(ω, c)-Periodicity and its Generalizations on Time Scales: Theory and Applications to Deterministic and Stochastic Dynamics

विधारत्नम् महद्धनम्

Thesis submitted in partial fulfilment

for the Award of Degree

Doctor of Philosophy

by

PUJA BHARTI

RAJIV GANDHI INSTITUTE OF PETROLEUM TECHNOLOGY

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21MA0101 2026

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It is certified that the work contained in the thesis titled ***"(ω, c)-Periodicity and its Generalizations on Time Scales: Theory and Applications to Deterministic and Stochastic Dynamics"*** by ***"Puja Bharti"*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Dedicated

To

My Sweet ***Grandparents***

Smt. Phulkumari Devi & Shri Jitnath Sahu

Late Sonamati Devi & Shri Sundar Saw

My Beloved ***Parents***

Smt. Anita Devi & Shri Ram Prasad Sahu

Smt. Basanti Devi & Shri Suresh Sahu

Smt. Suryamukhi Devi & Shri Naresh Sahu

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Puja Bharti

Contents

Notations and Abbreviations

$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{T}$	Time scale
σ	Forward jump operator on $\mathbb{T}$
μ	Graininess operator on $\mathbb{T}$
$\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$	Space of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$
$\mathrm{BC}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$	Space of bounded rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$
$x^{\Delta}(t)$	Delta derivative of x at $t\in\mathbb{T}^{\kappa}$
$\mathcal{R}$ $(\mathcal{R}^{+})$	Collection of regressive (positively regressive) functions $f:\mathbb{T}\to\mathbb{R}$
$e_p(t,s)$	Generalized exponential of p on $\mathbb{T}$ ($p\in\mathcal{R},\ s,t\in\mathbb{T}$)
$\Pi_{\mathbb{T}}$	$\{r\in\mathbb{R}: t\pm r\in\mathbb{T}$ for all $t\in\mathbb{T}\}$
$\mathbb{T}_{\omega}$	ω-periodic time scale, i.e., a time scale invariant under translation by ω, ($\omega\in\mathbb{R},\ \omega>0$)
$(\mathbb{B},\lVert\cdot\rVert_{\mathbb{B}})$	Banach space over the field $\mathbb{F}_{\mathbb{B}}$
$\mathbb{F}^{*}_{\mathbb{B}}$	$\mathbb{F}_{\mathbb{B}}\setminus\{0\}$
$\mathscr{L}(\mathbb{B})$	Space of linear operators on $\mathbb{B}$
$\mathbb{T}^{0+}_{\omega}$	$\mathbb{T}_{\omega}\cap[0,\infty)$
$\mathbb{T}^{+}_{\omega}$	$\mathbb{T}_{\omega}\cap(0,\infty)$
$\Pi^{0+}_{\mathbb{T}_{\omega}}$	$\Pi_{\mathbb{T}_{\omega}}\cap[0,\infty)$
$\Pi^{+}_{\mathbb{T}_{\omega}}$	$\Pi_{\mathbb{T}_{\omega}}\cap(0,\infty)$
$\mathbb{B}^{+}$	$\mathbb{B}\cap(0,\infty)$
$\mathbb{B}^{+}_{0}$	$\mