variables with respect to *mrl* order are compared.

Proposition 4.3.7. *(Misra, N. and Naqvi, S., 2017). Suppose $Z_1 \leqslant_{lr} Z_2$ and the following assumptions hold:*

(i) for every $\theta \geqslant 0$, X^θ has IMIT and $X^{\theta_2} \leqslant_{mit} X^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;
or,
for every $\theta \geqslant 0$, Y^θ has IMIT and $Y^{\theta_2} \leqslant_{mit} Y^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;

(ii) $X^\theta \leqslant_{mit} Y^\theta$, for all $\theta \geqslant 0$;

(iii) for every fixed $x > 0$, $\frac{\int_0^{\theta-x} G_\theta(u)du}{\int_0^{\theta-x} F_\theta(u)du}$ is increasing in $\theta \in (x, \infty)$.

Then, $X_{(Z_1)} \leqslant_{mrl} Y_{(Z_2)}$.

Under the assumption that $Z_1 \leqslant_{hr} Z_2$, the following proposition compares two ITRTs with respect to *mrl* order.

Proposition 4.3.8. *(Misra, N. and Naqvi, S., 2017). Suppose conditions (i)-(iii) of Proposition 4.3.7 hold. If $Z_1 \leqslant_{hr} Z_2$ and $\int_0^{\theta-x} G_\theta(u)du$ is increasing in $\theta \in (x, \infty)$, then $X_{(Z_1)} \leqslant_{mrl} Y_{(Z_2)}$.*

The above propositions exhibit that two RLRTs or ITRTs are stochastically comparable with respect to *lr, hr* and *mrl* orders. In the sequel we provide some results which present sufficient conditions for stochastic monotonicity of RLRT/ITRT in terms of the *vrl* order. From Theorems 4.3.1-4.3.4 we assume that $X^\theta \overset{\mathrm{d}}{=} Y^\theta$, for all $\theta > 0$ where $\overset{\mathrm{d}}{=}$ means equality in distribution. The first result is related to stochastic comparison of RLRT based on *vrl* order.

Theorem 4.3.1. *Suppose $Z_1 \leqslant_{hr} Z_2$ and the following assumptions are fulfilled:*

- *X^θ has IVRL for every $\theta > 0$;*
- *$X^{\theta_1} \leqslant_{vrl} X^{\theta_2}$ for all $0 < \theta_1 \leqslant \theta_2$*
- *and $\int_{x+\theta}^{\infty} \int_t^{\infty} \overline{F}_\theta(u)dudt$ is increasing in $\theta \in (0, \infty)$ for every fixed $x > 0$.*

Then, $X_{Z_1} \leqslant_{vrl} X_{Z_2}$.

Proof: Denote by $\overline{F}$, $\overline{H}_1$ and $\overline{H}_2$ the survival functions of X, Z_1 and Z_2, respectively. Since X^θ has IVRL, and $X^{\theta_1} \leqslant_{vrl} X^{\theta_2}$, so

$$\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{F}_\theta(u)dudt \text{ is TP}_2 \text{ in } (x,\theta) \in (0,\infty)\times(0,\infty). \tag{4.3.1}$$

On the other hand,

$$\overline{H}_i(x) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\}\times(0,\infty) \tag{4.3.2}$$

in accordance with $Z_1 \leqslant_{hr} Z_2$. Again,

$$\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{F}_\theta(u)dudt \text{ is increasing in } \theta \text{ for each fixed } x. \tag{4.3.3}$$

Therefore, on using (4.3.1), (4.3.2) and (4.3.3), we obtain from Lemma 4.2.1,

$$\int_{0}^{\infty}\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{F}_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\}\times(0,\infty).$$

Therefore,

$$\int_{0}^{\infty}\int_{x}^{\infty}\int_{t+\theta}^{\infty}\overline{F}_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\}\times(0,\infty).$$

This gives that

$$\int_{0}^{\infty}\int_{x}^{\infty}\int_{t}^{\infty}\overline{F}_\theta(\theta+u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\}\times(0,\infty).$$

Or equivalently,

$$\int_{x}^{\infty}\int_{0}^{\infty}\int_{t}^{\infty}\overline{F}_\theta(\theta+u)dudH_i(\theta)dt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\}\times(0,\infty),$$

which implies that $\int_x^\infty\int_t^\infty\int_0^\infty \overline{F}_\theta(\theta+u)dH_i(\theta)dudt$ is TP$_2$ in $(i,x) \in \{1,2\}\times(0,\infty)$. Hence,

$$\frac{\int_x^\infty\int_t^\infty\int_0^\infty \overline{F}_\theta(\theta+u)dH_2(\theta)dudt}{\int_x^\infty\int_t^\infty\int_0^\infty \overline{F}_\theta(\theta+u)dH_1(\theta)dudt} \text{ is increasing in } x.$$

Thus, $X_{Z_1} \leqslant_{vrl} X_{Z_2}$. □

Under the assumptions that X, Z_1 and X, Z_2 are independent random variables, the above theorem yields the following result as given in Theorem 2.3.4.

Corollary 4.3.1. *If $Z_1 \leqslant_{hr} Z_2$ and X is of IVRL, then $(Z_1)_{(X)} \leqslant_{vrl} (Z_2)_{(X)}$.*

Now we consider the following example to verify Theorem 4.3.1.

Example 4.3.1. *Suppose the joint pdf of (X, Z_1) is*

$$f(x,y) = \frac{1}{y}e^{-(\frac{x}{y}+y)},\ x,y>0$$

and that of (Y, Z_2) is

$$g(x,y) = e^{-(\frac{x}{y}+y)},\ x,y>0.$$

Now the marginal pdf of Z_1 is $f_{Z_1}(y) = e^{-y},\ y>0$ and the conditional pdf of X given that $Z_1=\theta$ is

$$f_{X|Z_1=\theta}(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}},\ x,\theta>0.$$

Also, the marginal pdf of Z_2 and the conditional pdf of Y given that $Z_2=\theta$ are

$$g_{Z_2}(y) = ye^{-y},\ y>0 \quad \text{and} \quad g_{Y|Z_2=\theta}(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}\ x,\theta>0.$$

Therefore, $X^\theta \stackrel{\mathrm{d}}{=} Y^\theta$ as $f_{X|Z_1=\theta} = g_{Y|Z_2=\theta}$. Now $f_{Z_1}(y)/g_{Z_2}(y) = \frac{1}{y}$ is decreasing in y giving that $Z_1 \leqslant_{lr} Z_2$, or equivalently, $Z_1 \leqslant_{hr} Z_2$. Also, it can be seen that X^θ has IVRL for every $\theta>0$. Since

$$\begin{aligned}
\frac{\overline{F}_{X|Z_1=\theta_1}(x)}{\overline{F}_{X|Z_1=\theta_2}(x)} &= \frac{e^{-\frac{x}{\theta_1}}}{e^{-\frac{x}{\theta_2}}}\\
&= e^{-\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)x}\\
&= e^{-\frac{\theta_2-\theta_1}{\theta_1\theta_2}x}
\end{aligned}$$

is decreasing in $x \in (0,\infty)$ for all $0<\theta_1 \leqslant \theta_2$, so $X^{\theta_1} \leqslant_{hr} X^{\theta_2}$ for all $0<\theta_1\leqslant\theta_2$, or equivalently, $X^{\theta_1} \leqslant_{vrl} X^{\theta_2}$ for all $0<\theta_1\leqslant\theta_2$. Further,

$$\begin{aligned}
\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt &= \int_{x+\theta}^{\infty}\int_t^{\infty}e^{-\frac{u}{\theta}}dudt\\
&= \int_{x+\theta}^{\infty}\theta e^{-\frac{t}{\theta}}dt\\
&= \theta^2 e^{-(\frac{x}{\theta}+1)},\ \text{which is increasing in } \theta.
\end{aligned}$$

Thus, all the conditions of Theorem 4.3.1 are satisfied. Now

$$\frac{\int_0^\infty \overline{F}_\theta(\theta+x)g_{z_2}(\theta)d\theta}{\int_0^\infty \overline{F}_\theta(\theta+x)f_{Z_1}(\theta)d\theta} = \frac{\int_0^\infty \theta e^{-(\frac{x}{\theta}+\theta)d\theta}}{\int_0^\infty e^{-(\frac{x}{\theta}+\theta)}d\theta} = \alpha(x),\ say$$

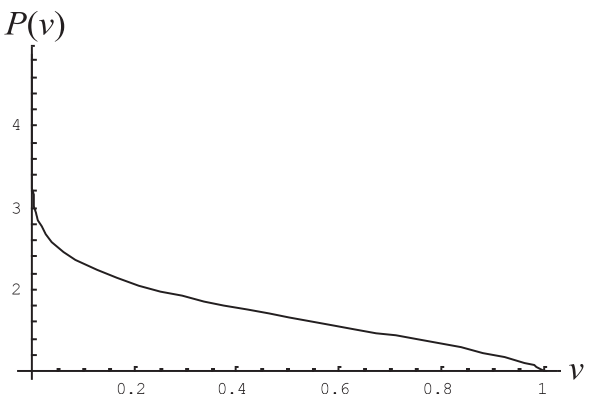

Figure 4.3.1: Plot of $P(v)$ against $v \in [0, 1]$ (Example 4.3.1)

is increasing in $x \in (0, \infty)$ *as shown in Figure 4.3.1. It is to be mentioned here that the substitution* $v = e^{-x}$ *has been used while plotting the curve so that* $\alpha(x) = P(v)$*, say. Hence,* $X_{Z_1} \leqslant_{hr} X_{Z_2}$*, which in turn gives that* $X_{Z_1} \leqslant_{vrl} X_{Z_2}$*.*

In the following theorem, we investigate sufficient conditions to establish *vrl* order between $X_{(Z_1)}$ and $X_{(Z_2)}$.

Theorem 4.3.2. *Let* X*,* Z_1 *and* Z_2 *be three nonnegative random variables where* X *is not necessarily independent of* Z_1 *and* Z_2*. Suppose* $Z_1 \leqslant_{hr} Z_2$ *and the following conditions hold:*

- X^θ *has IVIT for every* $\theta > 0$*;*
- $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ *for all* $0 < \theta_1 \leqslant \theta_2$
- *and* $\int_0^{\theta - x} \int_0^t F_\theta(u)dudt$ *is increasing in* $\theta \in (x, \infty)$ *for every fixed* $x > 0$*.*

Then, $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$*.*

Proof: Denote by F, H_1 and H_2 the *cdf*s of X, Z_1 and Z_2, respectively. Since X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ so $\int_0^{\theta-x} \int_0^t F_\theta(u)dudt$ is TP$_2$ in $(x, \theta) \in (0, \infty) \times (0, \infty)$. Now, $Z_1 \leqslant_{hr} Z_2$ implies that $\overline{H}_i(x)$ is TP$_2$ in $(i, x) \in \{1, 2\} \times (0, \infty)$ and again $\int_0^{\theta-x} \int_0^t F_\theta(u)dudt$ is increasing in θ for each fixed x. So, from Lemma 4.2.1, it follows that

$$\int_0^\infty \int_0^{\theta-x} \int_0^t F_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i, x) \in \{1, 2\} \times (0, \infty).$$

Or equivalently,

$$\int_x^\infty \int_x^\theta \int_0^{\theta-t} F_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty),$$

which in turn gives that

$$\int_x^\infty \int_t^\infty \int_0^{\theta-t} F_\theta(u)dudH_i(\theta)dt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

Therefore,

$$\int_x^\infty \int_t^\infty \int_t^\theta F_\theta(\theta-u)dudH_i(\theta)dt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty),$$

which is equivalent to $\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_i(\theta)dudt$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$. Or equivalently,

$$\frac{\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_2(\theta)dudt}{\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_1(\theta)dudt} \text{ is increasing in } x.$$

Hence, $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$. □

As an immediate consequence of Theorem 4.3.2, we have the following corollary by assuming X, Z_1 and X, Z_2 are independently distributed.

Corollary 4.3.2. *If $Z_1 \leqslant_{hr} Z_2$ and X is of IVIT, then $(Z_1)_X \leqslant_{vrl} (Z_2)_X$.*

Remark 4.3.1. *The result of Corollary 4.3.2 has been obtained in Theorem 2.3.3.*

We now present the following example to illustrate the application of the above result.

Example 4.3.2. *Suppose the joint pdf of (X, Z_1) is*

$$f(x,y) = \begin{cases} \frac{e^{-y}}{y+1}, & 0 < x < y \\ \frac{e^{-x}}{y+1}, & x > y \end{cases}$$

and that of (Y, Z_2) is

$$g(x,y) = \begin{cases} \frac{e^{-y}}{2}, & 0 < x < y \\ \frac{e^{-x}}{2}, & x > y \end{cases}$$

Now the marginal pdf of Z_1 and the conditional pdf of X given that $Z_1 = \theta$ are respectively

$$f_{Z_1}(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & otherwise \end{cases}$$

and

$$f_{X|Z_1=\theta}(x) = \begin{cases} \frac{1}{\theta+1}, & 0 < x < \theta \\ \frac{e^{\theta-x}}{\theta+1}, & x > \theta. \end{cases}$$

Also, the marginal pdf of Z_2 *is*

$$g_{Z_2}(y) = \begin{cases} \frac{e^{-y}(y+1)}{2}, & y > 0 \\ 0, & otherwise \end{cases}$$

and the conditional pdf of Y *given that* $Z_2 = \theta$ *is*

$$f(x,y) = \begin{cases} \frac{e^{-y}}{y+1}, & 0 < x < y \\ \frac{e^{-x}}{y+1}, & x > y \end{cases}$$

$$g_{Y|Z_2=\theta}(x) = \begin{cases} \frac{1}{\theta+1}, & 0 < x < \theta \\ \frac{e^{\theta-x}}{\theta+1}, & x > \theta. \end{cases}$$

Therefore, $X^\theta \overset{\mathrm{d}}{=} Y^\theta$ *as* $f_{X|Z_1=\theta} = g_{Y|Z_2=\theta}$. *It is clear that* $Z_1 \leqslant_{hr} Z_2$. *The cdf of* X^θ *is*

$$F_\theta(x) = \begin{cases} \frac{x}{\theta+1}, & 0 < \theta < y \\ 1 - \frac{e^{\theta-x}}{\theta+1}, & x > \theta \end{cases}$$

Hence, the hazard rate function of X^θ *is*

$$\begin{cases} \frac{1}{x}, & 0 < \theta < y \\ \frac{1}{(\theta+1)e^{x-\theta}-1}, & x > \theta, \end{cases}$$

which is decreasing in $x \in (0, \infty)$ *for every* $\theta > 0$. *Hence,* X^θ *has DRHR, which is turn gives that* X^θ *has IVIT. Since*

$$\frac{F_{X|Z_1=\theta_1}(x)}{F_{X|Z_1=\theta_2}(x)} = \begin{cases} \frac{\theta_2+1}{\theta_1+1}, & 0 < \theta < \theta_2 \\ \frac{1-\frac{e^{\theta_1-x}}{\theta_1+1}}{1-\frac{e^{\theta_2-x}}{\theta_2+1}}, & x > \theta_2 \end{cases}$$

is increasing in $x \in (0, \infty)$ *for all* $0 < \theta_1 \leqslant \theta_2$, *so* $X^{\theta_2} \leqslant_{rh} X^{\theta_1}$ *for all* $0 < \theta_1 \leqslant \theta_2$, *or equivalently,* $X^{\theta_2} \leqslant_{VIT} X^{\theta_1}$ *for all* $0 < \theta_1 \leqslant \theta_2$. *Further, for* $0 < x < \theta < \infty$,

$$\int_0^{\theta-x} \int_0^t F_\theta(u)dudt = \frac{(\theta-x)^3}{6(\theta+1)},$$

which is increasing in θ. *Therefore, all the conditions of Theorem 4.3.2 are satisfied. Hence, from Theorem 4.3.2, we have* $X_{(Z_1)} \leqslant_{VRL} X_{(Z_2)}$.

In the following theorem we perform stochastic comparison of RLRT based on *vrl* order extending a similar result given in Chapter 2 derived under independence of X and Z_1 (X and Z_2) to include situations where they may be dependent random variables.

Theorem 4.3.3. *Let $Z_1 \leqslant_{rh} Z_2$ and the following assumptions are fulfilled:*

- *X^θ has IVRL (DVRL) for every $\theta > 0$;*
- *$X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$ for all $0 < \theta_1 \leqslant \theta_2$*
- *and $\int_{x+\theta}^{\infty}\int_t^\infty \overline{F}_\theta(u)dudt$ is decreasing in $\theta \in (0,\infty)$ for every fixed $x > 0$.*

Then, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) X_{Z_2}$. Conversely, if $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) X_{Z_2}$ and $Z_1 \leqslant_{rh} Z_2$ then X^θ has IVRL (DVRL) and $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$ for all $0 < \theta_1 \leqslant \theta_2$.

Proof: Since X^θ has IVRL (DVRL) and $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$, so

$$\int_{x+\theta}^{\infty}\int_t^\infty \overline{F}_\theta(u)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Again, $Z_1 \leqslant_{rh} Z_2$ implies that $H_i(x)$ is TP$_2$ in $(i,x) \in \{1,2\}\times(0,\infty)$ and, also

$$\int_{x+\theta}^{\infty}\int_t^\infty \overline{F}_\theta(u)dudt \text{ is decreasing in } \theta \text{ for each fixed } x.$$

Therefore, from Lemma 4.2.2, it follows that

$$\int_0^\infty\int_{x+\theta}^{\infty}\int_t^\infty \overline{F}_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\}\times(0,\infty).$$

Or equivalently,

$$\int_0^\infty\int_x^\infty\int_{t+\theta}^\infty \overline{F}_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\}\times(0,\infty).$$

This gives that

$$\int_0^\infty\int_x^\infty\int_t^\infty \overline{F}_\theta(\theta+u)dudtdH_i(\theta) \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\}\times(0,\infty),$$

which yields

$$\int_x^\infty\int_0^\infty\int_t^\infty \overline{F}_\theta(\theta+u)dudH_i(\theta)dt \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\}\times(0,\infty).$$

Therefore,

$$\int_x^\infty \int_t^\infty \int_0^\infty \overline{F}_\theta(\theta+u)dH_i(\theta)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\} \times (0,\infty).$$

Or equivalently,

$$\frac{\int_x^\infty \int_t^\infty \int_0^\infty \overline{F}_\theta(\theta+u)dH_2(\theta)dudt}{\int_x^\infty \int_t^\infty \int_0^\infty \overline{F}_\theta(\theta+u)dH_1(\theta)dudt} \text{ is increasing (decreasing) in } x.$$

Hence, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) X_{Z_2}$.

Conversely, if $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) X_{Z_2}$, then

$$\int_x^\infty \int_t^\infty \int_0^\infty \overline{F}_\theta(\theta+u)dH_i(\theta)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\} \times (0,\infty).$$

Or equivalently,

$$\int_0^\infty \int_{x+\theta}^\infty \int_t^\infty \overline{F}_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\} \times (0,\infty),$$

and $Z_1 \leqslant_{rh} Z_2$ implies that $H_i(x)$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$ so from Lemma 4.2.2, it follows that

$$\int_{x+\theta}^\infty \int_t^\infty \overline{F}_\theta(u)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (x,\theta) \in (0,\infty) \times (0,\infty).$$

Hence, X^θ has IVRL (DVRL) and $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$. □

Assume that X is independent of Z_1 and Z_2 in Theorem 4.3.3 then we get the following corollary which was presented as Theorem 2.3.6 in Chapter 2.

Corollary 4.3.3. *If $Z_1 \leqslant_{rh} Z_2$ and X is of DVRL, then $(Z_1)_{(X)} \geqslant_{vrl} (Z_2)_{(X)}$.*

Consider the following example in support of Theorem 4.3.3.

Example 4.3.3. *Let the joint pdfs of (X, Z_1) and (Y, Z_2) be*

$$f(x,y) = 2(y+1)e^{-x-2y-xy} \text{ and } g(x,y) = (1+y)e^{-x-y-xy};\ x,y>0,$$

respectively. Then the marginal pdfs of Z_1 and Z_2 are

$$f_{Z_1}(y) = 2e^{-2y} \text{ and } g_{Z_2}(y) = e^{-y};\ y>0.$$

So the conditional pdf of X given that $Z_1 = \theta$ is

$$f_{X|Z_1=\theta}(x) = (1+\theta)e^{-(1+\theta)x};\ x, \theta > 0$$

and that of Y given that $Z_2 = \theta$ is

$$g_{Y|Z_2=\theta}(x) = (1+\theta)e^{-(1+\theta)x};\ x, \theta > 0.$$

Therefore, $X^\theta \stackrel{\mathrm{d}}{=} Y^\theta$ for all $\theta > 0$. Now, one can easily verify that $Z_1 \leqslant_{lr} Z_2$ which in turn implies that $Z_1 \leqslant_{rh} Z_2$. Also, the hazard rate function of X^θ is

$$\frac{f_{X|Z_1=\theta}(x)}{\overline{F}_{X|Z_1=\theta}(x)} = 1+\theta.$$

Hence, X^θ has IFR. Or equivalently, X^θ has DVRL for every $\theta > 0$. Now

$$\frac{\overline{F}_{X|Z_1=\theta_1}(x)}{\overline{F}_{X|Z_1=\theta_2}(x)} = \frac{e^{-(1+\theta_1)x}}{e^{-(1+\theta_2)x}} = e^{(\theta_2-\theta_1)x}$$

is increasing in $x \in (0,\infty)$ for all $0 < \theta_1 \leqslant \theta_2$. Thus, $X^{\theta_1} \geqslant_{vrl} X^{\theta_2}$ for all $0 < \theta_1 \leqslant \theta_2$. Further,

$$\begin{aligned}
\int_{x+\theta}^{\infty}\int_{t}^{\infty} \overline{F}_\theta(u)dudt &= \int_{x+\theta}^{\infty}\int_{t}^{\infty} e^{-(1+\theta)u}dudt \\
&= \frac{1}{1+\theta}\int_{x+\theta}^{\infty} e^{-(1+\theta)t}dt \\
&= \frac{1}{(1+\theta)^2}\left[-e^{-(1+\theta)t}\right]_{x+\theta}^{\infty} \\
&= \frac{e^{-(1+\theta)(x+\theta)}}{(1+\theta)^2}, \text{ which is decreasing in } \theta.
\end{aligned}$$

Therefore, all the conditions of Theorem 4.3.3 are satisfied. Now,

$$\alpha(x) = \frac{\int_0^\infty \overline{F}_\theta(\theta+x)g_{z_2}(\theta)d\theta}{\int_0^\infty \overline{F}_\theta(\theta+x)f_{Z_1}(\theta)d\theta} = \frac{\int_0^\infty e^{-(1+\theta)(x+\theta)}e^{-\theta}d\theta}{\int_0^\infty 2e^{-(1+\theta)(x+\theta)}e^{-2\theta}d\theta}$$

is decreasing in $x \in (0,\infty)$ as shown in Figure 4.3.2. Note that the substitution $v = e^{-x}$ has been used while plotting the curve so that $\alpha(x) = p(v)$, say. Hence, $X_{Z_1} \geqslant_{hr} X_{Z_2}$, which in turn gives that $X_{Z_1} \geqslant_{vrl} X_{Z_2}$.

In the following theorem, reversed hazard rate order is assumed between Z_1 and Z_2 and sufficient conditions are obtained to establish *vrl* order between $X_{(Z_1)}$ and $X_{(Z_2)}$.

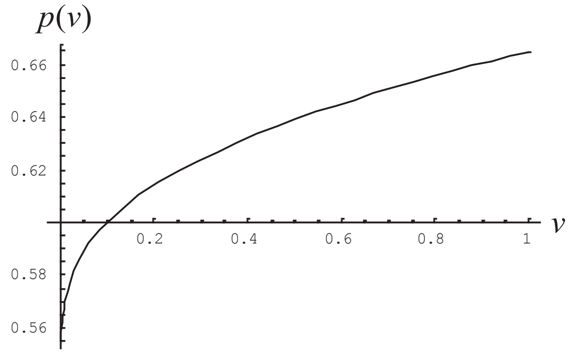

Figure 4.3.2: Plot of $p(v)$ against $v \in [0, 1]$ (Example 4.3.3)

Theorem 4.3.4. *Suppose $Z_1 \leqslant_{rh} Z_2$ and the following conditions are satisfied:*

- *X^θ has IVIT for every $\theta > 0$;*
- *$X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $0 < \theta_1 \leqslant \theta_2$*
- *and $\int_0^{\theta - x} \int_0^t F_\theta(u) dudt$ is decreasing in $\theta \in (x, \infty)$ for every fixed $x > 0$.*

Then, $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$. Conversely, if $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$ and $Z_1 \leqslant_{rh} Z_2$, then X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $0 < \theta_1 \leqslant \theta_2$.

Proof: Since X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$, so

$$\int_0^{\theta - x} \int_0^t F_\theta(u) dudt \text{ is TP}_2 \text{ in } (x, \theta) \in (0, \infty) \times (0, \infty).$$

Now $Z_1 \leqslant_{rh} Z_2$ implies that $H_i(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times (0, \infty)$ and again

$$\int_0^{\theta - x} \int_0^t F_\theta(u) dudt \text{ is decreasing in } \theta \text{ for each fixed } x.$$

Thus, from Lemma 4.2.2, it follows that

$$\int_0^\infty \int_0^{\theta - x} \int_0^t F_\theta(u) dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i, x) \in \{1, 2\} \times (0, \infty).$$

Or equivalently,

$$\int_x^\infty \int_x^\theta \int_0^{\theta - t} F_\theta(u) dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i, x) \in \{1, 2\} \times (0, \infty).$$

This gives that

$$\int_x^\infty \int_t^\infty \int_0^{\theta-t} F_\theta(u)dudH_i(\theta)dt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

Therefore,

$$\int_x^\infty \int_t^\infty \int_t^{\theta} F_\theta(\theta-u)dudH_i(\theta)dt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty),$$

which in turn implies that

$$\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_i(\theta)dudt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

Thus,

$$\frac{\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_2(\theta)dudt}{\int_x^\infty \int_t^\infty \int_u^\infty F_\theta(\theta-u)dH_1(\theta)dudt} \text{ is increasing in } x.$$

Hence, $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$.

Conversely, if $X_{(Z_1)} \leqslant_{vrl} X_{(Z_2)}$ then

$$\int_x^\infty \int_t^\infty \int_\theta^\infty F_\theta(\theta-u)dH_i(\theta)dudt \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

Or equivalently,

$$\int_0^\infty \int_0^{\theta-x} \int_0^t F_\theta(u)dudtdH_i(\theta) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

Again, $Z_1 \leqslant_{rh} Z_2$ implies that $H_i(x)$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$. So, from Lemma 4.2.2, it follows that

$$\int_0^{\theta-x} \int_0^t F_\theta(u)dudt \text{ is TP}_2 \text{ in } (x,\theta) \in (0,\infty) \times (0,\infty),$$

which is equivalent to X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $\theta_1 \leqslant \theta_2$. □

Stochastic comparisons of RLRT of two systems having different random ages are investigated in the upcoming theorems. The following result expands Theorem 2.3.5 of Chapter 2 to the case when X and Z_1 (Y and Z_2) are dependent random variables.

Theorem 4.3.5. *Let X, Z_1 and Y, Z_2 be nonnegative random variables not necessarily independent. Suppose $Z_1 \leqslant_{lr} Z_2$ and the following assumptions hold:*

(i) X^{θ} *has IVRL (DVRL) for every* $\theta > 0$ *and* $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$*, for all* $\theta_1 \leqslant \theta_2$*;*

or,

Y^{θ} *has IVRL (DVRL) for every* $\theta > 0$ *and* $Y^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) Y^{\theta_2}$*, for all* $\theta_1 \leqslant \theta_2$*;*

(ii) $X^{\theta} \leqslant_{vrl} (\geqslant_{vrl}) Y^{\theta}$*, for all* $\theta > 0$*;*

(iii) $\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt}$ *is increasing in* $\theta \in (0,\infty)$ *for every fixed* $x > 0$*.*

Then, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$*.*

Proof: Suppose X^{θ} has IVRL (DVRL) and $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$, so

$$\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Again, $X^{\theta} \leqslant_{vrl} (\geqslant_{vrl}) Y^{\theta}$ which implies that

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Also, $Z_1 \leqslant_{lr} Z_2$ and

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt} \text{ is increasing in } \theta \in (0,\infty).$$

Therefore, from Lemma 4.2.3(i) it follows that

$$\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt\right)h_i(\theta)d\theta \text{ is TP}_2 \text{ (RR}_2\text{) in } (i,x) \in \{1,2\}\times(0,\infty).$$

Thus,

$$\frac{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt\right)h_2(\theta)d\theta}{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Hence, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$.

Again, if Y^{θ} has IVRL (DVRL) and $Y^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) Y^{\theta_2}$, then

$$\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Now, $X^{\theta} \leqslant_{vrl} (\geqslant_{vrl}) Y^{\theta}$, which in turn gives that

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_{\theta}(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_{\theta}(u)dudt} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Also, $Z_1 \leqslant_{lr} Z_2$ and

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt} \text{ is increasing in } \theta \in (0,\infty).$$

Therefore, with the help of Lemma 4.2.3(i) it follows that

$$\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt\right)h_i(\theta)d\theta \text{ is TP}_2\text{ (RR}_2\text{) in } (i,x)\in\{1,2\}\times(0,\infty).$$

Thus,

$$\frac{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt\right)h_2(\theta)d\theta}{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing (decreasing) in } x\in(0,\infty).$$

Hence, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$. □

The next result provides some sufficient conditions for stochastic comparisons of RLRTs under the assumption of (reversed) hazard rate order.

Theorem 4.3.6. *Suppose conditions (i)-(iii) of Theorem 4.3.5 hold true and either of the following assumptions is fulfilled:*

(i) $Z_1 \leqslant_{hr} Z_2$ *and* $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt$ *or* $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt$ *is increasing in* $\theta\in(0,\infty)$;

(ii) $Z_1 \leqslant_{rh} Z_2$ *and* $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt$ *or* $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt$ *is decreasing in* $\theta\in(0,\infty)$.

Then, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$.

Proof: From condition (i) of Theorem 4.3.5 we have

$$\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt \text{ is TP}_2\text{ (RR}_2\text{) in } (x,\theta)\in(0,\infty)\times(0,\infty),$$

or

$$\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt \text{ is TP}_2\text{ (RR}_2\text{) in } (x,\theta)\in(0,\infty)\times(0,\infty).$$

Using conditions (ii) and (iii) of Theorem 4.3.5 we have

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt} \text{ is increasing (decreasing) in } x\in(0,\infty),$$

and

$$\frac{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt}{\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt} \text{ is increasing in } \theta \in (0,\infty).$$

Also, we have the given conditions $Z_1 \leqslant_{hr} Z_2$ and $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt$ or $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt$ is increasing in $\theta \in (0,\infty)$. Thus, on using Lemma 4.2.3(ii) we get

$$\frac{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt\right)h_2(\theta)d\theta}{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Hence, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$.

Again, if $Z_1 \leqslant_{rh} Z_2$ and $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt$ or $\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt$ is decreasing in $\theta \in (0,\infty)$. Then, on using Lemma 4.2.3(iii), we get

$$\frac{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{G}_\theta(u)dudt\right)h_2(\theta)d\theta}{\int_0^{\infty}\left(\int_{x+\theta}^{\infty}\int_t^{\infty}\overline{F}_\theta(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Hence, $X_{Z_1} \leqslant_{vrl} (\geqslant_{vrl}) Y_{Z_2}$. □

In what follows, we would like to perform stochastic comparisons of ITRT of two systems failed at different random times. Below we compare inactivity times of two systems, failed at two different random times, with respect to *vrl* order.

Theorem 4.3.7. *Suppose $Z_1 \leqslant_{lr} Z_2$ and the following assumptions hold:*

(i) for every $\theta \geqslant 0$, X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;

or,

for every $\theta \geqslant 0$, Y^θ has IVIT and $Y^{\theta_2} \leqslant_{vit} Y^{\theta_1}$, for all $\theta_1 \leqslant \theta_2$;

(ii) $X^\theta \leqslant_{vit} Y^\theta$, for all $\theta \geqslant 0$;

(iii) for every fixed $x > 0$, $\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt}$ is increasing in $\theta \in (x,\infty)$.

Then, $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$.

Proof: Since X^θ has IVIT and $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$, so

$$\int_0^{\theta-x}\int_0^t F_\theta(u)dudt \text{ is TP}_2 \text{ in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Again, $X^\theta \leqslant_{vit} Y^\theta$, which implies that

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } x \in (0,\theta).$$

Also, $Z_1 \leqslant_{lr} Z_2$ and

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } \theta \in (x,\infty).$$

Thus, from Lemma 4.2.5 it follows that

$$\frac{\int_0^\infty\left(\int_0^{\theta-x}\int_0^t G_\theta(u)dudt\right)h_2(\theta)d\theta}{\int_0^\infty\left(\int_0^{\theta-x}\int_0^t F_\theta(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing in } x \in (0,\infty).$$

Hence, $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$.

Again, if Y^θ has IVIT and $Y^{\theta_2} \leqslant_{vit} Y^{\theta_1}$ then

$$\int_0^{\theta-x}\int_0^t G_\theta(u)dudt \text{ is } \mathrm{TP}_2 \text{ in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Now, $X^\theta \leqslant_{vit} Y^\theta$, which in turn gives that

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } x \in (0,\theta).$$

Also, $Z_1 \leqslant_{lr} Z_2$ and

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } \theta \in (x,\infty)$$

yield from Lemma 4.2.5

$$\frac{\int_0^\infty\left(\int_0^{\theta-x}\int_0^t G_\theta(u)dudt\right)h_2(\theta)d\theta}{\int_0^\infty\left(\int_0^{\theta-x}\int_0^t F_\theta(u)dudt\right)h_1(\theta)d\theta} \text{ is increasing in } x \in (0,\infty).$$

Hence, $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$. □

Consider the following example to illustrate an application of the theorem.

Example 4.3.4. *Suppose the joint pdfs of* (X, Z_1) *and* (Y, Z_2) *are*

$$f(x,y) = 4(y+1)e^{-2x-2y-2xy} \quad \textit{and} \quad g(x,y) = (y+1)e^{-x-y-xy};\ x,y>0.$$

Now the marginal pdf of Z_1 *and the conditional pdf of* X *given that* $Z_1 = \theta$ *are*

$$f_{Z_1}(y) = 2e^{-2y},\ y > 0 \ \ \textit{and} \ \ f_{X|Z_1=\theta}(x) = 2(1+\theta)e^{-2(1+\theta)x};\ x, \theta > 0.$$

Similarly, the marginal pdf of Z_2 *and the conditional pdf of* Y *given that* $Z_2 = \theta$ *are*

$$g_{Z_2}(y) = e^{-y},\ y > 0 \ \ \textit{and} \ \ g_{Y|Z_2=\theta}(x) = (1+\theta)e^{-(1+\theta)x};\ x, \theta > 0.$$

It is clear that $Z_1 \leqslant_{lr} Z_2$. *Now the sf of* X^θ *is* $e^{-2(1+\theta)x}$. *Then the reversed hazard rate of* X^θ

$$\frac{2(1+\theta)e^{-2(1+\theta)x}}{1 - e^{-2(1+\theta)x}} = \frac{2(1+\theta)}{e^{2(1+\theta)x} - 1}, \ \textit{is decreasing in } x.$$

Thus, X^θ *is DRHR, which in turn gives that* X^θ *is IVIT. Now,*

$$\begin{aligned} \frac{f_{X|Z_1=\theta_1}(x)}{f_{X|Z_1=\theta_2}(x)} &= \frac{2(1+\theta_1)e^{-2(1+\theta_1)x}}{2(1+\theta_2)e^{-2(1+\theta_2)x}} \\ &= \frac{1+\theta_1}{1+\theta_2} e^{2(\theta_2-\theta_1)x} \end{aligned}$$

is increasing in x, *so* $X^{\theta_2} \leqslant_{lr} X^{\theta_1}$ *for all* $0 < \theta_1 \leqslant \theta_2$, *or equivalently,* $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ *for all* $0 < \theta_1 \leqslant \theta_2$. *Further,*

$$\frac{f_{X|Z_1=\theta}(x)}{g_{Y|Z_2=\theta}(x)} = \frac{2e^{-2(1+\theta)x}}{e^{-(1+\theta)x}} \ \textit{is dcreasing in } x.$$

Hence, $X^\theta \leqslant_{lr} Y^\theta$, *which implies that* $X^\theta \leqslant_{vit} Y^\theta$. *Now,*

$$\begin{aligned} \int_0^{\theta-x} \int_0^t G_\theta(u)dudt &= \int_0^{\theta-x} \int_0^t [1 - e^{-u(1+\theta)}]dudt \\ &= \int_0^{\theta-x} \left[u + \frac{e^{-u(1+\theta)}}{1+\theta} \right]_0^t dt \\ &= \int_0^{\theta-x} \left[t + \frac{e^{-t(1+\theta)}}{1+\theta} - \frac{1}{1+\theta} \right] dt \\ &= \left[\frac{t^2}{2} - \frac{e^{-t(1+\theta)}}{(1+\theta)^2} - \frac{t}{1+\theta} \right]_0^{\theta-x} \\ &= \frac{1}{(1+\theta)^2} + \frac{(\theta-x)^2}{2} - \frac{\theta-x}{1+\theta} - \frac{e^{-(\theta-x)(1+\theta)}}{(1+\theta)^2}. \end{aligned}$$

Again,

$$\begin{aligned}
\int_0^{\theta-x}\int_0^t F_\theta(u)dudt &= \int_0^{\theta-x}\int_0^t [1-e^{-2u(1+\theta)}]dudt \\
&= \int_0^{\theta-x}\left[u+\frac{e^{-2u(1+\theta)}}{2(1+\theta)}\right]_0^t dt \\
&= \int_0^{\theta-x}\left[t+\frac{e^{-2t(1+\theta)}}{2(1+\theta)}-\frac{1}{2(1+\theta)}\right] dt \\
&= \left[\frac{t^2}{2}-\frac{e^{-2t(1+\theta)}}{4(1+\theta)^2}-\frac{t}{2(1+\theta)}\right]_0^{\theta-x} \\
&= \frac{1}{4(1+\theta)^2}+\frac{(\theta-x)^2}{2}-\frac{\theta-x}{2(1+\theta)}-\frac{e^{-2(\theta-x)(1+\theta)}}{4(1+\theta)^2}.
\end{aligned}$$

Moreover,

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} = 2\frac{2+(\theta-x)^2(1+\theta)^2-2(\theta-x)(1+\theta)-2e^{-(\theta-x)(1+\theta)}}{1+2(\theta-x)^2(1+\theta)^2-2(\theta-x)(1+\theta)-e^{-2(\theta-x)(1+\theta)}},$$

which is increasing in θ. Therefore, all the conditions of Theorem 4.3.7 are satisfied. Hence, from Theorem 4.3.7 we have $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$.

The following theorem provides sufficient conditions for which $X_{(Z_1)}$ and $Y_{(Z_2)}$ are ordered under *vrl* order when hazard rate or reversed hazard rate order holds between Z_1 and Z_2.

Theorem 4.3.8. *Suppose conditions (i)-(iii) of Theorem 4.3.7 hold. If either of the following assumptions are fulfilled:*

(i) $Z_1 \leqslant_{hr} Z_2$ and $\int_0^{\theta-x}\int_0^t G_\theta(u)dudt$ is increasing in $\theta \in (x,\infty)$;

(ii) $Z_1 \leqslant_{rh} Z_2$ and $\int_0^{\theta-x}\int_0^t F_\theta(u)dudt$ is decreasing in $\theta \in (x,\infty)$

then $X_{(Z_1)} \leqslant_{VRL} Y_{(Z_2)}$.

Proof: Form conditions (ii) and (iii) of Theorem 4.3.7 it follows that

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } x \in (0,\theta),$$

and

$$\frac{\int_0^{\theta-x}\int_0^t G_\theta(u)dudt}{\int_0^{\theta-x}\int_0^t F_\theta(u)dudt} \text{ is increasing in } \theta \in (x,\infty).$$

(i) From condition (i) of Theorem 4.3.7 it follows that

$$\int_0^{\theta-x}\int_0^t G_\theta(u)dudt \text{ is TP}_2 \text{ in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Also, we have $\int_0^{\theta-x}\int_0^t G_\theta(u)dudt$ is increasing in $\theta \in (x,\infty)$ and $Z_1 \leqslant_{hr} Z_2$, so from Lemma 4.2.6 we obtain

$$\frac{\int_0^\infty \left(\int_0^{\theta-x}\int_0^t G_\theta(u)dudt\right) h_2(\theta)d\theta}{\int_0^\infty \left(\int_0^{\theta-x}\int_0^t F_\theta(u)dudt\right) h_1(\theta)d\theta} \text{ is increasing in } x \in (0,\infty).$$

Hence, $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$.

(ii) From condition (i) of Theorem 4.3.7 it follows that

$$\int_0^{\theta-x}\int_0^t F_\theta(u)dudt \text{ is TP}_2 \text{ in } (x,\theta) \in (0,\infty)\times(0,\infty).$$

Also, from given conditions we have $\int_0^{\theta-x}\int_0^t F_\theta(u)dudt$ is decreasing in $\theta \in (x,\infty)$ and $Z_1 \leqslant_{rh} Z_2$, so from Lemma 4.2.7 we obtain

$$\frac{\int_0^\infty \left(\int_0^{\theta-x}\int_0^t G_\theta(u)dudt\right) h_2(\theta)d\theta}{\int_0^\infty \left(\int_0^{\theta-x}\int_0^t F_\theta(u)dudt\right) h_1(\theta)d\theta} \text{ is increasing in } x \in (0,\infty).$$

Hence, $X_{(Z_1)} \leqslant_{vrl} Y_{(Z_2)}$.

4.4 Ageing Properties

In this section we study various ageing properties of RLRT/ITRT based on DVRL (IVRL) and IVIT classes. We investigate under what conditions the ageing property of X^θ is preserved for X_Z when X and Z are not necessarily independent. In the following theorem we show that DVRL class is preserved for RLRT under certain conditions.

Theorem 4.4.1. *Let (X, Z) be jointly distributed nonnegative random variables (not necessarily independent). If*

- *X^θ has IVRL (DVRL) for all $\theta > 0$;*
- *$X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$*
- *and, for fixed $x > 0$ and $y > 0$, $\frac{\int_{x+y}^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt}{\int_x^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt}$ is increasing in $\theta \in (x,\infty)$,*

then X_Z has IVRL (DVRL).

Proof: Since X^θ has IVRL (DVRL), so for every $\theta > 0$,

$$\frac{\int_{x+y}^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt}{\int_x^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Again,

$$\int_x^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt \text{ is TP}_2 \text{ (RR}_2\text{) in } (x,\theta) \in (0,\infty)\times(0,\infty)$$

since X^θ has IVRL (DVRL) and $X^{\theta_1} \leqslant_{vrl} (\geqslant_{vrl}) X^{\theta_2}$, for all $\theta_1 \leqslant \theta_2$. Also, for fixed $x > 0$ and $y > 0$,

$$\frac{\int_{x+y}^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt}{\int_x^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt} \text{ is increasing in } \theta \in (x,\infty).$$

So, from Lemma 4.2.3(i), it follows that

$$\frac{\int_0^{\infty}\left(\int_{x+y}^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt\right)h_1(\theta)d\theta}{\int_0^{\infty}\left(\int_x^{\infty}\int_t^{\infty}\overline{F}_\theta(u+\theta)dudt\right)h_1(\theta)d\theta} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Therefore,

$$\frac{\int_{x+y}^{\infty}\int_t^{\infty}\int_0^{\infty}\overline{F}_\theta(u+\theta)h_1(\theta)d\theta dudt}{\int_x^{\infty}\int_t^{\infty}\int_0^{\infty}\overline{F}_\theta(u+\theta)h_1(\theta)d\theta dudt} \text{ is increasing (decreasing) in } x \in (0,\infty).$$

Hence, X_Z has IVRL (DVRL). □

In the following theorems we assume that $X^\theta \stackrel{\text{d}}{=} Y^\theta$ where we take $Z_1 = Z_z$ and $Z_2 = Z$ such that $Z_z = (Z - z|Z > z)$ is the residual life at fixed time z. In Theorem 2.3.7, we have investigated how the ageing properties of X and Z affect ageing characteristic of ITRT under the assumption that X and Z are independently distributed. Below we consider the situation when X and Z are not independently distributed.

Theorem 4.4.2. *Suppose X and Z are jointly distributed nonnegative random variables (not necessarily independent). If Z is IFR, for every $\theta > 0$, X^{θ} has IVIT, $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $0 < \theta_1 \leqslant \theta_2$ and $\int_0^{\theta - x} \int_0^t F_\theta(u)dudt$ is increasing in $\theta \in (x, \infty)$ for every fixed $x > 0$, then $X_{(Z)}$ has DVRL.*

Proof: According to Theorem 1.B.38 of Shaked and Shanthikumar (2007), Z is IFR if and only if $Z_z \leqslant_{hr} Z$ for all $z \geqslant 0$. On using Theorem 4.3.2 we obtain $X_{(Z_z)} \leqslant_{vrl} X_{(Z)}$ for all $z \geqslant 0$. Note that $X_{(Z_z)} =_{st} \left[X_{(Z)}\right]_z$ for all $z \geqslant 0$. Thus, $\left[X_{(Z)}\right]_z \leqslant_{vrl} X_{(Z)}$ for all $z \geqslant 0$. Now from Theorem 2.3.1, it follows immediately that $X_{(Z)}$ is of DVRL. □

We provide below an example to show that all the conditions of the above theorem are satisfied.

Example 4.4.1. *Suppose the joint pdf of (X, Z) is*

$$f(x,y) = \begin{cases} \frac{e^{-y}}{2}, & 0 < x < y \\ \frac{e^{-x}}{2}, & x > y. \end{cases}$$

Now the marginal pdf of Z and the conditional pdf of X given that $Z = \theta$ are

$$f_Z(y) = \begin{cases} \frac{e^{-y}(y+1)}{2}, & y > 0 \\ 0, & otherwise \end{cases}$$

and

$$f_{X|Z=\theta}(x) = \begin{cases} \frac{1}{\theta+1}, & 0 < x < \theta \\ \frac{e^{\theta - x}}{\theta+1}, & x > \theta. \end{cases}$$

Thus, the survival function of Z is $e^{-y}(y+2)/2$, $y > 0$. Then the hazard rate of Z is $(y+1)/(y+2)$, which is increasing in $y > 0$. Hence Z is IFR. It can also be readily seen from Example 4.3.2 that all the remaining conditions of Theorem 4.4.2 are satisfied. Hence, from Theorem 4.4.2, $X_{(Z)}$ has DVRL.

In the following theorem we characterize the DVRL (IVIT) class based on ITRT.

Theorem 4.4.3. *If $Z_z \leqslant_{rh} Z$ for all $z \geqslant 0$, X^{θ} has IVIT, $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $0 < \theta_1 \leqslant \theta_2$ and $\int_0^{\theta - x} \int_0^t F_\theta(u)dudt$ is decreasing in $\theta \in (x, \infty)$ for every fixed $x > 0$, then $X_{(Z)}$ has DVRL. Conversely, if $X_{(Z)}$ has DVRL and $Z_z \leqslant_{rh} Z$ then X^{θ} has IVIT.*

Proof: Since $Z_z \leqslant_{rh} Z$ for all $z \geqslant 0$, X^θ has IVIT, $X^{\theta_2} \leqslant_{vit} X^{\theta_1}$ for all $0 < \theta_1 \leqslant \theta_2$ and $\int_0^{\theta-x} \int_0^t F_\theta(u)dudt$ is decreasing in θ, so by Theorem 4.3.4, $X_{(Z_z)} \leqslant_{vrl} X_{(Z)}$ for all $z \geqslant 0$. Or equivalently, $\left[X_{(Z)}\right]_z \leqslant_{vrl} X_{(Z)}$ for all $z \geqslant 0$. Now, from Theorem 2.3.1, it follows immediately that $X_{(Z)}$ is of DVRL.
Conversely, if $X_{(Z)}$ has DVRL then $\left[X_{(Z)}\right]_z \leqslant_{vrl} X_{(Z)}$, which is equivalent to $X_{(Z_z)} \leqslant_{vrl} X_{(Z)}$. Again, $Z_z \leqslant_{rh} Z$, so from Theorem 4.3.4, it follows that X^θ has IVIT. $\square$

To conclude, we consider the following ageing related result on RLRT.

Theorem 4.4.4. *Let (X, Z) be jointly distributed nonnegative random variables (not necessarily independent). If X_Z has DVRL (IVRL) and $Z_z \leqslant_{rh} Z$ then X^θ has IVRL (DVRL).*

Proof: Since X_Z has DVRL (IVRL) so $[X_Z]_z \leqslant_{vrl} (\geqslant_{vrl}) X_Z$, which is equivalent to $X_{Z_z} \leqslant_{vrl} (\geqslant_{vrl}) X_Z$. Again, $Z_z \leqslant_{rh} Z$, so from Theorem 4.3.3, we get X^θ has IVRL (DVRL).

4.5 On Some Applications and Examples

In many situations, stochastic comparisons of RLRTs and ITRTs, defined in Section 4.3, may be useful. To get a feel of the same, in this section, we present two such scenarios (one is for RLRT and another is for ITRT) where comparison of RLRTs and ITRTs may be of use. For some more scenarios on stochastic comparisons of RLRT and an application of ITRT for the study of robustness of a system one may refer to Misra and Naqvi (2018*a*, 2017). Although Theorem 4.3.5 is apparently interesting just from a theoretical point of view, we apply it to show that remaining lifetimes of two different model alarm trip are stochastically comparable.

Example 4.5.1. *Consider two different models (say, Model A and Model B) of a small size EMU (electric multiple unit) train, each having two engines, where the failure of one engine usually does not prevent the train from continuing on its journey. Let Model A* *have engines* e_1 *and* e_2 *with random lifetimes* X *and* Z_1 *and Model B have engines* e_3 *and* e_4 *with random lifetimes* Y *and* Z_2*, respectively. Now the remaining life of the Model A (Model B) EMU train after failure of engine* e_2 (e_4) *is* X_{Z_1} (Y_{Z_2})*. In this scenario, it would be of interest to compare the residual lives* X_{Z_1} *of Model A and* Y_{Z_2} *of Model B under vrl order to account for the dispersion of* X_{Z_1} *and* Y_{Z_2}*.*

Example 4.5.2. *In clinical trials, it often happens that the time at which a person goes to clinic for examination of a disease is actually different from the time he got infected. Let* X *and* Θ *be the respective times of infection and onset of a disease. Then* $X_{(\Theta)}$ *represents the 'incubation period' of the disease. Consider a hospital where patients from two different localities (say, Locality A and Locality B) arrive for the treatment of the disease. When a patient arrives at the hospital the time elapsed since infection (or initial time of infection) of the disease, which is unknown, varies between patients and also between two localities. Suppose that the inception of infection of the disease in a typical patient arriving at the hospital from Locality A (Locality B) is described by a random variable* X (Y) *and the corresponding time of beginning of the disease by* Z_1 (Z_2)*. Due to inter-individual variation, the specific incubation period of a disease is unknown and always expressed as a range. Therefore, it may be of interest to compare* $X_{(Z_1)}$ *and* $Y_{(Z_2)}$ *for Locality A and Locality B, respectively in terms of variability.*

In the following example, we provide an application of Theorem 4.3.5.

Example 4.5.3. *Consider a company that sells alarm trip 1-out-of-2 voting of two different models (say, Model A and Model B), where Model A is made up of components* c_1 *and* c_2 *with random lifetimes* X *and* Z_1 *and Model B is made up of components* c_3 *and* c_4 *with random lifetimes* Y *and* Z_2*, respectively. Now the remaining life of the Model A (Model B) after failure of component* c_2 (c_4) *is* X_{Z_1} (Y_{Z_2})*, without replacing the failed component. Suppose the joint pdfs of* (X, Z_1) *and* (Y, Z_2) *are*

$$f(x,y) = 2(y+1)e^{-x-2y-xy} \quad \textit{and} \quad g(x,y) = \frac{1}{y}e^{-(\frac{x}{y}+y)}; \;\; x,y > 0.$$

Needless to say that in mean residual life order we compare the means of their associated residual lifetimes. However, the means sometimes may be equal and therefore are often not *very informative. In many instances in applications one has more detailed information, for the purpose of comparison of two residual lives that have equal means, one is usually interested in the comparison of the dispersion of these random variables. As a result, stochastic comparison based on variance residual life order have been investigated in the literature. Now the marginal pdf of* Z_1 *and the conditional pdf of* X *given that* $Z_1 = \theta$ *are*

$$f_{Z_1}(y) = 2e^{-2y},\ y > 0 \ \ \text{and} \ \ f_{X|Z_1=\theta}(x) = (1+\theta)e^{-(1+\theta)x};\ x, \theta > 0.$$

Again, the marginal pdf of Z_2 *and the conditional pdf of* Y *given that* $Z_2 = \theta$ *are*

$$g_{Z_2}(y) = e^{-y},\ y > 0 \ \ \text{and} \ \ g_{Y|Z_2=\theta}(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}};\ x, \theta > 0.$$

Here $Z_1 \leqslant_{lr} Z_2$ *as* $f_{Z_1}(y)/g_{Z_2}(y) = \frac{1}{1+y}$ *is decreasing in* y*. Since,*

$$\frac{\overline{G}_{X|Z_1=\theta}(x)}{\overline{G}_{X|Z_2=\theta}(x)} = e^{-\left(\frac{\theta_2-\theta_1}{\theta_1\theta_2}\right)x}$$

is decreasing in x*, so* $Y^{\theta_1} \leqslant_{hr} Y^{\theta_2}$*, which in turn implies that* $Y^{\theta_1} \leqslant_{vrl} Y^{\theta_2}$ *for all* $0 < \theta_1 \leqslant \theta_2$*. It can be verified that* Y^{θ} *has IVRL for every* $\theta > 0$*. On noting that*

$$\frac{f_{X|Z_1=\theta}(x)}{g_{Y|Z_2=\theta}(x)} = \theta(1+\theta)e^{-\left(1+\theta-\frac{1}{\theta}\right)x} \ \text{decreasing in } x$$

we have $X^{\theta} \leqslant_{lr} Y^{\theta}$*, which in turn implies that* $X^{\theta} \leqslant_{vrl} Y^{\theta}$*. Now,*

$$\begin{aligned}
\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{G}_{Y|Z_2=\theta}(u)dudt &= \int_{x+\theta}^{\infty}\int_{t}^{\infty} e^{-\frac{u}{\theta}}dudt \\
&= \int_{x+\theta}^{\infty}\theta[-e^{-\frac{u}{\theta}}]_{t}^{\infty}dt \\
&= \int_{x+\theta}^{\infty}\theta e^{-\frac{t}{\theta}}dt \\
&= \theta^2 e^{-\left(\frac{x+\theta}{\theta}\right)}.
\end{aligned}$$

Also, from Example 4.3.3

$$\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{F}_{Y|Z_1=\theta}(u)dudt = \frac{e^{-(x+\theta)(1+\theta)}}{(1+\theta)^2}.$$

So,

$$\frac{\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{G}_{\theta}(u)dudt}{\int_{x+\theta}^{\infty}\int_{t}^{\infty}\overline{F}_{\theta}(u)dudt} = \theta^2(1+\theta)^2 e^{\left(1+\theta-\frac{1}{\theta}\right)(x+\theta)},$$

which is increasing in θ. *Thus, conditions (i)-(iii) of Theorem 4.3.5 are satisfied. Hence, from Theorem 4.3.5, we obtain* $X_{Z_1} \leqslant_{vrl} Y_{Z_2}$. *Thus the variance of the remaining lifetime for Model A alarm trip is less than the variance of the remaining lifetime for Model B alarm trip, which shows that the uncertainty is smaller for the case of Model A alarm trip than that of Model B alarm trip. Therefore, if we are interested in avoiding uncertainty or variability, we should choose the Model A alarm trip.*

Chapter 5

Generalized Orderings and Ageing Classes for RLRT and ITRT[1]

In this chapter, we enhance the study of ageing classes and stochastic comparisons of residual life at random time (RLRT) and inactivity time at random time (ITRT). We provide some new preservation properties of generalized ageing classes (viz. s-IFR, s-DFR) and generalized stochastic ordering (s-FR) for RLRT and ITRT, where s is a nonnegative integer. An application in reliability theory is also investigated. The results strengthen some results available in the literature and are expected to be useful in reliability theory, forensic science, econometrics, queueing theory and actuarial science.

5.1 Introduction

Let X be an absolutely continuous nonnegative random variable representing the lifetime of a unit/system. The residual life and the inactivity time of X at a fixed time $t > 0$ are defined as the random variables $X_t = (X - t|X > t)$ and $X_{(t)} = (t - X|X < t)$. If we replace t with a random variable Y, independent of X, then $X_Y = (X - Y|X > Y)$ denotes the residual life of X and $X_{(Y)} = (Y - X|X < Y)$ denotes the inactivity time of X at a random time Y. The RLRT is one of the important notions in reliability and

[1]Part of the work done in this chapter has been published in *Metrika*, 2019, 82, 691-704.

queuing theory (see Stoyan, 1983, for more details). For example, in the classical GI/G/1 queuing system, the idle period is a RLRT (see Marshall, 1968). ITRT is used in medical science to describe the dormant season of a disease, i.e., the time between infection and the beginning of a disease. Stochastic comparison results and ageing properties of RLRT and ITRT have been investigated by Yue and Cao (2000), Li and Zuo (2004), Li and Xu (2006), Misra et al. (2008), Cai and Zheng (2012), Dewan and Khaledi (2014) and Misra and Naqvi (2017). For some discussions on variance residual life and variance inactivity time one may refer to Gupta (2006), Mahdy (2012) and Kayid and Izadkhah (2016).

There is a proliferation of generalized partial orderings and generalized ageing classes in the literature over the recent years. They are used in reliability, economics, queues, inventory, actuarial science, applied probability and stochastic process contexts. Several of these definitions gave rise to new ageing classes. Fagiuoli and Pellerey (1993) obtained several preservation results on generalized orderings under Poisson shock models. Kass et al. (1994) used generalized orderings in actuarial sciences. Denuit et al. (1998) used some generalized orderings (s-convex order) in queues and insurance. Nanda et al. (1996a, b) studied different properties of generalized orderings such as moments, closure under mixtures etc. Nanda (1997) used generalized orderings in minimal repair policy. Fagiuoli and Pellerey (1994) gave a different type of classification of generalized ageing classes in a unified way depending on the generalized orderings. Several results on generalized orderings and generalized ageing classes with their implications were obtained by Hu et al. (2001, 2004), Navarro and Hernandez (2004) and Belzunce et al. (2008). Recently, Cai and Zheng (2009) characterized generalized ageing classes of inter-arrival times by the excess lifetime of a renewal process.

In this chapter we focus on the study of generalized ageing classes (s-IFR, s-DFR) and generalized stochastic ordering (s-FR) for RLRT and ITRT. First, we carry out stochastic comparison of RLRT and ITRT under s-FR ordering in two sample problems having same as well as different random ages or observed to fail at same/different random times. Later, the preservation properties and some characterizations of s-DFR ageing class and its dual are derived. Finally, an application in reliability theory is also investigated. From this general discussion, many known results on RLRT/ITRT can be obtained as particular cases of our general results. To be more specific, we obtain some interesting results which are generalizations of those by Yue and Cao (2000), Li and Zuo (2004), Misra et al. (2008) and Dewan and Khaledi (2014).

For the random variable X, let $f_X(x)$, $F_X(x)$, $\overline{F}_X(x)$, $r_X(x) = f_X(x)/\overline{F}_X(x)$ and $\tilde{r}_X(x) = f_X(x)/F_X(x)$ denote its density function, distribution function, survival function, hazard rate function and reversed hazard rate function, respectively. Let $\overline{T}_0(X,x) = \overline{\Phi}_0(X,x) = f_X(x)$, $\overline{\Phi}_1(X,x) = \overline{F}_X(x)$, $\forall x \geqslant 0$, $s \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ where $\mathbb{N} = \{0,1,2,3,\ldots\}$,

$$\overline{\Phi}_s(X,x) = \int_x^\infty \overline{\Phi}_{s-1}(X,u)du,\ \overline{T}_s(X,x) = \frac{\overline{\Phi}_s(X,x)}{\overline{\Phi}_s(X,0)}. \tag{5.1.1}$$

Then

$$r_s(X,x) = \frac{\overline{T}_{s-1}(X,x)}{\int_x^\infty \overline{T}_{s-1}(X,u)du}. \tag{5.1.2}$$

Clearly, for $s = 1,2$, $r_1(X,x) = r_X(x)$ and $r_2^{-1}(X,x) = E(X_x)$ are the hazard rate and the mean residual life of X, respectively. Also, take $X^{(1)} = X$ and for $k = 2,3,\ldots$, let $X^{(k)}$ be a random variable with survival function $\overline{T}_k(X,x)$ and hazard rate function $r_k(X,x)$, respectively. For $t > 0$, let $(X^{(k)})_t = (X^{(k)} - t|X^{(k)} > t)$ and $E(X^{(k)})_t = E(X^{(k)} - t|X^{(k)} > t)$. From (5.1.2), we have

$$\overline{T}_k(X,x) = \exp\left(-\int_0^x r_k(X,u)du\right), \quad r_{k+1}^{-1}(X,t) = E[X^{(k)}]_t. \tag{5.1.3}$$

In order to make the presentation self-contained we restate below the definition of some stochastic orders and ageing classes that are closely related to our main theme. For a comprehensive discussion on them, one may refer to Barlow and Proschan (1981), Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Belzunce et al. (2015), among others.

Definition 5.1.1. *Let X and Y be two nonnegative random variables. X is said to be smaller than Y in*

(a) likelihood ratio order (written as $X \leqslant_{lr} Y$), if $f_X(x)/f_Y(x)$ is decreasing in x;

(b) hazard rate order (written as $X \leqslant_{hr} Y$), if $r_X(x) \geq r_Y(x)$ for all $x \geqslant 0$;

(c) *reverse hazard rate order (written as* $X \leqslant_{rh} Y$*), if* $\tilde{r}_X(x) \leqslant \tilde{r}_Y(x)$ *for all* $x \geqslant 0$*;*

(d) *mean residual life order (written as* $X \leqslant_{mrl} Y$*), if* $E(X_t) \leqslant E(Y_t)$ *for all* $t \geqslant 0$*;*

(e) *variance residual life order (written as* $X \leqslant_{vrl} Y$*) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \leqslant \frac{\int_t^\infty \int_x^\infty \overline{G}(u)dudx}{\overline{G}(t)} \; for\ all\ t \geqslant 0;$$

(f) *s-FR order (written as* $X \leqslant_{s-FR} Y$*)* $\Leftrightarrow r_s(X,x) \geqslant r_s(Y,x)$ *for all* $x \geqslant 0$ *and* $s \in \mathbb{N}$*, or equivalently, if*

$$\frac{\overline{T}_s(X,x)}{\overline{T}_s(Y,x)} \text{ is decreasing in } x \geqslant 0 \text{ for all } s \in \mathbb{N}.$$

Definition 5.1.2. *X is said to be*

(a) *increasing (resp. decreasing) likelihood ratio (ILR (resp. DLR)) if for any* $a > 0$*,* $f(t+a)/f(t)$ *is decreasing (resp. increasing) in* $t \geqslant 0$*;*

(b) *increasing (resp. decreasing) failure rate (IFR (resp. DFR)) if* $r_X(x)$ *is increasing (decreasing) in* $x \geqslant 0$*;*

(c) *decreasing reverse hazard rate (DRHR) if* $\tilde{r}_X(x)$ *is decreasing in* $x \geqslant 0$*;*

(d) *decreasing (resp. increasing) mean residual life (DMRL (resp. IMRL)) if* $E(X_t)$ *is decreasing (increasing) in* $t \geqslant 0$*;*

(e) *decreasing (resp. increasing) variance residual life (DVRL (resp. IVRL)) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \overline{F}(t)} \text{ is decreasing (increasing) in } t \geqslant 0;$$

(f) *s-IFR (s-DFR) if*

$$\frac{\overline{\Phi}_s(X,x+t)}{\overline{\Phi}_s(X,x)} \text{ is decreasing (increasing) in } x \text{ for all } x,t \geqslant 0 \text{ and } s \in \mathbb{N},$$

or equivalently, if $r_s(X,x)$ *is increasing (decreasing) in* $x \geqslant 0$*.*

It is easy to see that

- $X \leqslant_{0-FR} Y \Leftrightarrow X \leqslant_{lr} Y$, $X \leqslant_{1-FR} Y \Leftrightarrow X \leqslant_{hr} Y$, $X \leqslant_{2-FR} Y \Leftrightarrow X \leqslant_{mrl} Y$, $X \leqslant_{3-FR} Y \Leftrightarrow X \leqslant_{vrl} Y$.

- X is 0-IFR (0-DFR) $\Leftrightarrow$ X is ILR (DLR), X is 1-IFR (1-DFR) $\Leftrightarrow$ X is IFR (DFR), X is 2-IFR (2-DFR) $\Leftrightarrow$ X is DMRL (IMRL), X is 3-IFR (3-DFR) $\Leftrightarrow$ X is DVRL (IVRL).

The following implications are well known:

- X is ILR (DLR) $\Rightarrow$ X is IFR (DFR) $\Rightarrow$ X is DMRL (IMRL) $\Rightarrow$ X is DVRL (IVRL).

- $X \leqslant_{lr} Y \Rightarrow X \leqslant_{hr} Y \Rightarrow X \leqslant_{mrl} Y \Rightarrow X \leqslant_{vrl} Y$.

By (5.1.1), (5.1.3) and simple calculation, one can prove the following.

Proposition 5.1.1. *Suppose $s \in \mathbb{N}_+$, then*

(a) X is s-IFR (s-DFR) $\Rightarrow X^{(2)}$ is s-IFR (s-DFR) $\Leftrightarrow X$ is $(s+1)$-IFR ($(s+1)$-DFR).

(b) X is s-IFR (s-DFR) $\Leftrightarrow X^{(k)}$ is $(s-k+1)$-IFR ($(s-k+1)$-DFR) for $k = 1, 2, 3, \ldots, s$. Specially, X is s-IFR (s-DFR) $\Leftrightarrow X^{(s)}$ is IFR (DFR).

(c) $X \leqslant_{(s+2)-FR} Y \Leftrightarrow X^{(s+2)} \leqslant_{hr} Y^{(s+2)} \Leftrightarrow X^{(s+1)} \leqslant_{mrl} Y^{(s+1)} \Leftrightarrow X^{(s)} \leqslant_{vrl} Y^{(s)}$.

(d) $X \leqslant_{s-FR} Y \Rightarrow X \leqslant_{(s+1)-FR} Y$.

(e) X is s-IFR (s-DFR) $\Leftrightarrow X_t \leqslant (\geqslant)_{s-FR} X$, for all $t \geqslant 0$.

5.2 Results on RLRT and ITRT

In this section, under s-FR ordering, we deal with stochastic comparisons of RLRT and ITRT of two systems having same as well as different random ages or observed to fail at the same/different random times. First we derive a result on stochastic comparison of $(X_1)_Y$ and $(X_2)_Y$, the residual lifetime of X_1 and X_2 at a random time Y, and then extend it for $(X_1)_{Y_1}$ and $(X_2)_{Y_2}$. Stochastic comparisons between $X_{(Z)}$ and $Y_{(Z)}$, the inactivity time of X and Y at a random time Z, are also focused on. Later we discuss the preservation properties and some characterizations of s-DFR and s-IFR ageing classes by means of RLRT. Further, it is shown that ILR (DLR) ageing class is preserved for RLRT and ITRT. In view of $X_Y = Y_{(X)}$ for continuous distributions, each result for either RLRT or ITRT can be translated into a result for the other by exchanging the roles of X and Y.

Let X and Y be two nonnegative and mutually independent random variables, and let $U = X_Y$ be the residual life of X at a random time Y. Then

$$\overline{\Phi}_1(U,t) = \overline{F}_U(t) = \frac{\int_0^\infty \overline{F}_X(t+y)dF_Y(y)}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}$$

and

$$f_U(t) = -\frac{d}{dt}\overline{F}_U(t) = \frac{\int_0^\infty f_X(t+y)dF_Y(y)}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}, \ \forall t \geqslant 0.$$

It is easy to verify by induction that $\forall t \geqslant 0$ and $k = 1, 2, 3\ldots$,

$$\overline{\Phi}_k(U,t) = \frac{\int_0^\infty \overline{\Phi}_k(X, t+y)dF_Y(y)}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}.$$

The following lemma is due to Dewan and Khaledi (2014) which will be used to prove the next theorem. First recall from Karlin (1968) that a nonnegative function $\psi : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, the set of real numbers, is said to be TP_2 (*totally positive of order 2*) if $\psi(x,y)\psi(x^*,y^*) \geqslant \psi(x,y^*)\psi(x^*,y)$ for all $x, x^* \in \mathbb{X}$ and $y, y^* \in \mathbb{Y}$ such that $x \leqslant x^*$ and $y \leqslant y^*$, where $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the real line.

Lemma 5.2.1. *Let $h_i(x,y)$, $i = 1, 2$, be a nonnegative real valued function on $\mathbb{R} \times \mathbb{X}$, where $\mathbb{X}$ is a subset of real line. If*
(i) $h_2(x,y)/h_1(x,y)$ is increasing in x and y and
(ii) if either $h_1(x,y)$ or $h_2(x,y)$ is TP_2 in (x,y),
then

$$s_i(x) = \int_{\mathbb{X}} h_i(x,y)l(y)dy$$

is TP_2 in (i,x), where l is a continuous function with $\int_{\mathbb{X}} l(y)dy < \infty$.

Theorem 5.2.1. *Let X_i, $i = 1, 2$ be two nonnegative random variables having density function f_{X_i}, distribution function F_{X_i}, survival function $\overline{F}_{X_i}$. Let Y be a random variable independent of X_1 and X_2 with density function f_Y and distribution function F_Y. Suppose that $s \in \mathbb{N}_+$. If $X_1 \leqslant_{s-FR} X_2$ and either X_1 or X_2 is s-DFR then $(X_1)_Y \leqslant_{s-FR} (X_2)_Y$.*

Proof: X_i is s-DFR if and only if

$$\overline{T}_s(X_i, x+y) \text{ is TP}_2 \text{ in } (x,y) \in (0,\infty) \times (0,\infty).$$

That is,

$$\overline{\Phi}_s(X_i, x+y) \text{ is TP}_2 \text{ in } (x,y) \in (0,\infty) \times (0,\infty).$$

On the other hand, $X_1 \leqslant_{s-FR} X_2$ if and only if

$$\frac{\overline{T}_s(X_2, u)}{\overline{T}_s(X_1, u)} \text{ is increasing in } u > 0,$$

which in turn implies that

$$\frac{\overline{T}_s(X_2, x+y)}{\overline{T}_s(X_1, x+y)} \text{ is increasing in } x > 0 \text{ as well as } y > 0.$$

Or equivalently,

$$\frac{\overline{\Phi}_s(X_2, x+y)}{\overline{\Phi}_s(X_1, x+y)} \text{ is increasing in } x > 0 \text{ as well as } y > 0.$$

Hence, the conditions of Lemma 5.2.1 are satisfied by replacing the function $l(y)$ with $f_Y(y)$ and $h_i(x,y)$ with $\overline{\Phi}_s(X_i, x+y)$, $i = 1, 2$. Therefore,

$$\overline{\Phi}_s((X_i)_Y, x) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty).$$

This gives that

$$\overline{T}_s((X_i)_Y, x) \text{ is TP}_2 \text{ in } (i,x) \in \{1,2\} \times (0,\infty),$$

which is equivalent to

$$\frac{\overline{T}_s((X_1)_Y, x)}{\overline{T}_s((X_2)_Y, x)} \text{ is decreasing in } x \geqslant 0.$$

Therefore, $(X_1)_Y \leqslant_{s-FR} (X_2)_Y$. $\square$

The above theorem is somewhat related to the following theorem of Hu et al. (2001) on preservation of s-FR ordering for X_t, the residual life of X at a fixed time $t > 0$, to include situation where t is a random variable independent of X.

Theorem 5.2.2. *For* $s \in \mathbb{N}$, $X \leqslant_{s-FR} Y \Longleftrightarrow X_t \leqslant_{s-FR} Y_t, \forall t \geqslant 0$.

Taking $s = 1, 2$ in the Theorem 5.2.1, the following results of Dewan and Khaledi (2014) are obtained easily.

Corollary 5.2.1. *Let X_1, X_2 and Y be three nonnegative random variables and Y be independent of X_1 and X_2. If*

- *$X_1 \leqslant_{hr} X_2$ and either X_1 or X_2 is DFR then $(X_1)_Y \leqslant_{hr} (X_2)_Y$;*
- *$X_1 \leqslant_{mrl} X_2$ and either X_1 or X_2 is IMRL then $(X_1)_Y \leqslant_{mrl} (X_2)_Y$.*

Taking $s = 3$ in Theorem 5.2.1, the following result is obtained.

Corollary 5.2.2. *If $X_1 \leqslant_{vrl} X_2$ and either X_1 or X_2 is IVRL then $(X_1)_Y \leqslant_{vrl} (X_2)_Y$.*

Now we consider the stochastic comparisons of RLRT of two system having different random ages. Now we recall the following lemma from Chapter 2 which will be used to prove the next theorem.

Lemma 5.2.2. *Let $h_i(x, y)$, $i = 1, 2$, be a nonnegative real valued function on $\mathbb{R} \times \mathbb{X}$, where $\mathbb{X}$ is a subset of real line. If*
(i) $h_2(x, y)/h_1(x, y)$ is increasing in x and y,
(ii) $l_2(y)/l_1(y)$ is increasing in y and
(iii) if either $h_1(x, y)$ or $h_2(x, y)$ is TP_2 in (x, y),
then

$$s_i(x) = \int_{\mathbb{X}} h_i(x, y) l_i(y) dy$$

is TP_2 in (i, x), where l_i is a continuous function with $\int_{\mathbb{X}} l_i(y)dy < \infty$.

Theorem 5.2.3. *Let X_i, $i = 1, 2$ be two nonnegative random variables having density function f_{X_i}, distribution function F_{X_i}, survival function $\overline{F}_{X_i}$. Let Y_i, $i = 1, 2$ be another two nonnegative random variables independent of X_1 and X_2, respectively with density function f_{Y_i} and distribution function F_{Y_i}. Suppose $Y_1 \leqslant_{lr} Y_2$ and the following assumptions hold:*

- *$X_1 \leqslant_{s-FR} X_2$;*
- *X_1 or X_2 is s-DFR.*

Then, $(X_1)_{Y_1} \leqslant_{s-FR} (X_2)_{Y_2}$, *for all* $s \in \mathbb{N}_+$.

Proof: X_i is s-DFR if and only if

$$\overline{T}_s(X_i, x+y) \text{ is TP}_2 \text{ in } (x, y) \in (0, \infty) \times (0, \infty).$$

Or equivalently,

$$\overline{\Phi}_s(X_i, x+y) \text{ is TP}_2 \text{ in } (x, y) \in (0, \infty) \times (0, \infty).$$

On the other hand, $X_1 \leqslant_{s-FR} X_2$ if and only if

$$\frac{\overline{T}_s(X_2, u)}{\overline{T}_s(X_1, u)} \text{ is increasing in } u > 0,$$

which in turn implies that

$$\frac{\overline{T}_s(X_2, x+y)}{\overline{T}_s(X_1, x+y)} \text{ is increasing in } x > 0 \text{ as well as } y > 0.$$

That is,

$$\frac{\overline{\Phi}_s(X_2, x+y)}{\overline{\Phi}_s(X_1, x+y)} \text{ is increasing in } x > 0 \text{ as well as } y > 0.$$

Again, $Y_1 \leqslant_{lr} Y_2$ if and only if

$$\frac{f_{Y_2}(u)}{f_{Y_1}(u)} \text{ is increasing in } u > 0.$$

Hence, the conditions of Lemma 5.2.2 are satisfied by replacing the function $l_i(y)$ with $f_{Y_i}(y)$ and $h_i(x, y)$ with $\overline{\Phi}_s(X_i, x+y)$, $i = 1, 2$. Therefore,

$$\overline{\Phi}_s((X_i)_{Y_i}, x) \text{ is TP}_2 \text{ in } (i, x) \in \{1, 2\} \times (0, \infty),$$

which implies that

$$\overline{T}_s((X_i)_{Y_i}, x) \text{ is TP}_2 \text{ in } (i, x) \in \{1, 2\} \times (0, \infty).$$

Or equivalently,

$$\frac{\overline{T}_s((X_1)_{Y_1}, x)}{\overline{T}_s((X_2)_{Y_2}, x)} \text{ is decreasing in } x \geqslant 0.$$

Hence, the result follows. □

Taking $s = 1, 2, 3$ in the above theorem, the following results of Chapter 2 are obtained easily on using $X_Y = Y_{(X)}$.

Corollary 5.2.3. *Let X_1, Y_1 and X_2, Y_2 be independent nonnegative random variables. If*

- *$Y_1 \leqslant_{lr} Y_2$, $X_1 \leqslant_{hr} X_2$ and either X_1 or X_2 is DFR then $(X_1)_{Y_1} \leqslant_{hr} (X_2)_{Y_2}$;*
- *$Y_1 \leqslant_{lr} Y_2$, $X_1 \leqslant_{mrl} X_2$ and either X_1 or X_2 is IMRL then $(X_1)_{Y_1} \leqslant_{mrl} (X_2)_{Y_2}$;*
- *$Y_1 \leqslant_{lr} Y_2$, $X_1 \leqslant_{vrl} X_2$ and either X_1 or X_2 is IVRL then $(X_1)_{Y_1} \leqslant_{vrl} (X_2)_{Y_2}$.*

In continuation with Theorem 5.2.1 we state the following two theorems on stochastic comparison of ITRT of two different systems observed to fail at the same random time. The proof of the following result is immediate from Nanda and Kundu (2009) and hence omitted.

Theorem 5.2.4. *Let X and Y be two nonnegative random variables representing the lifetimes of two systems failed at random time Z. Let Z be independent of X and Y. If $X \leqslant_{rh} Y$ and Z is s-IFR (s-DFR) then $X_{(Z)} \geqslant_{s-FR} (\leqslant_{s-FR}) Y_{(Z)}$ for all $s \in \mathbb{N}_+$.*

For $s = 1, 2$ we obtain the following results of Chapter 2.

Corollary 5.2.4.

- *If $X \leqslant_{rh} Y$ and Z is DFR (IFR), then $X_{(Z)} \leqslant_{hr} (\geqslant_{hr}) Y_{(Z)}$.*
- *If $X \leqslant_{rh} Y$ and Z is IMRL (DMRL), then $X_{(Z)} \leqslant_{mrl} (\geqslant_{mrl}) Y_{(Z)}$.*

Consider the following lemma which will be used to prove the upcoming theorem.

Lemma 5.2.3. *(Joag-Dev et al., 1995). Let $\psi(x, y)$ be any TP_2 function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each $i \in \{1, 2\}$. Denote*

$$H_i(y) = \int_{\mathbb{X}} \psi(x, y) dF_i(x).$$

If $\overline{F}_i(x)$ is TP_2 in $i \in \{1, 2\}$ and $x \in \mathbb{X}$ and if $\psi(x, y)$ is increasing in $x \in \mathbb{X}$ for each $y \in \mathbb{Y}$, then $H_i(y)$ is TP_2 in $y \in \mathbb{Y}$ and $i \in \{1, 2\}$.

The following theorem gives another stochastic comparison of ITRT. Here we provide some sufficient conditions under which two ITRTs are stochastically comparable in $(s+1)$-FR sense.

Theorem 5.2.5. *Suppose that Z is independent of X and Y. If $X \leqslant_{hr} Y$ and Z is of $(s+1)$-DFR then $X_{(Z)} \leqslant_{(s+1)-FR} Y_{(Z)}$, for all $s \in \mathbb{N}_+$.*

Proof: Denote by $F_1,\ F_2$ and H the distribution functions of $X,\ Y$ and Z, respectively. Since Z is of $(s+1)$-DFR, we have for all $y \geqslant 0$ and $\triangle > 0$,

$$\frac{\int_{y+\triangle}^{\infty} \overline{\Phi}_s(Z,u)du}{\overline{\Phi}_s(Z, y+\triangle)} \geqslant \frac{\int_{y}^{\infty} \overline{\Phi}_s(Z,u)du}{\overline{\Phi}_s(Z, y)}.$$

Now,

$$\begin{aligned} \frac{d}{dy}\left(\frac{\int_{y+\triangle}^{\infty} \overline{\Phi}_s(Z,u)du}{\int_{y}^{\infty} \overline{\Phi}_s(Z,u)du}\right) &= -\frac{\overline{\Phi}_s(Z, y+\triangle)}{\int_{y}^{\infty} \overline{\Phi}_s(Z,u)du} + \frac{\overline{\Phi}_s(Z,y)(\int_{y+\triangle}^{\infty} \overline{\Phi}_s(Z,u)du)}{(\int_{y}^{\infty} \overline{\Phi}_s(Z,u)du)^2} \\ &\geqslant 0. \end{aligned}$$

Hence,

$$\frac{\int_{y+\triangle}^{\infty} \overline{\Phi}_s(Z,u)du}{\int_{y}^{\infty} \overline{\Phi}_s(Z,u)du} \text{ is increasing in } y \geqslant 0.$$

Thus,

$$\frac{\int_{y_2+t_2}^{\infty} \overline{\Phi}_s(Z,u)du}{\int_{y_1+t_2}^{\infty} \overline{\Phi}_s(Z,u)du} \geqslant \frac{\int_{y_2+t_1}^{\infty} \overline{\Phi}_s(Z,u)du}{\int_{y_1+t_1}^{\infty} \overline{\Phi}_s(Z,u)du}, \tag{5.2.1}$$

for all $0 < t_1 \leqslant t_2 < y_1 \leqslant y_2$. Denote

$$\psi(y,t) = \begin{cases} \int_{y+t}^{\infty} \overline{\Phi}_s(Z,u)du, & y > 0 \\ 0, & y \leqslant 0. \end{cases}$$

Then (5.2.1) gives that

$$\psi(y_1,t_1)\psi(y_2,t_2) \geqslant \psi(y_1,t_2)\psi(y_2,t_1), \tag{5.2.2}$$

for all $(t_1,t_2,y_1,y_2) \in S = \{(t_1,t_2,y_1,y_2) : 0 < t_1 \leqslant t_2 < y_1 \leqslant y_2\}$. It can be verified that (5.2.2) is also valid for those $(t_1,t_2,y_1,y_2) \in \{(t_1,t_2,y_1,y_2) : 0 < t_1 \leqslant t_2;\ 0 < y_1 \leqslant y_2\} - S$. Thus $\psi(y,t)$ is TP_2 in $(y,t) \in (0,\infty) \times (0,\infty)$. Let

$$\mathcal{H}_i(t) = \frac{\int_0^{\infty} \psi(y,t)dF_i(y)}{\int_0^{\infty} \overline{\Phi}_s(Z,y)dF_i(y)}.$$

Now $X \leqslant_{hr} Y$ gives that $\overline{F}_i(x)$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$ and $\psi(y,t)$ is increasing in y for each fixed t. From Lemma 5.2.3 it follows that $\mathcal{H}_i(t)$ is TP_2 in $(i,t) \in \{1,2\} \times (0,\infty)$. Then

$$\begin{aligned}
\frac{\mathcal{H}_2(t)}{\mathcal{H}_1(t)} &= \frac{\int_0^\infty \psi(y,t)dF_2(y)}{\int_0^\infty \psi(y,t)dF_1(y)} \times \frac{\int_0^\infty \overline{\Phi}_s(Z,y)dF_1(y)}{\int_0^\infty \overline{\Phi}_s(Z,y)dF_2(y)} \\
&= \frac{\int_0^\infty \int_{y+t}^\infty \overline{\Phi}_s(Z,u)dudF_2(y)}{\int_0^\infty \int_{y+t}^\infty \overline{\Phi}_s(Z,u)dudF_1(y)} \times \frac{\int_0^\infty \overline{\Phi}_s(Z,y)dF_1(y)}{\int_0^\infty \overline{\Phi}_s(Z,y)dF_2(y)} \\
&= \frac{\int_0^\infty \int_t^\infty \overline{\Phi}_s(Z,y+u)dudF_2(y)}{\int_0^\infty \int_t^\infty \overline{\Phi}_s(Z,y+u)dudF_1(y)} \times \frac{\int_0^\infty \overline{\Phi}_s(Z,y)dF_1(y)}{\int_0^\infty \overline{\Phi}_s(Z,y)dF_2(y)} \\
&= \frac{\int_t^\infty \int_0^\infty \overline{\Phi}_s(Z,y+u)dF_2(y)du}{\int_t^\infty \int_0^\infty \overline{\Phi}_s(Z,y+u)dF_1(y)du} \times \frac{\int_0^\infty \overline{\Phi}_s(Z,y)dF_1(y)}{\int_0^\infty \overline{\Phi}_s(Z,y)dF_2(y)} \\
&= \frac{\int_t^\infty \overline{\Phi}_s(Y_{(Z)},u)du}{\int_t^\infty \overline{\Phi}_s(X_{(Z)},u)du},
\end{aligned}$$

is increasing in $t \geqslant 0$. Hence the result follows. □

Taking $s = 1$ in Theorem 5.2.5, we get the following result.

Corollary 5.2.5. *If Z is IMRL and $X \leqslant_{hr} Y$ then $X_{(Z)} \leqslant_{mrl} Y_{(Z)}$.*

Taking $s = 2$ in Theorem 5.2.5, we obtain Theorem 2.3.4.

Corollary 5.2.6. *If Z is IVRL and $X \leqslant_{hr} Y$ then $X_{(Z)} \leqslant_{vrl} Y_{(Z)}$.*

The following theorem is from Cai and Zheng (2012).

Theorem 5.2.6. *Suppose that $s \in \mathbb{N}_+$. Let X and Y be two independent nonnegative random variables. If Y is DRHR and X is s-IFR (s-DFR) then X_Y is s-IFR (s-DFR).*

The following theorem is due to Nanda and Kundu (2009).

Theorem 5.2.7. *Let X and Y be two independent nonnegative random variables and Y be DRHR. Then X is s-IFR if and only if X_Y is s-IFR.*

Now we discuss some new properties of s-DFR ageing class on the RLRT. The results are interesting in the sense that they give some existing results with less sufficient conditions. First we study the preservation properties of s-DFR class for RLRT. In Theorem

5.2.6 it has been shown that s-IFR (s-DFR) class is preserved for X_Y provided Y has DRHR. The same result for s-IFR class was obtained in Theorem 5.2.7. The following theorem strengthens Theorem 5.2.6 for s-DFR class in the sense that the DRHR property on Y has been relaxed here. A natural question, therefore, may arise, whether the DRHR property on Y for s-IFR class can be relaxed. We address this question in Example 5.2.2.

Theorem 5.2.8. *Suppose that $s \in \mathbb{N}_+$. Let X and Y be two independent nonnegative random variables. If X is s-DFR then X_Y is s-DFR.*

Proof: According to Proposition 5.1.1(e), X is s-DFR if and only if $X_t \leqslant_{s-FR} X$ for all $t \geqslant 0$. Hence, by Theorem 5.2.1, $(X_t)_Y \leqslant_{s-FR} X_Y$ for all $t \geqslant 0$. In view of $(X_Y)_t \stackrel{\text{st}}{=} (X_t)_Y$ for all $t \geqslant 0$, we obtain $(X_Y)_t \leqslant_{s-FR} X_Y$ for all $t \geqslant 0$. Now from Proposition 5.1.1(e), it follows immediately that X_Y is of s-DFR. $\square$

Taking $s = 1, 2, 3$ in the above theorem we get the following results.

Corollary 5.2.7. *For two independent nonnegative random variables X and Y*

- *if X is DFR then X_Y is DFR;*
- *if X is IMRL then X_Y is IMRL;*
- *if X is IVRL then X_Y is IVRL.*

Consider the following example in support of the above result.

Example 5.2.1. *Let X and Y be two mutually independent random variables. Let $\overline{F}_X(x) = \frac{1}{1+x}$, $x \geqslant 0$ be the survival function of X with $r_X(x) = \frac{1}{1+x}$, $x \geqslant 0$. Clearly X is DFR. Also, let*

$$F_Y(x) = \begin{cases} \frac{x^2}{2}, & 0 \leqslant x < 1 \\ \frac{x^2+2}{6}, & 1 \leqslant x < 2 \\ 1, & x \geqslant 2 \end{cases}$$

and

$$\tilde{r}_Y(x) = \begin{cases} \frac{2}{x}, & 0 \leqslant x < 1 \\ \frac{2x}{x^2+2}, & 1 \leqslant x < 2 \\ 1, & x \geqslant 2 \end{cases}$$

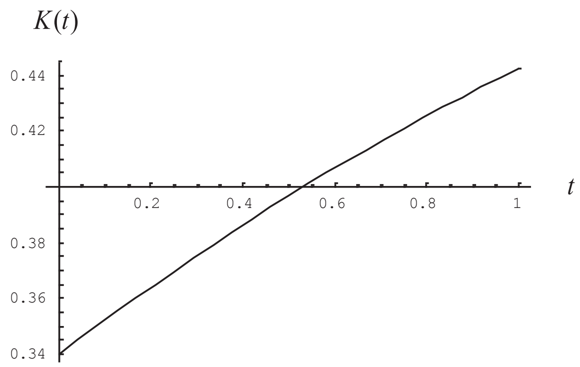

Figure 5.2.1: Plot of $K(t)$ against $t \in [0,1]$ (Example 5.2.1)

be the distribution function and reversed hazard rate function of Y *respectively. Since* $\frac{2x}{x^2+2}$ *is non decreasing in* $1 \leqslant x < 2$ *so* Y *is not DRHR. Then the survival function of* $U = X_Y$ *is given by*

$$
\begin{aligned}
\overline{F}_U(x) &= \frac{\int_0^\infty \overline{F}_X(x+y)dF_Y(y)}{\int_0^\infty \overline{F}_X(y)dF_Y(y)} \\
&= \frac{1}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}\left[\int_0^1 \frac{y}{1+x+y}dy + \frac{1}{3}\int_1^2 \frac{y}{1+x+y}dy\right] \\
&= \frac{1}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}\left[\int_{1+x}^{2+x} \frac{z-x-1}{z}dz + \frac{1}{3}\int_{2+x}^{3+x} \frac{z-x-1}{z}dz\right] \\
&= \frac{1}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}\left[[z-(x+1)\ln z]_{1+x}^{2+x} + \frac{1}{3}[z-(x+1)\ln z]_{2+x}^{3+x}\right] \\
&= \frac{1}{\int_0^\infty \overline{F}_X(y)dF_Y(y)}\left[\frac{4}{3} - (x+1)\ln\left(\frac{2+x}{1+x}\right) - \frac{1}{3}(x+1)\ln\left(\frac{3+x}{2+x}\right)\right], \ x \geqslant 0.
\end{aligned}
$$

Again,

$$
r_U(x) = \frac{\left[\ln\left(\frac{2+x}{1+x}\right) + \frac{1}{3}\ln\left(\frac{3+x}{2+x}\right)\right] + (x+1)\left[\frac{1}{2+x} - \frac{1}{1+x} + \frac{1}{3(3+x)} - \frac{1}{3(2+x)}\right]}{\frac{4}{3} - (x+1)\left[\ln\left(\frac{2+x}{1+x}\right) + \frac{1}{3}\ln\left(\frac{3+x}{2+x}\right)\right]}, \quad x \geqslant 0
$$

which is decreasing in x *as shown in Figure 5.2.1. Note that we use substitution* $t = e^{-x}$ *while plotting the curve so that* $r_U(x) = K(t)$*, say. Hence* X_Y *is DFR.*

In the following example we show that, for s-IFR class, the DRHR property on Y in Theorem 5.2.6 cannot be relaxed.

Example 5.2.2. *Let X and Y be two mutually independent random variables. Let*

$$\overline{F}_X(x) = \begin{cases} 1, & 0 \leqslant x < 1 \\ 1 - \ln x, & 1 \leqslant x < e \\ 0, & x \geqslant e \end{cases}$$

be the survival function of X, and

$$F_Y(x) = \begin{cases} \frac{x^2}{2}, & 0 \leqslant x < 1 \\ \frac{x^2+2}{6}, & 1 \leqslant x < 2 \\ 1, & x \geqslant 2 \end{cases}$$

be the distribution function of Y . Here X is IFR but Y is not DRHR. Then the survival function of X_Y is given by

$$\overline{F}_U(x) = \frac{1}{k} \times \begin{cases} \int_0^{1-x} y dy + \int_{1-x}^{1} [1 - \ln(x+y)] y dy + \frac{1}{3} \int_1^2 [1 - \ln(x+y)] y dy, & 0 \leqslant x < e-2 \\ \int_0^{1-x} y dy + \int_{1-x}^{1} [1 - \ln(x+y)] y dy + \frac{1}{3} \int_1^{e-x} [1 - \ln(x+y)] y dy, & e-2 \leqslant x < 1 \\ \int_0^1 [1 - \ln(x+y)] y dy + \frac{1}{3} \int_1^{e-x} [1 - \ln(x+y)] y dy, & 1 \leqslant x < e-1 \\ \int_0^{e-x} [1 - \ln(x+y)] y dy, & e-1 \leqslant x < e \\ 0, & x \geqslant e. \end{cases}$$

Or equivalently,

$$\overline{F}_U(x) = \frac{1}{k} \times \begin{cases} 1 + \frac{1}{2}(2x - x^2 - \frac{5}{3}) \ln(x+1) - \frac{1}{6} \ln(x+2), & 0 \leqslant x < e-2 \\ \frac{1}{2}(2x - x^2 - \frac{1}{3}) \ln(x+1) + \frac{1}{6} + \frac{1}{6}(e-x)^2 [1 - \ln(x+1)], & e-2 \leqslant x < 1 \\ \frac{1}{6}(e-x)^2 [1 - \ln(x+1)] + \frac{1}{6} + \frac{1}{3} \ln(x+1) - \frac{1}{2} \ln x, & 1 \leqslant x < e-1 \\ \frac{1}{2}(e-x)^2 (1 - \ln x), & e-1 \leqslant x < e \\ 0, & x \geqslant e, \end{cases}$$

where $k = 1 - \frac{1}{6} \ln 2$. Again,

$$r_U(x) = \begin{cases} \dfrac{-(1-x)\ln(x+1) - \frac{[2x-x^2-5/3]}{2(x+1)} + \frac{1}{6(x+2)}}{1 + \frac{1}{2}(2x - x^2 - \frac{5}{3})\ln(x+1) - \frac{1}{6}\ln(x+2)}, & 0 \leqslant x < e-2 \\ \dfrac{-(1-x)\ln(x+1) - \frac{[2x-x^2-1/3]}{x+1} + \frac{1}{3}(e-x)[1 - \ln(x+1)] + \frac{(e-x)^2}{6(x+1)}}{\frac{1}{2}(2x - x^2 - \frac{1}{3})\ln(x+1) + \frac{1}{6} + \frac{1}{6}(e-x)^2[1 - \ln(x+1)]}, & e-2 \leqslant x < 1 \\ \dfrac{\frac{1}{3}(e-x)[1 - \ln(x+1)] + \frac{(e-x)^2}{6(x+1)} - \frac{1}{3(x+1)} + \frac{1}{2x}}{\frac{1}{6}(e-x)^2[1 - \ln(x+1)] + \frac{1}{6} + \frac{1}{3}\ln(x+1) - \frac{1}{2}\ln x}, & 1 \leqslant x < e-1 \\ \dfrac{(e-x)[1 - \ln x] + \frac{(e-x)^2}{2x}}{\frac{1}{2}(e-x)^2(1 - \ln x)}, & e-1 \leqslant x < e, \end{cases}$$

which is not increasing in x as shown in Figure 5.2.2. Hence X_Y is not IFR.

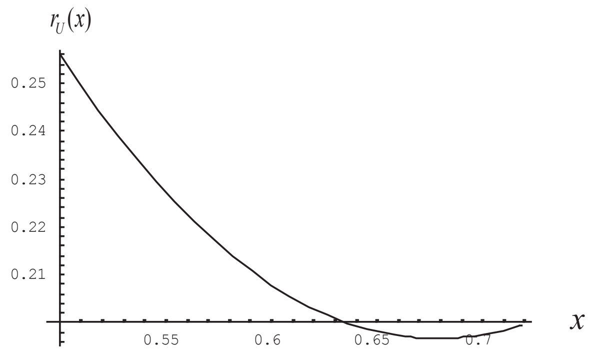

Figure 5.2.2: Plot of $r_U(x)$ against $x \in [.5, e-2]$ (Example 5.2.2)

The following theorem is due to Nanda and Kundu (2009).

Theorem 5.2.9. *Let X_1 and X_2 be two nonnegative random variables and Y be a DRHR random variable which is independent of X_1 and X_2. Then $X_1 \leqslant_{s-FR} X_2$ if and only if $(X_1)_Y \leqslant_{s-FR} (X_2)_Y$.*

In the following theorem we show that the converse of Theorem 5.2.6 for s-DFR class is also true. The same result also holds for s-IFR class as shown in Theorem 5.2.7. It is to be mentioned here that Counterexample 2.2 of Cai and Zheng (2012) in negation of this claim is erroneous.

Theorem 5.2.10. *Suppose that $s \in \mathbb{N}_+$. Let X and Y be two nonnegative independent random variables. If Y is DRHR and X_Y is s-DFR then X is s-DFR.*

Proof: Since X_Y is s-DFR, it follows that $X_Y \leqslant_{s-FR} (X_Y)_t$ for all $t \geqslant 0$. Or equivalently, $X_Y \leqslant_{s-FR} (X_t)_Y$ for all $t \geqslant 0$. Since Y is DRHR and $X_Y \leqslant_{s-FR} (X_t)_Y$ for all $t \geqslant 0$ so from Theorem 5.2.9 we obtain $X \leqslant_{s-FR} X_t$ for all $t \geqslant 0$. Hence, it follows that X is s-DFR. $\square$

Taking $s = 1, 2, 3$ in the above theorem we get the following results.

Corollary 5.2.8. *Let X and Y be two independent nonnegative random variables with Y DRHR.*

- *If X_Y is DFR then X is DFR.*
- *If X_Y is IMRL then X is IMRL.*
- *If X_Y is IVRL then X is IVRL.*

Following theorem is from Cai and Zheng (2012).

Theorem 5.2.11. *Suppose that Y is DRHR. If X is ILR (DLR) and has a decreasing density function, then X_Y is ILR (DLR).*

Following theorem is from Dewan and Khaledi (2014) which will be used to prove the upcoming theorem.

Theorem 5.2.12. *If $X_1 \leqslant_{lr} X_2$ and either X_1 or X_2 is ILR, then $(X_1)_Y \leqslant_{lr} (X_2)_Y$.*

The following result strengthens Theorem 5.2.11 for ILR class where an extra ageing property on Y and monotonicity on the density of X have been imposed. Here we provide a shorter proof of their theorem too.

Theorem 5.2.13. *Let X and Y be two nonnegative independent random variables. If X is ILR then X_Y is ILR.*

Proof: According to Theorem 1.C.52 of Shaked and Shanthikumar (2007), X is ILR if and only if $X_t \leqslant_{lr} X$ for all $t \geqslant 0$. Then from Theorem 5.2.12 it follows that $(X_t)_Y \leqslant_{lr} X_Y$ for all $t \geqslant 0$. This is equivalent to $(X_Y)_t \leqslant_{lr} X_Y$ for all $t \geqslant 0$. Hence, it follows that X_Y is ILR. $\square$

Example 5.2.3. *Let X be a Gamma random variable with density function*

$$f_X(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0; \ \alpha, \beta > 0.$$

Here X is ILR. Therefore, from Theorem 5.2.13 it follows that for any nonnegative random variable Y independent with X, X_Y is ILR.

The following example shows that in Theorem 5.2.11 the DRHR property on Y for the dual class (DLR class) can also be relaxed.

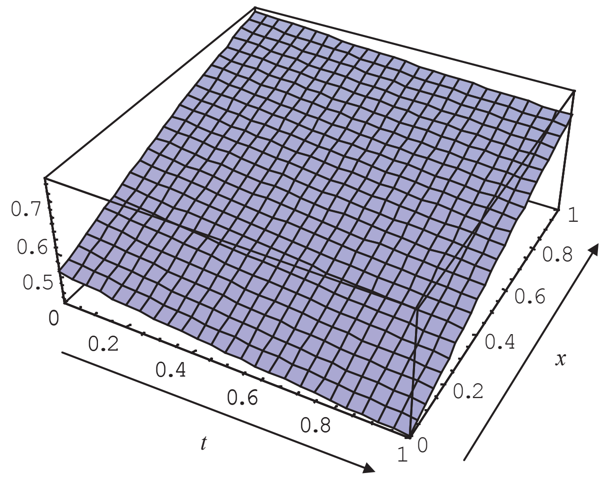

Figure 5.2.3: Plot of $\frac{f_U(x+t)}{f_U(t)}$ against $t \in [0,1]$ and $x \in [0,1]$ (Example 5.2.4)

Example 5.2.4. *Let X and Y follow the distributions as given in Example 5.2.1. Then X is DLR but Y is not DRHR. Now, $\frac{f_U(a+u_1)}{f_U(u_1)} \leqslant \frac{f_U(a+u_2)}{f_U(u_2)}$ for all $u_1 \leqslant u_2$, since $\frac{f_U(a+u)}{f_U(u)}$ is increasing in u for all $a > 0$ as shown in Figure 5.2.3. It is to be mentioned here that the substitutions $t = e^{-u}$ and $x = e^{-a}$ have been used while plotting the curve so that $\frac{f_U(a+u)}{f_U(u)} = \frac{f_U(x+t)}{f_U(t)}$, say. Hence X_Y is DLR.*

To conclude, we show that ILR (DLR) class is preserved for ITRT under certain conditions. Before giving the theorem, let us state a lemma which is due to Shaked and Shanthikumar (2007) and will be used in the sequel.

Lemma 5.2.4. *Let X and Y be two independent random variables, let ϕ_1 and ϕ_2 be two bivariate functions. Denote $\triangle\phi_{21}(x,y) = \phi_2(x,y) - \phi_1(x,y)$. Then $X \leqslant_{hr} Y$ if and only if $E\phi_1(X,Y) \leqslant E\phi_2(X,Y)$ for all ϕ_1 and ϕ_2 that satisfy the conditions:*
(i) for each x, $\triangle\phi_{21}(x,y)$ increases in y on $\{y \geqslant x\}$; and (ii) $\triangle\phi_{21}(x,y) \geqslant -\triangle\phi_{21}(y,x)$ whenever $x \leqslant y$.

Theorem 5.2.14. *Let X and Y be two independent nonnegative random variables. Suppose that Y is DFR. If X is ILR (DLR) and has an increasing density function, then $X_{(Y)}$ is ILR (DLR).*

Proof: Recall that, for $V = X_{(Y)}$,

$$f_V(t) = \frac{\int_0^\infty f_X(y-t)dF_Y(y)}{\int_0^\infty F_X(y)dF_Y(y)}.$$

Then, for $a > 0$,

$$\begin{aligned}\frac{f_V(a+t)}{f_V(t)} &= \frac{\int_0^\infty f_X(a+y-t)dF_Y(y)}{\int_0^\infty f_X(y-t)dF_Y(y)}\\ &= \frac{\int_0^\infty f_X(a+v)dF_Y(v+t)}{\int_0^\infty f_X(v)dF_Y(v+t)}.\end{aligned}$$

For the random variable Y, there exists two independent identical distributed random variables Z_1 and Z_2 with the common distribution function F_Y. For $0 < t_1 < t_2$, let $Y_i = Z_i - t_i,\ i = 1, 2$. Note that Y is DFR, so $Y_1 \leqslant_{hr} Y_2$. Assume that Y_i has distribution function G_i, then we have

$$\begin{aligned}\frac{f_V(a+t_1)}{f_V(t_1)} &= \frac{\int_0^\infty f_X(a+v)dG_1(v)}{\int_0^\infty f_X(v)dG_1(v)}\\ &= \frac{Ef_X(a+Y_1)}{Ef_X(Y_1)}.\end{aligned}$$

Let $\phi_1(x, y) = f_X(x)f_X(a + y)$, $\phi_2(x, y) = f_X(a + x)f_X(y)$ and $\Delta\phi_{21}(x, y) = \phi_2(x, y) - \phi_1(x, y)$. Note that

$$\Delta\phi_{21}(x,y) = -\Delta\phi_{21}(y,x) = f_X(x)f_X(y)\left(\frac{f_X(a+x)}{f_X(x)} - \frac{f_X(a+y)}{f_X(y)}\right).$$

Since X is ILR and $f_X(x)$ is increasing, so $\Delta\phi_{21}(x, y)$ increasing in x for $x \leqslant y$. Again Y is DFR, which gives that $Y_1 \leqslant_{hr} Y_2$. Applying Lemma 5.2.4 we have $E\phi_1(Y_1, Y_2) \leqslant E\phi_2(Y_1, Y_2)$, i.e., $E[f_X(Y_1)f_X(a + Y_2)] \leqslant E[f_X(Y_2)f_X(a + Y_1)]$. Note that $X_{(Y)}$ is ILR if and only if $\frac{f_V(a+t_1)}{f_V(t_1)} \geqslant \frac{f_V(a+t_2)}{f_V(t_2)}$ for all $t_1 \leqslant t_2$, by the equality and inequality above, the assertion follows.

5.3 An Application in Reliability Theory

Here we provide an application of Theorem 5.2.3 (for $s = 1$) to compare the lifetimes of two parallel systems. Let A and B be two parallel systems, where system A is made up of components C_1 and C_2 with lifetimes X_1 and Y_1 and system B is made up of components

C_3 and C_4 having lifetimes X_2 and Y_2, respectively. Let Y_i, $i = 1, 2$ be a gamma random variable with density function

$$f_{Y_i}(y; \alpha_i, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha_i)} y^{\alpha_i - 1} e^{-\beta y}, \ y > 0; \ \alpha_i, \beta > 0.$$

Let X_1 be a random variable with density function

$$f_{X_1}(x) = \left(\frac{1}{\sqrt{x}} + 1 \right) \exp(-2\sqrt{x} - x), \ x > 0$$

and X_2 be another random variable with density function

$$f_{X_2}(x) = \left(\frac{1}{\sqrt{x}} + \frac{1}{2} \right) \exp(-2\sqrt{x} - \frac{x}{2}), \ x > 0.$$

Note that X_1 and Y_1 (X_2 and Y_2) are exchangeable random variables. In this framework, it may be of interest to know the time for which the systems A and B operate without load-sharing, i.e., it may be of interest to study the random variables

$$T_1 = \max\{X_1, Y_1\} - \min\{X_1, Y_1\}$$

$$\text{and} \ \ T_2 = \max\{X_2, Y_2\} - \min\{X_2, Y_2\}.$$

Using exchangeability of X_1 and Y_1, we have, for $t > 0$,

$$P(T_1 > t) = 2P(X_1 - Y_1 > t | X_1 > Y_1) = 2 \frac{\int_0^{\infty} \overline{F}_{X_1}(t + y) dF_{Y_1}(y)}{\int_0^{\infty} \overline{F}_{X_1}(y) dF_{Y_1}(y)},$$

i.e., the survival function of T_1 is same as that of $(X_1)_{Y_1}$. Similarly, the survival function of T_2 is same as that of $(X_2)_{Y_2}$. In reliability theory, the system which survives for a longer time period without load-sharing is considered to be more robust. It may be of interest to compare T_1 and T_2 (equivalently, $(X_1)_{Y_1}$ and $(X_2)_{Y_2}$) in order to determine which system is more robust. If $\alpha_1 < 1$ and $\alpha_1 \leq \alpha_2$, then $Y_1 \leq_{lr} Y_2$ from Dewan and Khaledi (2014). Also from Dewan and Khaledi (2014) we show that $X_1 \leq_{hr} X_2$ and both X_1 and X_2 are DFR. Thus condition of Theorem 5.2.3 are satisfied for $s = 1$ and, consequently, $(X_1)_{Y_1} \leq_{hr} (X_2)_{Y_2}$. Hence system B is more robust than system A.

Chapter 6

Stochastic Properties of Residual Lifetime Mixture Models[1]

In this chapter, we enhance the study of stochastic comparisons and ageing properties of residual lifetime mixture models. To this aim, first we perform stochastic comparisons of two different mixture models under *lr, hr, mrl* and *vrl* orders having different baseline distributions as well as two different mixing distributions. Then, we develop some sufficient conditions which lead to the stochastic comparisons of these mixture models based on *rh, mit* and *vit* orders. Furthermore, we show that ILR, IFR, DMRL, DVRL and IVRL classes are preserved under the formation of the model. Few applications in reliability engineering are also investigated.

6.1 Introduction

Mixture models arise in a number of applications and statistical settings when the population of lifetimes is not homogeneous. They are widely used when all the items in the population do not have exactly the same distribution rather data from several populations are mixed and information about which subpopulation gave rise to individual data points

[1]A manuscript based on this chapter has been accepted in *Mathematical Methods of Operations Research*, 2021.

is unavailable. In practical situations, it is hard to find data that are not some kind of a mixture, because there is almost always some relevant covariate that is not observed. For example, measurements of life lengths of a device may be gathered without regard to the manufacturer, or data may be gathered on humans without regard, say, to blood type. If the ignored variable (manufacturer or blood type) has a bearing on the characteristic being measured, then the data are said to come from a mixture (cf. Marshall and Olkin, 2007). Formally, a mixture model corresponds to the mixture distribution that represents the probability distribution of observation in the overall population. Let $\mathcal{F} = \{F(\cdot\ |\theta) : \theta \in \Theta\}$ be a family of distributions indexed by a parameter θ which takes values in a set Θ. In reliability engineering applications, the past age parameter θ exhibits random behaviour. When θ can be regarded as a random variable with a distribution function (*df*) H, then $F^*(x) = \int_\Theta F(x|\theta)dH(\theta)$ is the mixture of $\mathcal{F}$ with respect to H, and H is called the mixing distribution. The corresponding survival function (*sf*) is given by $\overline{F}^*(x) = \int_\Theta \overline{F}(x|\theta)dH(\theta)$. Mixtures also play a central role in Bayesian statistics, not from a physical mixing of several populations but from a lack of precise knowledge of the exact distribution from which data are obtained.

Study of duration of a system or some living organism is a subject of interest specially in reliability, survival analysis, actuary, economics, biology and many other fields. Consider a system which has survived up to time t and is still working. The distribution of remaining life for an unfailed item of age t is often of interest and plays a recurring role in what follows. Let the lifetime of a fresh item be represented by a random variable X, having an absolutely continuous *df* F and *sf* $\overline{F} = 1 - F$, with distributional support of X as $[0, \infty)$. Then $X_t = (X - t|X > t)$ is known as the residual life of X at age $t > 0$ and its *sf* is given by

$$\overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(t)},\ \forall\ x, t > 0. \tag{6.1.1}$$

Customary study of the residual lifetime often concerns the scenario in which the system or component has already survived to a certain age and is still working. However, in many practical circumstances the age parameter t may not be constant and the occurrence of heterogeneity is sometimes unpredictable and unexplained. Also, in reliability engineering, survival analysis and many other applied areas, researcher often encounter the scenario in which a system or a component has survived one unknown age. Usually this unknown age can be modeled by a nonnegative mixing random variable. To be more specific, consider a population composed of lifetime devices of various ages that are still working. Suppose that a device is randomly taken from the population in which its age is naturally unknown. For evaluating the residual life of this device after the time up which it has already survived, the parametric residual life distribution with a constant parameter does not work. This is because the age of the selected device is indeed a random variable (cf. Kayid and Izadkhah, 2015*b*). Suppose that the random behaviour of the age is described by a random variable Y with *df* G. For simplicity, assume that the support of Y is also $[0, \infty)$. To account the influence of the random ages on the residual lifetime distribution and to handle the heterogeneity of the age parameter t in residual lifetime family of distributions, Kayid and Izadkhah (2015*b*) introduced an extended mixture model with *sf*

$$\overline{F}^{Y}(x) = \int_0^\infty \frac{\overline{F}(x+y)}{\overline{F}(y)} dG(y), \tag{6.1.2}$$

which can be interpreted as the average survival probability of X_t with respect to the random age Y. Denote by X^Y, the random variable that has the *sf* (6.1.2) with baseline random variable X and mixing random age Y. Then, the *df* of X^Y is given by

$$F^{Y}(x) = \int_0^\infty \frac{F(x+y) - F(y)}{\overline{F}(y)} dG(y).$$

The notion of residual lifetime mixture model (6.1.2) has been considered in the literature as residual lifetime with a random age (RLRA) (see Finkelstein, 2002*b*; Finkelstein and Vaupel, 2015; Cha and Finkelstein, 2018). For example, when an item is selected from a large number of statistically identical items having different (unknown) ages running through an ageing process, then the remaining life of that item is represented by RLRA. Random age naturally arises also in population biology and demography when population of organisms are described by its age distribution at each chronological instant of time. Recently, Hazra et al. (2017), introduced this concept of RLRA for the residual lifetime of manufactured components with a random age. In real life, technical items can be incepted into operation having already some random age. Assume that we do not know when an operating item has been incepted into operation. In order to model its unknown initial age we recall (6.1.2).

In the literature, one can find another notion of random age called residual lifetime with random time (RLRT) represented by the random variable $X_Y = (X - Y|X > Y)$ where the remaining lifetime is defined after an item has survived in $[0, Y]$. For RLRT, if X and Y are independent, the *sf* is defined as

$$\overline{F}_Y(x) = \frac{\int_0^\infty \overline{F}(x+y)dG(y)}{\int_0^\infty \overline{F}(y)dG(y)},$$

which gives the conditional survival probability of the remaining lifetime given that the lifetime survives a random time. To get a feel on the difference between RLRT and RLRA recall the example due to Misra and Naqvi (2018*b*). Consider the case of buying a second hand car from a company that sells used cars. Suppose the random variable Y denotes the time for which cars have been used (initial age) before being put on sale and X is the total life of those cars. Now, if a car is directly purchased from a potential seller in the locality, then the remaining lifetime of that car would be defined by RLRT X_Y. But, when the car is purchased (picked up at random) from a lot/mixture of used cars available with the car selling company, then the remaining lifetime of the car would be defined by RLRA X^Y. For a detailed discussion on the connection between RLRT and RLRA one may refer to Li and Fang (2018).

The simplest and the most popular method of comparing the magnitudes of two random variables X_1 and X_2 are through their means and medians. It may happen that in some cases the median of X_1 is larger than that of X_2, while the mean of X_1 is smaller than the mean of X_2. However, this confusion will not arise if the random variables are stochastically ordered. Similarly, the same may happen if one would like to compare the variability of X_1 with that of X_2 based on only numerical measures like variance, standard deviation, and so forth. Besides, these characteristics of distributions might not exist in some cases. In most cases one can express various forms of knowledge about the underlying distributions in terms of their survival, quantile, hazard rate, mean/variance residual life functions and other suitable functions of probability distributions in reversed time. These methods are much more informative than those based only on few numerical characteristics of distributions. Comparisons of random variables based on such functions usually establish partial orders among them. We call them as stochastic orders. Recently, stochastic comparisons and ageing notions of mixture model have received much attention due to their important role in risk theory, reliability and various areas of applied probability and engineering. If two mixture models have the different mixing distributions and same as well as different generic distributions, then it might be of interest to compare the residual lifetimes of these models. For the RLRA, Finkelstein and Vaupel (2015) and Cha and Finkelstein (2018) investigated the stochastic comparisons. In past, stochastic properties of mixture models with respect to *lr, hr, rh* and *mrl* orders have been studied by Kayid and Izadkhah (2015) and Hazra et al. (2017). More recently, following the same spirit, Misra and Naqvi (2018*b*) further provided several stochastic comparison results on the RLRAs. But, another context where the stochastic orders arise is in the comparison of random variables in terms of their variability or dispersion. This study aims to further explore the stochastic comparisons on the residual lifetime mixture models in the sense of *mrl, mit, vrl* and *vit* orders for which the current literature seems to have been silent. For some discussions on variance residual life (inactivity time) one may refer to Gupta (2006), Mahdy (2012) and Kayid and Izadkhah (2016). For different stochastic properties of RLRT see, for example, the works of Yue and Cao (2000), Li and Zuo (2004), Li and Xu (2006), Misra et al. (2008), Nanda and Kundu (2009), Cai and Zheng (2012), Dewan and Khaledi (2014), Misra and Naqvi (2018*a*), to mention a few.

In this chapter, following Hazra et al. (2017) and Misra and Naqvi (2018*b*), we consider some further stochastic comparisons of residual lifetime mixture models arising out of different base line distributions and/or different mixing distributions. First we recall definitions of some stochastic orders and ageing classes that will be used in this chapter. For details one may refer to the famous books by Shaked and Shanthikumar (2007), Barlow and Proschan (1981), Belzunce et al. (2015) and Müller and Stoyan (2002), among others.

Definition 6.1.1. *For two random variables X and Y, X is said to be smaller than Y in*

(a) usual stochastic order (denoted by $X \leqslant_{st} Y$) if $\overline{F}(t) \leqslant \overline{G}(t)$ for all $t \geqslant 0$;

(b) hazard rate order (denoted by $X \leqslant_{hr} Y$) if $\overline{F}(t)/\overline{G}(t)$ is decreasing in $t \geqslant 0$;

(c) *reversed hazard rate order (denoted by $X \leqslant_{rh} Y$) if $F(t)/G(t)$ is decreasing in $t \geqslant 0$;*

(d) *likelihood ratio order (denoted by $X \leqslant_{lr} Y$) if $f(t)/g(t)$ is decreasing in $t \geqslant 0$;*

(e) *mean residual life order (denoted by $X \leqslant_{mrl} Y$) if*

$$\frac{\int_t^\infty \overline{F}(x)dx}{\int_t^\infty \overline{G}(x)dx} \text{ is decreasing in } t \geqslant 0;$$

(f) *mean inactivity time order (denoted by $X \leqslant_{mit} Y$) if*

$$\frac{\int_0^t F(x)dx}{\int_0^t G(x)dx} \text{ is decreasing in } t \geqslant 0;$$

(g) *variance residual life order (denoted by $X \leqslant_{vrl} Y$) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \int_x^\infty \overline{G}(u)dudx} \text{ is decreasing in } t \geqslant 0;$$

(h) *variance inactivity time order (denoted by $X \leqslant_{vit} Y$) if*

$$\frac{\int_0^t \int_0^x F(u)dudx}{\int_0^t \int_0^x G(u)dudx} \text{ is decreasing in } t \geqslant 0;$$

(i) *convex order (denoted by $X \leqslant_{cx} Y$) if $\int_t^\infty \overline{F}(u)du \leqslant \int_t^\infty \overline{G}(u)du$ for all $t \geqslant 0$;*

(j) *increasing concave order (denoted by $X \leqslant_{icv} Y$) if $\int_0^t F(u)du \geqslant \int_0^t G(u)du$ for all $t \geqslant 0$.*

The following well-known ageing classes are closely related to our discussion.

Definition 6.1.2. *A random variable X is said to have an*

(a) *increasing (resp. decreasing) likelihood ratio (ILR (resp. DLR)) if for any $a > 0$, $f(t+a)/f(t)$ is decreasing (resp. increasing) in $t \geqslant 0$;*

(b) *increasing (resp. decreasing) failure rate (IFR (resp. DFR)) if X_t is stochastically decreasing (resp. increasing) in $t \geqslant 0$;*

(c) *increasing (resp. decreasing) mean residual life (IMRL (resp. DMRL)) if*

$$\frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0;$$

(d) *increasing (resp. decreasing) variance residual life (IVRL (resp. DVRL)) if*

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\int_t^\infty \overline{F}(x)dx} \text{ is increasing (resp. decreasing) in } t \geqslant 0,$$

or equivalently,

$$\frac{\int_t^\infty \int_x^\infty \overline{F}(u)dudx}{\overline{F}(t)} \text{ is increasing (resp. decreasing) in } t \geqslant 0;$$

(e) *decreasing reverse hazard rate (DRHR) if $X_{(t)}$ is stochastically increasing in $t > 0$;*

(f) *new better (resp. worse) than used (NBU (resp. NWU)) if $\overline{F}(x+y) \leqslant (\geqslant)\overline{F}(x)\overline{F}(y)$, for all $x, y \geqslant 0$;*

(g) *new better (resp. worse) than used in convex order (NBUCX (resp. NWUCX)) if $\int_x^\infty \overline{F}(u+y)du \leqslant (\geqslant)\overline{F}(y)\int_x^\infty \overline{F}(u)du$, for all $x, y \geqslant 0$.*

This chapter is organized as follows. In Section 6.2, first we provide few simple characterization results and then establish some useful stochastic ordering relations of two different mixture models under *lr, hr, mrl* and *vrl* orders having different baseline and mixing distributions. Later, in view of the proposed model, we perform some stochastic comparisons based on *rh, mit* and *vit* orders. We also show that ILR, IFR, DMRL, DVRL and IVRL classes are preserved for this model under certain conditions. In Section 6.3, we provide some examples to illustrate the applications of the results derived in this chapter to guaranteed lead times and reliability engineering. Throughout this chapter, the random variables are assumed to be nonnegative and absolutely continuous.

6.2 Stochastic Comparison Results and Ageing Properties

Stochastic comparisons of residual lifetime mixture model (X^Y) based on *st, lr, hr, rh* and *mrl* orders are studied by kayid and Izadkhah (2015), Hazra et al. (2017), Misra and Naqvi (2018*b*) and Li and Fang (2018), but to the best of our knowledge, till now, no work seems to have been done based on *mit, vrl* and *vit* orders. Here we perform stochastic comparisons of $X_1^{Y_1}$ and $X_2^{Y_2}$, the residual lifetime mixture models having different mixing distributions, with respect to *lr, hr, mrl* and *vrl* orders with assuming no sufficient conditions on mixing random age in contrast to Misra and Naqvi (2018*b*). Later we discuss stochastic comparisons of X^{Y_1} and X^{Y_2} , the residual lifetime mixture models with the same generic lifetime, under *rh, mit* and *vit* orders. In addition, we provide some ageing related results. Before coming to the main discussion let us start with some simple results on stochastic comparison of the average remaining lifetime of an item with its generic lifetime having some specific ageing properties.

Theorem 6.2.1. *Let X be the lifetime of an item with random age Y. Then*

i. $X^Y \leqslant_{st} (\geqslant_{st})X$ *if X is NBU (NWU);*

ii. $X^Y \leqslant_{hr} (\geqslant_{hr})X$ *if X is IFR (DFR);*

iii. $X^Y \leqslant_{mrl} (\geqslant_{mrl})X$ *if X is DMRL (IMRL);*

iv. $X^Y \leqslant_{cx} (\geqslant_{cx})X$ *if X is NBUCX (NWUCX);*

v. $X^Y \leqslant_{vrl} (\geqslant_{vrl})X$ *if X is DVRL (IVRL).*

Proof: *(i)* Suppose that X is NBU (NWU). Then it can easily be seen that

$$\int_0^\infty \frac{\overline{F}(x+y)}{\overline{F}(y)}dG(y) \leqslant (\geqslant) \int_0^\infty \overline{F}(x)dG(y)$$

which implies that $X^Y \leqslant_{st} (\geqslant_{st})X$.

To prove *(ii)*, suppose that X is IFR (DFR). Now

$$\begin{aligned}\frac{\overline{F}^Y(x)}{\overline{F}(x)} &= \frac{1}{\overline{F}(x)}\int_0^\infty \frac{\overline{F}(x+y)}{\overline{F}(y)}dG(y)\\ &= \int_0^\infty \frac{\overline{F}(x+y)}{\overline{F}(x)}\frac{1}{\overline{F}(y)}dG(y)\end{aligned}$$

is decreasing (increasing) in x, since $\frac{\overline{F}(x+y)}{\overline{F}(x)}$ is decreasing (increasing) in x for X being IFR (DFR). Hence $X^Y \leqslant_{hr} (\geqslant_{hr})X$.

For *(iii)*, let X be DMRL (IMRL). Then $\frac{\int_x^\infty \overline{F}(u+y)du}{\int_x^\infty \overline{F}(u)du}$ is decreasing (increasing) in x. Now

$$\begin{aligned}
\frac{\int_x^\infty \overline{F}^Y(u)du}{\int_x^\infty \overline{F}(u)du} &= \frac{1}{\int_x^\infty \overline{F}(u)du}\int_x^\infty \int_0^\infty \frac{\overline{F}(u+y)}{\overline{F}(y)}dG(y)du \\
&= \int_0^\infty \frac{\int_x^\infty \overline{F}(u+y)du}{\int_x^\infty \overline{F}(u)du}\frac{1}{\overline{F}(y)}dG(y),
\end{aligned}$$

which is decreasing (increasing) in x. Hence $X^Y \leqslant_{mrl} (\geqslant_{mrl})X$.

(iv) Suppose that X is NBUCX (NWUCX). Then, for all $x \geqslant 0$,

$$\begin{aligned}
& \int_x^\infty \overline{F}(u+y)du \leqslant (\geqslant)\overline{F}(y)\int_x^\infty \overline{F}(u)du \\
\Rightarrow\ & \int_0^\infty \int_x^\infty \frac{\overline{F}(u+y)}{\overline{F}(y)}dudG(y) \leqslant (\geqslant) \int_0^\infty dG(y)\int_x^\infty \overline{F}(u)du \\
\Rightarrow\ & \int_x^\infty \int_0^\infty \frac{\overline{F}(u+y)}{\overline{F}(y)}dG(y)du \leqslant (\geqslant) 1 \times \int_x^\infty \overline{F}(u)du \\
\Rightarrow\ & X^Y \leqslant_{cx} (\geqslant_{cx})X.
\end{aligned}$$

For *(v)*, let X be DVRL (IVRL). Then $\frac{\int_x^\infty\int_v^\infty \overline{F}(u+y)dudv}{\int_x^\infty\int_v^\infty \overline{F}(u)dudv}$ is decreasing (increasing) in x. Now

$$\begin{aligned}
\frac{\int_x^\infty\int_v^\infty \overline{F}^Y(u)dudv}{\int_x^\infty\int_v^\infty \overline{F}(u)dudv} &= \frac{1}{\int_x^\infty\int_v^\infty \overline{F}(u)dudv}\int_x^\infty \int_v^\infty \int_0^\infty \frac{\overline{F}(u+y)}{\overline{F}(y)}dG(y)dudv \\
&= \int_0^\infty \frac{\int_x^\infty\int_v^\infty \overline{F}(u+y)dudv}{\int_x^\infty\int_v^\infty \overline{F}(u)dudv}\frac{1}{\overline{F}(y)}dG(y),
\end{aligned}$$

which is decreasing (increasing) in x. Hence $X^Y \leqslant_{vrl} (\geqslant_{vrl})X$. □

Now we consider the stochastic comparisons of two different mixture models having different baseline distributions as well as two different mixing distributions. We present some sufficient conditions under which *lr, hr, mrl* and *vrl* orders between X_1 and X_2 are preserved for the mixture models.

Theorem 6.2.2. *Let $X_i,\ i = 1, 2$ be two nonnegative random variables having probability density functions (pdf) f_i, sfs $\overline{F}_i$. Let Y_i be another two nonnegative random variables with dfs $G_i,\ i = 1, 2$. If* *i.* $X_1 \leqslant_{lr} X_2$, X_2 *is DLR and* X_1 *is ILR, then* $X_1^{Y_1} \leqslant_{lr} X_2^{Y_2}$;

ii. $X_1 \leqslant_{hr} X_2$, X_2 *is DFR and* X_1 *is IFR, then* $X_1^{Y_1} \leqslant_{hr} X_2^{Y_2}$;

iii. $X_1 \leqslant_{mrl} X_2$, X_2 *is IMRL and* X_1 *is DMRL, then* $X_1^{Y_1} \leqslant_{mrl} X_2^{Y_2}$;

iv. $X_1 \leqslant_{vrl} X_2$, X_2 *is IVRL and* X_1 *is DVRL, then* $X_1^{Y_1} \leqslant_{vrl} X_2^{Y_2}$.

Proof: *(i)* Since X_2 is DLR,

$$\frac{f_2(x+y)}{f_2(x)} \text{ is increasing in } x \geqslant 0. \tag{6.2.1}$$

Also,

$$\frac{f_1(x+y)}{f_1(x)} \text{ is decreasing in } x \geqslant 0 \tag{6.2.2}$$

as X_1 is ILR. Again, $X_1 \leqslant_{lr} X_2$ gives

$$\frac{f_1(x)}{f_2(x)} \text{ is decreasing in } x \geqslant 0. \tag{6.2.3}$$

Thus, on using (6.2.1), (6.2.2) and (6.2.3), we arrive at

$$\begin{aligned}
\frac{f_2^{Y_2}(x)}{f_1^{Y_1}(x)} &= \frac{\int_0^\infty \frac{f_2(x+y)}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{f_1(x+y)}{\overline{F}_1(y)} dG_1(y)} \\
&= \frac{f_2(x)}{f_1(x)} \times \frac{\int_0^\infty \frac{f_2(x+y)}{f_2(x)} \frac{1}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{f_1(x+y)}{f_1(x)} \frac{1}{\overline{F}_1(y)} dG_1(y)}
\end{aligned}$$

is increasing in $x \geqslant 0$. Hence $X_1^{Y_1} \leqslant_{lr} X_2^{Y_2}$.

(ii)

$$\frac{\overline{F}_2(x+y)}{\overline{F}_2(x)} \text{ is increasing in } x \geqslant 0 \tag{6.2.4}$$

as X_2 is DFR. Again, since X_1 is IFR which gives

$$\frac{\overline{F}_1(x+y)}{\overline{F}_1(x)} \text{ is decreasing in } x \geqslant 0. \tag{6.2.5}$$

On the other hand,

$$\frac{\overline{F}_1(x)}{\overline{F}_2(x)} \text{ is decreasing in } x \geqslant 0 \tag{6.2.6}$$

in accordance with $X_1 \leqslant_{hr} X_2$. Thus, in view of (6.2.4), (6.2.5) and (6.2.6) we have

$$\begin{aligned} \frac{\overline{F}_2^{Y_2}(x)}{\overline{F}_1^{Y_1}(x)} &= \frac{\int_0^\infty \frac{\overline{F}_2(x+y)}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{\overline{F}_1(x+y)}{\overline{F}_1(y)} dG_1(y)} \\ &= \frac{\overline{F}_2(x)}{\overline{F}_1(x)} \times \frac{\int_0^\infty \frac{\overline{F}_2(x+y)}{\overline{F}_2(x)} \frac{1}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{\overline{F}_1(x+y)}{\overline{F}_1(x)} \frac{1}{\overline{F}_1(y)} dG_1(y)} \end{aligned}$$

is increasing in $x \geqslant 0$. Hence $X_1^{Y_1} \leqslant_{hr} X_2^{Y_2}$.

(iii) Since X_2 is IMRL,

$$\frac{\int_x^\infty \overline{F}_2(u+y)du}{\int_x^\infty \overline{F}_2(u)du} \text{ is increasing in } x \geqslant 0.$$

Also, X_1 DMRL yields

$$\frac{\int_x^\infty \overline{F}_1(u+y)du}{\int_x^\infty \overline{F}_1(u)du} \text{ is decreasing in } x \geqslant 0.$$

Again $X_1 \leqslant_{mrl} X_2$ which in turn gives that

$$\frac{\int_x^\infty \overline{F}_1(u)du}{\int_x^\infty \overline{F}_2(u)du} \text{ is decreasing in } x \geqslant 0.$$

Therefore, on using these facts we arrive at

$$\begin{aligned} \frac{\int_x^\infty \overline{F}_2^{Y_2}(u)du}{\int_x^\infty \overline{F}_1^{Y_1}(u)du} &= \frac{\int_x^\infty \int_0^\infty \frac{\overline{F}_2(u+y)}{\overline{F}_2(y)} dG_2(y)du}{\int_x^\infty \int_0^\infty \frac{\overline{F}_1(u+y)}{\overline{F}_1(y)} dG_1(y)du} \\ &= \frac{\int_x^\infty \overline{F}_2(u)du}{\int_x^\infty \overline{F}_1(u)du} \times \frac{\int_0^\infty \frac{\int_x^\infty \overline{F}_2(u+y)du}{\int_x^\infty \overline{F}_2(u)du} \frac{1}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{\int_x^\infty \overline{F}_1(u+y)du}{\int_x^\infty \overline{F}_1(u)du} \frac{1}{\overline{F}_1(y)} dG_1(y)} \end{aligned}$$

which is increasing in $x \geqslant 0$. Hence $X_1^{Y_1} \leqslant_{mrl} X_2^{Y_2}$.

(iv)

$$\frac{\int_x^\infty \int_v^\infty \overline{F}_2(u+y)dudv}{\int_x^\infty \int_v^\infty \overline{F}_2(u)dudv} \text{ is increasing in } x \geqslant 0$$

as X_2 is IVRL. Also, the DVRL property of X_1 implies that

$$\frac{\int_x^\infty \int_v^\infty \overline{F}_1(u+y)dudv}{\int_x^\infty \int_v^\infty \overline{F}_1(u)dudv} \text{ is decreasing in } x \geqslant 0.$$

Since $X_1 \leqslant_{vrl} X_2$ so

$$\frac{\int_x^\infty \int_v^\infty \overline{F}_1(u)dudv}{\int_x^\infty \int_v^\infty \overline{F}_2(u)dudv} \text{ is decreasing in } x \geqslant 0.$$

Now

$$\frac{\int_x^\infty \int_v^\infty \overline{F}_2^{Y_2}(u)dudv}{\int_x^\infty \int_v^\infty \overline{F}_1^{Y_1}(u)dudv} = \frac{\int_x^\infty \int_v^\infty \int_0^\infty \frac{\overline{F}_2(u+y)}{\overline{F}_2(y)} dG_2(y)dudv}{\int_x^\infty \int_v^\infty \int_0^\infty \frac{\overline{F}_1(u+y)}{\overline{F}_1(y)} dG_1(y)dudv}$$

$$= \frac{\int_x^\infty \int_v^\infty \overline{F}_2(u)dudv}{\int_x^\infty \int_v^\infty \overline{F}_1(u)dudv} \times \frac{\int_0^\infty \frac{\int_x^\infty \int_v^\infty \overline{F}_2(u+y)dudv}{\int_x^\infty \int_v^\infty \overline{F}_2(u)dudv} \frac{1}{\overline{F}_2(y)} dG_2(y)}{\int_0^\infty \frac{\int_x^\infty \int_v^\infty \overline{F}_1(u+y)dudv}{\int_x^\infty \int_v^\infty \overline{F}_1(u)dudv} \frac{1}{\overline{F}_1(y)} dG_1(y)}$$

is increasing in $x \geqslant 0$. Hence $X_1^{Y_1} \leqslant_{vrl} X_2^{Y_2}$. □

Note that Theorem 6.2.2(iii) can be compared with Theorem 2.13 of Misra and Naqvi (2018*b*) where the same stochastic comparison result has been obtained with an extra sufficient condition $Y_1 \leqslant_{hr} Y_2$ which is relaxed here. The following example illustrates the above theorem.

Example 6.2.1. *(i) Suppose that random variables X_1 and X_2 have pdfs $f_1(x) = 2e^{-2x}$, $x \geqslant 0$ and $f_2(x) = 1/(1+x)^2$, $x \geqslant 0$, respectively and the pdf of Y_1 and Y_2 are $g_1(x) = xe^{-x}$, $x \geqslant 0$ and $g_2(x) = e^{-x}$, $x \geqslant 0$, respectively. Now $f_1(x)/f_2(x) = 2(1+x)^2/e^{2x}$ is decreasing in $x > 0$. hence, $X_1 \leqslant_{lr} X_2$. Clearly X_2 is DLR and X_1 is ILR. Thus, all the conditions of Theorem 6.2.2(i) are satisfied. Now*

$$\frac{f_2^{Y_2}(x)}{f_1^{Y_1}(x)} = 4(1 + x + 0.3x^2)e^{2x}$$

is increasing in $x \geqslant 0$. Hence $X_1^{Y_1} \leqslant_{lr} X_2^{Y_2}$.
(ii) Suppose that X_1 and X_2 have sfs $\overline{F}_1(x) = e^{-(x+x^2/2)}$, $x \geqslant 0$ and $\overline{F}_2(x) = e^{-x}$, $x \geqslant 0$, respectively. Now $\overline{F}_1(x)/\overline{F}_2(x) = e^{-x^2/2}$ is decreasing in $x > 0$. Hence, $X_1 \leqslant_{hr} X_2$. Clearly, X_2 is DFR and X_1 is IFR. Therefore, it follows from Theorem 6.2.2(ii) that for any nonnegative random variables Y_1 and Y_2, $X_1^{Y_1} \leqslant_{hr} X_2^{Y_2}$.

The following lemma will be used to prove the upcoming theorems. Before discussing the lemma we provide a definition from Karlin (1968) that a nonnegative function $\psi : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, the set of real numbers, is said to be TP$_2$ (*totally positive of order 2*) if $\psi(x,y)\psi(x^*,y^*) \geqslant \psi(x,y^*)\psi(x^*,y)$ for all $x, x^* \in \mathbb{X}$ and $y, y^* \in \mathbb{Y}$ such that $x \leqslant x^*$ and $y \leqslant y^*$, where $\mathbb{X}$ and $\mathbb{Y}$ are subsets of the real line. ψ is said to be RR$_2$ (*reverse regular of order 2*) if the inequality is reversed.

Lemma 6.2.1. *(Khaledi and Shaked, 2010). Let $\psi(x,y)$ be any TP_2 (resp. RR_2) function (not necessarily a reliability function) in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ and $F_i(x)$ be a distribution function for each i. Denote*

$$H_i(y) = \int_{\mathbb{X}} \psi(x,y) dF_i(x).$$

If $F_i(x)$ is TP_2 in $i \in \{1,2\}$ and $x \in \mathbb{X}$ and if $\psi(x,y)$ is decreasing in x for each y, then $H_i(y)$ is TP_2 (resp. RR_2) in $y \in \mathbb{Y}$ and $i \in \{1,2\}$.

In some practical situations, it may be of interest to compare the residual life mixture models concerning two populations with the same generic lifetime but different random ages. In the sequel, we provide some additional results on stochastic comparisons of $X_1^{Y_1}$ and $X_2^{Y_2}$ under the assumption that $X_1 \overset{\mathrm{d}}{=} X_2$ with respect to *rh, mit* and *vit* orders, where $\overset{\mathrm{d}}{=}$ means equality in distribution. The following result provides stochastic comparisons of X^{Y_1} and X^{Y_2} under *rh* ordering.

Theorem 6.2.3. *Suppose that $Y_1 \leqslant_{rh} Y_2$. If $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $0 < t_1 \leqslant t_2$, then $X^{Y_1} \leqslant_{rh} X^{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $0 < t_1 \leqslant t_2$, then

$$\frac{\frac{F(t_2+x)-F(t_2)}{\overline{F}(t_2)}}{\frac{F(t_1+x)-F(t_1)}{\overline{F}(t_1)}} \text{ is increasing in } x > 0.$$

This implies that

$$\frac{F(y+x)-F(y)}{\overline{F}(y)} \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}. \tag{6.2.7}$$

On the other hand,

$$G_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } y \in \mathbb{Y} \tag{6.2.8}$$

if $Y_1 \leqslant_{rh} Y_2$. Again, $X_{t_1} \leqslant_{rh} X_{t_2}$ implies that $X_{t_1} \leqslant_{st} X_{t_2}$ for all $0 < t_1 \leqslant t_2$. Or equivalently, we can say that

$$\frac{F(y+x)-F(y)}{\overline{F}(y)} \text{ is decreasing in } y > 0. \tag{6.2.9}$$

Therefore, on using (6.2.7), (6.2.8) and (6.2.9), in Lemma 6.2.1, we arrive at

$$\int_0^\infty \frac{F(y+x)-F(y)}{\overline{F}(y)} dG_i(y) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^\infty \frac{F(y+x)-F(y)}{\overline{F}(y)} dG_2(y)}{\int_0^\infty \frac{F(y+x)-F(y)}{\overline{F}(y)} dG_1(y)} \text{ is increasing in } x > 0.$$

Hence $X^{Y_1} \leqslant_{rh} X^{Y_2}$. □

Consider the following example in support of Theorem 6.2.3.

Example 6.2.2. *Let X follow the distribution*

$$F(x) = 1 - \frac{1}{(x+1)}, \ x > 0.$$

Now, $\frac{F(t_2+x)-F(t_2)}{F(t_1+x)-F(t_1)} = \frac{(t_1+1)(x+t_1+1)}{(t_2+1)(x+t_2+1)}$ *is increasing in $x > 0$ giving that $X_{t_1} \leqslant_{rh} X_{t_2}$ for all $t_1 \leqslant t_2$. Further, let the dfs of Y_1 and Y_2 be*

$$G_1(y) = 1 - e^{-y} \ \text{ and } G_2(y) = 1 - \left(1 + \frac{y}{2}\right) e^{-y}.$$

It is easy to verify that, $Y_1 \leqslant_{rh} Y_2$. Now

$$\frac{\int_0^\infty \frac{F(y+x)-F(y)}{\overline{F}(y)} dG_2(y)}{\int_0^\infty \frac{F(y+x)-F(y)}{\overline{F}(y)} dG_1(y)} = \frac{\int_0^\infty \frac{y+1}{2(x+y+1)} e^{-y} dy}{\int_0^\infty \frac{1}{x+y+1} e^{-y} dy} = \alpha(x), \ say$$

is increasing in $x \in (0, \infty)$ as shown in Figure 6.2.1. It is to be mentioned here that the substitution $v = e^{-x}$ has been used while plotting the curve so that $\alpha(x) = Q(v)$, say. Hence $X^{Y_1} \leqslant_{rh} X^{Y_2}$.

The following theorem provides some sufficient conditions for stochastic monotonicity in terms of the *mit* order.

Theorem 6.2.4. *Assume that $Y_1 \leqslant_{rh} Y_2$. If $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$, then $X^{Y_1} \leqslant_{mit} X^{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$. Then

$$\frac{\frac{\int_0^x [F(t_2+u)-F(t_2)] du}{\overline{F}(t_2)}}{\frac{\int_0^x [F(t_1+u)-F(t_1)] du}{\overline{F}(t_1)}} \text{ is increasing in } x > 0,$$

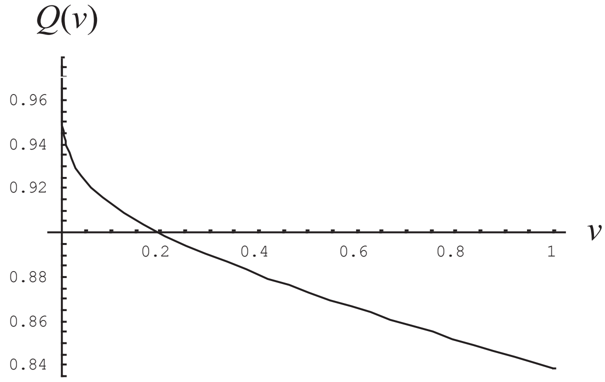

Figure 6.2.1: Plot of $Q(v)$ against $v \in [0, 1]$ (Example 6.2.2)

which in turn gives that $\frac{\int_0^x [F(y+u)-F(y)]du}{\overline{F}(y)}$ is TP_2 in $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. Again, $G_i(y)$ is TP_2 in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$ when $Y_1 \leqslant_{rh} Y_2$. Further, $X_{t_1} \leqslant_{mit} X_{t_2}$ implies that $X_{t_1} \leqslant_{icv} X_{t_2}$ for all $t_1 \leqslant t_2$. Or equivalently, $\frac{\int_0^x [F(y+u)-F(y)]du}{\overline{F}(y)}$ is decreasing in $y > 0$. Therefore, from Lemma 6.2.1, we have

$$\int_0^\infty \frac{\int_0^x [F(y+u) - F(y)]du}{\overline{F}(y)} dG_i(y) \text{ is TP}_2 \text{ in } i \in \{1, 2\} \textit{ and } x \in \mathbb{X}.$$

Which implies that

$$\int_0^x \int_0^\infty \frac{F(y+u) - F(y)}{\overline{F}(y)} dG_i(y)du \text{ is TP}_2 \text{ in } i \in \{1, 2\} \textit{ and } x \in \mathbb{X}.$$

Or equivalently,

$$\frac{\int_0^x \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_2(y)du}{\int_0^x \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_1(y)du} \text{ is increasing in } x > 0.$$

Hence $X^{Y_1} \leqslant_{mit} X^{Y_2}$. □

In the following example we verify the above theorem.

Example 6.2.3. *Let X have the df*

$$F(x) = 1 - \frac{1}{(x+1)^2}, \ x > 0.$$

Now,

$$\frac{\int_0^x [F(t_2+u) - F(t_2)]du}{\int_0^x [F(t_1+u) - F(t_1)]du} = \left(\frac{t_1+1}{t_2+1}\right)^2 \left(\frac{t_1+x+1}{t_2+x+1}\right)$$

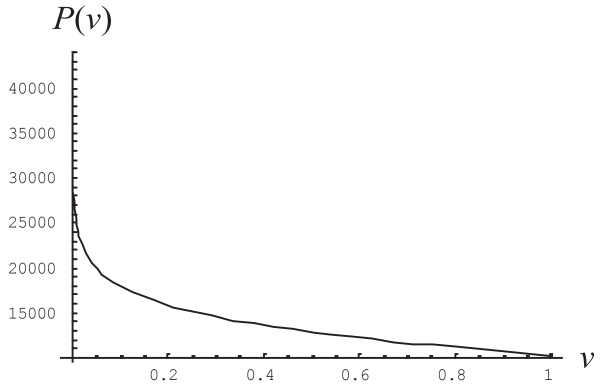

Figure 6.2.2: Plot of $P(v)$ against $v \in [0,1]$ (Example 6.2.3)

is increasing in $x > 0$ giving that $X_{t_1} \leqslant_{mit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$. Further, let the dfs of Y_1 and Y_2 be

$$G_1(y) = 1 - e^{-y} \;\text{ and } G_2(y) = 1 - \left(1 + \frac{y}{2}\right) e^{-y}.$$

It is easy to verify that, $Y_1 \leqslant_{rh} Y_2$. Now

$$\frac{\int_0^x \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_2(y)du}{\int_0^x \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_1(y)du} = \frac{\int_0^\infty \frac{y+1}{2(x+y+1)} e^{-y}dy}{\int_0^\infty \frac{1}{x+y+1} e^{-y}dy} = \beta(x), \;\; say$$

is increasing in $x \in (0, \infty)$ as shown in Figure 6.2.1. Note that the substitution $v = e^{-x}$ has been used while plotting the curve so that $\beta(x) = P(v)$, say. Hence $X^{Y_1} \leqslant_{mit} X^{Y_2}$.

Now we provide some sufficient conditions under which the VIT order holds between two mixture models having different mixing distributions.

Theorem 6.2.5. *If $Y_1 \leqslant_{rh} Y_2$ and $X_{t_1} \leqslant_{vit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$, then $X^{Y_1} \leqslant_{vit} X^{Y_2}$.*

Proof: Let $X_{t_1} \leqslant_{vit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$. Then

$$\frac{\frac{\int_0^x \int_0^v [F(t_2+u)-F(t_2)]dudv}{\overline{F}(t_2)}}{\frac{\int_0^x \int_0^v [F(t_1+u)-F(t_1)]dudv}{\overline{F}(t_1)}} \text{ is increasing in } x > 0,$$

which in turn gives that

$$\frac{\int_0^x \int_0^v [F(y+u) - F(y)]dudv}{\overline{F}(y)} \text{ is TP}_2 \text{ in } x \in \mathbb{X} \text{ and } y \in \mathbb{Y}.$$

Again, $G_i(y)$ is TP_2 in $i \in \{1, 2\}$ and $y \in \mathbb{Y}$ when $Y_1 \leqslant_{rh} Y_2$. According to Lemma 1.(ii) of Kayid and Izadkhah (2016), $\int_0^x \int_0^v (F(u) - G(u))dudv \geqslant 0$, for all $x > 0$ if $X \leqslant_{vit} Y$. Thus,

$$\int_0^x \int_0^v \left[\frac{F(t_1 + u) - F(t_1)}{\overline{F}(t_1)} - \frac{F(t_2 + u) - F(t_2)}{\overline{F}(t_2)}\right] dudv \geqslant 0$$

as $X_{t_1} \leqslant_{vit} X_{t_2}$, which implies that,

$$\frac{\int_0^x \int_0^v [F(y + u) - F(y)]dudv}{\overline{F}(y)} \text{ is decreasing in } y > 0.$$

Combining these observations, from Lemma 6.2.1, we have

$$\int_0^\infty \frac{\int_0^x \int_0^v [F(y + u) - F(y)]dudv}{\overline{F}(y)} dG_i(y) \text{ is } \text{TP}_2 \text{ in } i \in \{1, 2\} \textit{ and } x \in \mathbb{X},$$

which in turn gives that

$$\int_0^x \int_0^v \int_0^\infty \frac{F(y + u) - F(y)}{\overline{F}(y)} dG_i(y)dudv \text{ is } \text{TP}_2 \text{ in } i \in \{1, 2\} \textit{ and } x \in \mathbb{X}.$$

This can equivalently be written as

$$\frac{\int_0^x \int_0^v \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_2(y)dudv}{\int_0^x \int_0^v \int_0^\infty \frac{F(y+u)-F(y)}{\overline{F}(y)} dG_1(y)dudv} \text{ is increasing in } x > 0.$$

Hence $X^{Y_1} \leqslant_{vit} X^{Y_2}$. □

The following example provides an application of the above theorem.

Example 6.2.4. *Let the random variable X follow the distribution*

$$F(x) = 1 - \frac{1}{(x + 1)^3}, \; x > 0.$$

Then, it can be seen that $X_{t_1} \leqslant_{vit} X_{t_2}$ for all $0 < t_1 \leqslant t_2$. Therefore, from Theorem 6.2.5, $X^{Y_1} \leqslant_{vit} X^{Y_2}$ for any $Y_1 \leqslant_{rh} Y_2$.

We conclude this section with the preservation properties of some ageing classes of life distributions for residual lifetime mixture model. If one has the information on ageing properties of the baseline distribution then it is useful to study the ageing properties of the corresponding mixture model. In the context of mixture model, Lynch (1999) provided some conditions so that a mixture of distributions with increasing failure rates has increasing failure rate. Later, Block et al. (2003) showed that similar preservation properties are also possible for increasing failure rate average, NBU and DMRL classes. Recently, it has been shown by Kayid and Izadkhah (2015*b*) that DLR, DFR and IMRL classes are preserved under the formation of residual lifetime mixture model. Here we address the question of their dual classes. In the upcoming theorems, we show that under certain conditions ILR, IFR, DMRL, IVRL and DVRL classes are also preserved for this model.

Theorem 6.2.6. *Let X and Y be two nonnegative random variables with Y DRHR. If*

i. X is IFR then X^Y is IFR;

ii. X is DMRL then X^Y is DMRL;

iii. X is DVRL then X^Y is DVRL.

Proof: *(i)* Consider the following

$$\frac{\overline{F}^Y(x+t)}{\overline{F}^Y(t)} = \frac{\int_0^\infty \frac{\overline{F}(x+t+y)}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{\overline{F}(t+y)}{\overline{F}(y)} dG(y)}.$$

This can further be written as

$$\frac{\overline{F}^Y(x+t)}{\overline{F}^Y(t)} = \frac{\int_0^\infty \frac{\overline{F}(x+\theta)}{\overline{F}(\theta-t)} dG(\theta-t)}{\int_0^\infty \frac{\overline{F}(\theta)}{\overline{F}(\theta-t)} dG(\theta-t)}.$$

Let us fix a x and then write $X_1 \stackrel{\text{st}}{=} X - x$ and $X_2 \stackrel{\text{st}}{=} X$, so the above equality can be restated as

$$\frac{\overline{F}^Y(x+t)}{\overline{F}^Y(t)} = \frac{\int_0^\infty \frac{\overline{F}_1(\theta)}{\overline{F}(\theta-t)} dG(\theta-t)}{\int_0^\infty \frac{\overline{F}_2(\theta)}{\overline{F}(\theta-t)} dG(\theta-t)}. \tag{6.2.10}$$

Now X is IFR if and only if

$$\frac{\overline{F}(x+\theta)}{\overline{F}(\theta)} \text{ is decreasing in } \theta,$$

which can be restated as

$$\frac{\overline{F}_1(\theta)}{\overline{F}_2(\theta)} \text{ is decreasing in } \theta.$$

Or equivalently,

$$\overline{F}_i(\theta) \text{ is } \mathrm{TP}_2 \text{ in } i \in \{1,2\} \text{ and } \theta \in \mathbb{X},$$

which in turn implies that

$$\frac{\overline{F}_i(\theta)}{\overline{F}(\theta - t)} \text{ is } \mathrm{TP}_2 \text{ in } i \in \{1,2\} \text{ and } \theta \in \mathbb{X}. \tag{6.2.11}$$

Again,

$$\frac{\overline{F}_i(\theta)}{\overline{F}(\theta - t)} \text{ is decreasing in } \theta > t, \text{ for all } i = 1,2 \tag{6.2.12}$$

since X is IFR. Also,

$$G(\theta - t) \ \text{ is } \mathrm{TP}_2 \ \text{ in } (\theta, t) \in (\mathbb{X}, \mathbb{Y}) \tag{6.2.13}$$

as Y is DRHR. Thus, on using (6.2.11), (6.2.12) and (6.2.13) in Lemma 6.2.1, we arrive at

$$\int_0^\infty \frac{\overline{F}_i(\theta)}{\overline{F}(\theta - t)} dG(\theta - t) \text{ is } \mathrm{TP}_2 \text{ in } i \in \{1,2\} \text{ and } t \in \mathbb{Y}.$$

Or equivalently,

$$\frac{\int_0^\infty \frac{\overline{F}_1(\theta)}{\overline{F}(\theta - t)} dG(\theta - t)}{\int_0^\infty \frac{\overline{F}_2(\theta)}{\overline{F}(\theta - t)} dG(\theta - t)} \text{ is decreasing in } t.$$

Thus, recalling (6.2.10), we have

$$\frac{\overline{F}^Y(x+t)}{\overline{F}^Y(t)} \text{ is decreasing in } t.$$

Hence X^Y is IFR.

(ii) Let

$$\overline{\Phi}^X(t) = \int_t^\infty \overline{F}(u) du.$$

Now, X^Y is DMRL if and only if

$$\frac{\overline{\Phi}^{X^Y}(x+t)}{\overline{\Phi}^{X^Y}(t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

The above statement can be rewritten as,

$$\frac{\int_0^\infty \frac{\overline{\Phi}^X(t+x+y)}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{\overline{\Phi}^X(t+y)}{\overline{F}(y)} dG(y)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

Or equivalently,

$$\frac{\int_0^\infty \frac{\overline{\Phi}^X(\theta+x)}{\overline{F}(\theta-t)}dG(\theta-t)}{\int_0^\infty \frac{\overline{\Phi}^X(\theta)}{\overline{F}(\theta-t)}dG(\theta-t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

This can further be written as

$$\frac{\int_0^\infty \int_\theta^\infty \frac{\overline{F}(x+u)}{\overline{F}(\theta-t)}dudG(\theta-t)}{\int_0^\infty \int_\theta^\infty \frac{\overline{F}(u)}{\overline{F}(\theta-t)}dudG(\theta-t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0. \tag{6.2.14}$$

Let us fix a x and then write $X_1 \overset{\text{st}}{=} X - x$ and $X_2 \overset{\text{st}}{=} X$, so (6.2.14) can be restated as

$$\frac{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_1(u)}{\overline{F}(\theta-t)}dudG(\theta-t)}{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_2(u)}{\overline{F}(\theta-t)}dudG(\theta-t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0. \tag{6.2.15}$$

Now X is DMRL if and only if

$$\frac{\int_\theta^\infty \overline{F}(x+u)du}{\int_\theta^\infty \overline{F}(u)du} \text{ is decreasing in } \theta,$$

which can be restated as

$$\frac{\int_\theta^\infty \overline{F}_1(u)du}{\int_\theta^\infty \overline{F}_2(u)du} \text{ is decreasing in } \theta.$$

Or equivalently,

$$\int_\theta^\infty \overline{F}_i(u)du \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } \theta \in \mathbb{X},$$

which in turn gives that,

$$\frac{\int_\theta^\infty \overline{F}_i(u)du}{\overline{F}(\theta-t)} \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } \theta \in \mathbb{X}. \tag{6.2.16}$$

Again, the DMRL property of X implies that

$$\frac{\int_\theta^\infty \overline{F}_i(u)du}{\overline{F}(\theta-t)} \text{ is decreasing in } \theta > t, \text{ for all } i = 1,2. \tag{6.2.17}$$

Further, as Y is DRHR,

$$G(\theta-t) \text{ is TP}_2 \text{ in } (\theta,t) \in (\mathbb{X},\mathbb{Y}). \tag{6.2.18}$$

Therefore, on using (6.2.16), (6.2.17) and (6.2.18) in Lemma 6.2.1, we have

$$\int_0^\infty \int_\theta^\infty \frac{\overline{F}_i(u)}{\overline{F}(\theta-t)}dudG(\theta-t) \text{ is TP}_2 \text{ in } i \in \{1,2\} \text{ and } t \in \mathbb{Y}.$$

Or equivalently,

$$\frac{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_1(u)}{\overline{F}(\theta-t)} du dG(\theta - t)}{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_2(u)}{\overline{F}(\theta-t)} du dG(\theta - t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

Thus, from (6.2.15) it can be seen that, X^Y is DMRL. □

(iii) Let

$$\overline{\Phi}^X(t) = \int_t^\infty \int_v^\infty \overline{F}(u) du dv.$$

Now, X^Y is DVRL if and only if

$$\frac{\overline{\Phi}^{X^Y}(x+t)}{\overline{\Phi}^{X^Y}(t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0,$$

which can be restated as,

$$\frac{\int_0^\infty \frac{\overline{\Phi}^X(t+x+y)}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{\overline{\Phi}^X(t+y)}{\overline{F}(y)} dG(y)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

Or equivalently,

$$\frac{\int_0^\infty \frac{\overline{\Phi}^X(\theta+x)}{\overline{F}(\theta-t)} dG(\theta - t)}{\int_0^\infty \frac{\overline{\Phi}^X(\theta)}{\overline{F}(\theta-t)} dG(\theta - t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0.$$

This can further be written as

$$\frac{\int_0^\infty \int_\theta^\infty \int_v^\infty \frac{\overline{F}(x+u)}{\overline{F}(\theta-t)} du dv dG(\theta - t)}{\int_0^\infty \int_\theta^\infty \int_v^\infty \frac{\overline{F}(u)}{\overline{F}(\theta-t)} du dv dG(\theta - t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0. \tag{6.2.19}$$

Let us fix a x and then write $X_1 \stackrel{\text{st}}{=} X - x$ and $X_2 \stackrel{\text{st}}{=} X$, so that (6.2.19) can be restated as

$$\frac{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_1(u)}{\overline{F}(\theta-t)} du dG(\theta - t)}{\int_0^\infty \int_\theta^\infty \frac{\overline{F}_2(u)}{\overline{F}(\theta-t)} du dG(\theta - t)} \text{ is decreasing in } t, \text{ for all } x \geqslant 0. \tag{6.2.20}$$

Now X is DVRL if and only if

$$\frac{\int_\theta^\infty \int_v^\infty \overline{F}(x+u) du dv}{\int_\theta^\infty \int_v^\infty \overline{F}(u) du dv} \text{ is decreasing in } \theta,$$

which can be restated as

$$\frac{\int_\theta^\infty \int_v^\infty \overline{F}_1(u) du dv}{\int_\theta^\infty \int_v^\infty \overline{F}_2(u) du dv} \text{ is decreasing in } \theta.$$

Or equivalently,

$$\int_{\theta}^{\infty}\int_{v}^{\infty}\overline{F}_i(u)dudv \text{ is TP}_2 \text{ in } i\in\{1,2\} \text{ and } \theta\in\mathbb{X},$$

which in turn implies that,

$$\frac{\int_{\theta}^{\infty}\int_{v}^{\infty}\overline{F}_i(u)dudv}{\overline{F}(\theta-t)} \text{ is TP}_2 \text{ in } i\in\{1,2\} \text{ and } \theta\in\mathbb{X}. \tag{6.2.21}$$

Again,

$$\frac{\int_{\theta}^{\infty}\int_{v}^{\infty}\overline{F}_i(u)dudv}{\overline{F}(\theta-t)} \text{ is decreasing in } \theta>t, \text{ for all } i=1,2 \tag{6.2.22}$$

since X is DVRL. On the other hand,

$$G(\theta-t) \;\text{ is TP}_2\; \text{ in } (\theta,t)\in(\mathbb{X},\mathbb{Y}) \tag{6.2.23}$$

as Y is DRHR. Therefore, on using (6.2.21), (6.2.22) and (6.2.23) in Lemma 6.2.1, we arrive at

$$\int_{0}^{\infty}\int_{\theta}^{\infty}\int_{v}^{\infty}\frac{\overline{F}_i(u)}{\overline{F}(\theta-t)}dudvdG(\theta-t) \text{ is TP}_2 \text{ in } i\in\{1,2\} \text{ and } t\in\mathbb{Y}.$$

Or equivalently,

$$\frac{\int_{0}^{\infty}\int_{\theta}^{\infty}\int_{v}^{\infty}\frac{\overline{F}_1(u)}{\overline{F}(\theta-t)}dudvdG(\theta-t)}{\int_{0}^{\infty}\int_{\theta}^{\infty}\int_{v}^{\infty}\frac{\overline{F}_2(u)}{\overline{F}(\theta-t)}dudvdG(\theta-t)} \text{ is decreasing in } t, \text{ for all } x\geqslant 0.$$

This in turn gives from (6.2.20) that X^Y is DVRL. □

The following lemma is due to Dewan and Khaledi (2014) which will be used to prove the next theorem.

Lemma 6.2.2. *Let $h_i(x,y)$, $i=1,2$, be a nonnegative real valued function on $\mathbb{R}\times\mathbb{X}$, where $\mathbb{X}$ is a subset of real line. If*

- *$h_2(x,y)/h_1(x,y)$ is increasing in x and y;*
- *either $h_1(x,y)$ or $h_2(x,y)$ is TP_2 in (x,y),*

then

$$\frac{s_2(x)}{s_1(x)} = \frac{\int_{\mathbb{X}} h_2(x,y)l(y)dy}{\int_{\mathbb{X}} h_1(x,y)l(y)dy}$$

is increasing in x*, where* l *is a continuous function with* $\int_{\mathbb{X}} l(y)dy < \infty$*.*

Theorem 6.2.7. *Let* X *and* Y *be two nonnegative random variables. If*

i. X *is ILR then* X^Y *is ILR;*

ii. X *is IVRL then* X^Y *is IVRL.*

Proof: *(i)* The proof will be validated if we prove that

$$\frac{f^Y(t)}{f^Y(x+t)} = \frac{\int_0^\infty \frac{f(t+y)}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{f(x+t+y)}{\overline{F}(y)} dG(y)}$$

is increasing in t. Now X is ILR if and only if

$$\frac{f(\theta)}{f(x+\theta)} \text{ is increasing in } \theta,$$

which in turn implies that

$$\frac{f(t+y)}{f(x+t+y)} \text{ is increasing in } y \text{ as well as } t.$$

Also, $f(t+y)$ is TP_2 in (y,t) as X is ILR. Therefore, follows from Lemma 6.2.2, we arrive at

$$\frac{\int_0^\infty \frac{f(t+y)}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{f(x+t+y)}{\overline{F}(y)} dG(y)} \text{ is increasing in } t.$$

Hence X^Y is ILR.

(ii) To prove the result it is sufficient to show that

$$\begin{aligned} \frac{\int_t^\infty \int_v^\infty \overline{F}^Y(x+u)dudv}{\int_t^\infty \int_v^\infty \overline{F}^Y(u)dudv} &= \frac{\int_t^\infty \int_v^\infty \int_0^\infty \frac{\overline{F}(x+u+y)}{\overline{F}(y)} dG(y)dudv}{\int_t^\infty \int_v^\infty \int_0^\infty \frac{\overline{F}(u+y)}{\overline{F}(y)} dG(y)dudv} \\ &= \frac{\int_0^\infty \frac{\int_t^\infty \int_v^\infty \overline{F}(x+u+y)dudv}{\overline{F}(y)} dG(y)}{\int_0^\infty \frac{\int_t^\infty \int_v^\infty \overline{F}(u+y)dudv}{\overline{F}(y)} dG(y)} \end{aligned}$$

is increasing in t. Now X is IVRL if and only if

$$\frac{\int_\theta^\infty \int_v^\infty \overline{F}(x+u)dudv}{\int_\theta^\infty \int_v^\infty \overline{F}(u)dudv} \text{ is increasing in } \theta,$$

which in turn implies that

$$\frac{\int_{y+t}^{\infty}\int_{v}^{\infty}\overline{F}(x+u)dudv}{\int_{y+t}^{\infty}\int_{v}^{\infty}\overline{F}(u)dudv} \text{ is increasing in } y \text{ as well as } t.$$

Or equivalently,

$$\frac{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(x+u+y)dudv}{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(u+y)dudv} \text{ is increasing in } y \text{ as well as } t. \tag{6.2.24}$$

On the other hand

$$\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(u+y)dudv \text{ is } \mathrm{TP}_2 \text{ in } (y,t),$$

if and only if X is IVRL. Or equivalently,

$$\frac{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(u+y)dudv}{\overline{F}(y)} \text{ is } \mathrm{TP}_2 \text{ in } (y,t). \tag{6.2.25}$$

Therefore, on using (6.2.24) and (6.2.25) in Lemma 6.2.2, we have

$$\frac{\int_{0}^{\infty}\frac{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(x+u+y)dudv}{\overline{F}(y)}dG(y)}{\int_{0}^{\infty}\frac{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}(u+y)dudv}{\overline{F}(y)}dG(y)} \text{ is increasing in } t.$$

Thus

$$\frac{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}^{Y}(x+u)dudv}{\int_{t}^{\infty}\int_{v}^{\infty}\overline{F}^{Y}(u)dudv} \text{ is increasing in } t.$$

Hence X^Y is IVRL.

6.3 On Some Applications

In this section, we discuss the effectiveness of the results developed in the previous section. Some illustrative applications in the context of statistics and reliability theory are also included. Theorem 6.2.2 is used to make a comparison between two bids. In context of Theorems 6.2.3-6.2.5, we also present a scenario where the stochastic comparisons of residual lifetime mixture models in past life may be of use. In addition, using Theorem 6.2.6(i) we provide two results based on series system.

6.3.1 Guaranteed Lead Times

Consider an industrial plant which plans to purchase two used identical instruments (say, Instrument 1 and Instrument 2) from two different bids (say, Bid 1 and Bid 2). The age of Instrument 1 and Instrument 2 are represented by the random variables Y_1 and Y_2, respectively. Let the lifetime of Instrument 1 and Instrument 2 are represented by the random variables X_1 and X_2, respectively. The average remaining lifetimes of Instrument 1 and Instrument 2 will be described by random variables $X_1^{Y_1}$ and $X_2^{Y_2}$, respectively. The question that may arise is which of the two bids to accept. Obviously, if the industrial plant is only interested in the remaining lifetime of these instruments and moreover if $X_1^{Y_1} \leqslant_{st} X_2^{Y_2}$, then it is clear that Bid 2 is preferable. However, the comparisons based on $\leqslant_{lr}$ and $\leqslant_{hr}$ orders are more powerful than the comparison based on $\leqslant_{st}$ order. Hence it is of interest to find conditions under which $X_1^{Y_1}$ and $X_2^{Y_2}$ can be compared with respect to the orders $\leqslant_{lr}$ and $\leqslant_{hr}$. In this context, we have provided Theorem 6.2.2 (i) and (ii) which yield such more powerful comparisons.

6.3.2 Lifetime devices

Suppose that two devices (say, Device A and Device B) with lifetimes X_1 and X_2 which have survived unknown ages Y_1 and Y_2, respectively are randomly taken from the population composed of used devices of various ages that are still working. Then we recall $X_1^{Y_1}$ and $X_2^{Y_2}$ to model the average remaining lifetimes of these devices in the total population after the time up which they have already survived. However, it is reasonable to infer that in many realistic situations the random lifetime is related to the past not to the future. For instance, suppose the states of the devices are observed only at certain preassigned inspection times. If at time t the devices are inspected for the first time and both are found to be down, then the failure relies on the past i.e., on which instant in $(0, t)$ they have failed. It is thus quite natural to study a notion that refers to past time and not to future. Given that both the devices are known to have failed at time t, it might be of interest to compare $X_1^{Y_1}$ and $X_2^{Y_2}$ by stochastic orders defined on the basis of past life. Then, under the assumption of equal generic lifetime, the stochastic comparison results (Theorems 6.2.3-6.2.5) based on past life will be useful. To be more specific, from Theorem 6.2.3 we infer that Device B is better than Device A in reversed hazard rate order which mean that the probability that Device B has survived up to time $t - \Delta t$ is greater than the probability that Device A has survived up to time $t - \Delta t$ (for a small $\Delta t > 0$). Theorems 6.2.4-6.2.5 may also be used to study the times that have elapsed since the failure of the devices and to take into account the dispersion/variability of these elapsed interval of times.

6.3.3 Series System

Another important application of mixture model of series system arises in reliability theory. Consider a two component series system where the system is made up of components C_1 and C_2, say with lifetimes X_1 and X_2, respectively. A series system functions if and only if all of its components function. Study of reliability properties of series system is of great importance to reliability engineers. Clearly, $\min(X_1, X_2)$ represents the lifetime of the series system comprising of components C_1 and C_2. Here $\min(X_1, X_2)$ denotes the minimum of X_1 and X_2, and $(\min(X_1, X_2))^Y$ represents the residual life of the series system at a mixing random age Y. Also, $\min(X_1^Y, X_2^Y)$ denotes the lifetime of a series system having components lives X_1^Y and X_2^Y, where X_1^Y and X_2^Y are the average residual life random variables at a mixing random age Y. Now we study the reliability properties of $(\min(X_1, X_2))^Y$ and $\min(X_1^Y, X_2^Y)$. If the lifetimes of C_1 and C_2 are in IFR class, then the following theorem shows that the average residual life of the series system at a mixing random age is also in IFR.

Theorem 6.3.1. *Suppose that the random variables X_1 and X_2 have IFR and Y has DRHR then $(\min(X_1, X_2))^Y$ has IFR.*

Proof: If random variables X_1 and X_2 have IFR, then $\min(X_1, X_2)$ also has IFR. Also from Theorem 6.2.6(i), if X is IFR and Y is DRHR then X^Y is IFR. Hence $(\min(X_1, X_2))^Y$ has IFR. □

Now we show that if the generic lifetime of the components are in IFR class then the lifetime of the series system having components lives as mixture models is also in IFR.

Theorem 6.3.2. *Suppose that the random variables X_1 and X_2 have IFR and Y has DRHR then* $\min(X_1^Y, X_2^Y)$ *has IFR.*

Proof: If random variables X_1 and X_2 have IFR, then $\min(X_1, X_2)$ also has IFR. Also from Theorem 6.2.6(i), X_1^Y and X_2^Y are IFR as X_1, X_2 are IFR and Y is DRHR. Hence $\min(X_1^Y, X_2^Y)$ has IFR.

Remark 6.3.1. *Both the above theorems are comparable with the results given in Property 5 and Property 6 of Gupta et al. (2012), and can be thought of as an extension of their properties for fixed age to the case of random age.*

Chapter 7

Conclusion and Future Scope of Study

In this Chapter we summarize the findings of the work carried throughout the thesis with an emphasis on key points and novelties. During the present investigation, several ideas were originated which have the potential to extend the study further. We also give scope for further study which may be undertaken based on this research work.

7.1 Summary of the Reported Work

Stochastic orders and the (ageing) classes of life distributions are being used at an accelerated rate in many diverse areas of probability and statistics. In the literature, different types of stochastic ordering and (ageing) classes of life distributions are available. These orderings are effective in comparing stochastic models, establishing bounds and inequalities in reliability and queueing theory. They are useful in hypothesis testing, multiple decision problems and simultaneous comparisons in statistics, deducing probability inequalities in probability and taking decisions under risk in economics. In reliability, another important application of stochastic orders is the characterization of the classes of life distributions, where with the help of these stochastic orders it is easy to study the notion of ageing. This type of characterization is based on different type of stochastic orders defined on residual life and inactivity time. The applications of stochastic orders and ageing notions in the field of reliability theory, queueing theory, statistics, probability and statistical decision theory have attracted the attention of many researchers. Most of the stochastic orders and life distribution classes are defined in terms of the reliability measures based on residual life and inactivity time.

The present thesis has considered various aspects of residual life and inactivity time for continuous cases. Emphasis is given to establish results related to relationships among reliability characteristics, stochastic orders and ageing criteria of RLRT, ITRT and residual lifetime mixture model.

Chapter 1 was intended to make a bird's eye view of the thesis, with appropriate references to previous work, which focus on the usefulness of the residual life, inactivity time, some associated measures, stochastic orderings, classes of life distributions, residual life at random time, inactivity time at random time, residual lifetime mixture model in real life scenarios, particularly, in reliability engineering.

We obtained some further results on stochastic comparisons and ageing properties of RLRT and ITRT in Chapters 2-3 under the assumption that the generic lifetime and random time are independent. In Chapter 2, we compared two ITRTs in terms of *hr, mrl, lr* and *vrl* orderings, by choosing different concerned total lifetimes and observed failure times. Then two RLRTs/ITRTs have been compared based on *vrl* order. Furthermore, we discussed the ageing properties of DVRL class for RLRT. In Chapter 3, we obtained stochastic comparisons between two RLRTs/ITRTs under *rh, mit* and *vit* orderings. Finally, DRHR, IMIT and IVIT classes of life distributions have been investigated for ITRT.

Several stochastic comparisons and ageing properties of RLRT/ITRT based on *vrl* order, taking dependency between generic lifetime and random time have been investigated in Chapter 4. We compared two RLRTs/ITRTs in terms of *vrl* ordering for one sample problem, by choosing *hr/rh* order between the random times. Stochastic comparisons of RLRT/ITRT of two systems failed at two different random times or having different random ages based on *vrl* order are also investigated. Further, we studied the IVRL, DVRL ageing classes for RLRT and ITRT. Finally, some applications have been provided.

In Chapter 5, we studied generalized stochastic ordering (s-FR) and extended some preservation properties of generalized ageing classes (viz. s-IFR, s-DFR) for RLRT and ITRT, where s is a nonnegative integer. We carried out stochastic comparisons of ITRTs under s-FR ordering, by choosing different generic lifetimes and random times. The preservation properties and some characterizations of s-DFR ageing class and its dual were also derived. Further, we provided an application in reliability theory. The results strengthen some results available in the literature.

In Chapter 6, we provided stochastic comparisons of two different residual lifetime mixture models under *lr, hr, mrl* and *vrl* orders having different baseline distributions as well as two different mixing distributions. Then, we compared two mixture models based on *rh, mit* and *vit* orders. Finally, we investigated the preservation properties of ILR, IFR, DMRL, DVRL and IVRL classes under the formation of the model.

7.2 Future Research Directions

In this thesis, we endeavor to develop a theoretical framework for stochastic properties of RLRT (including RLMM) and ITRT with applications in reliability theory under dependence/independence structure. The present compilation is of course not complete. It rather unfolded several problems which needs further investigation. Further research can start from several aspects and particularly valuable future additions would be as follows:

- The work reported in this thesis is yet to be examined for discrete random variable.
- Recently, Amini-Seresht et al. (2020) have studied several stochastic orderings and ageing properties of RLRT and ITRT for a coherent system. In the same vein, one can study stochastic orderings and ageing properties of RLRT and ITRT for a series system, parallel system, k-out-of-n system, as well as a coherent system.
- The problem of comparing two RLRTs or ITRTs in two sample problems (with different random times) based on *rh, mit* and *vit* orders needs to be studied in detail.
- Assuming dependence between the random life and random time, comparisons of

two RLRTs and ITRTs in terms of *rh, mit* and *vit* orders is an area of further research.

- Comparisons of two RLRTs or ITRTs by assuming *mit/vit* orders between their system lifetimes is an area of special interest.
- As a future course of work, one may look upon the extension of the results based on s-FR ordering under dependence structure.
- Some more stochastic order results between RLMMs can also be obtained.

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List of Publications from the Thesis

(1) Patra, A. and Kundu, C. (2021), Stochastic comparisons and ageing properties of residual lifetime mixture models. *Mathematical Methods of Operations Research*. Accepted.

(2) Patra, A. and Kundu, C. (2020), Stochastic comparisons and ageing properties of RLRT (ITRT) based on variance residual life. *Communications in Statistics- Theory & Methods*, DOI: 10.1080/03610926.2020.1812655. Online First.

(3) Patra, A. and Kundu, C. (2020), Further results on residual life and inactivity time at random time. *Communications in Statistics- Theory & Methods*, **49(5)**, 1261-1271.

(4) Patra, A. and Kundu, C. (2019), On generalized orderings and ageing classes for residual life and inactivity time at random time. *Metrika*, **82**, 691-704.

(5) Kundu, C. and Patra, A. (2018), Some results on residual life and inactivity time at random time. *Communications in Statistics- Theory & Methods*, **47**, 372-384.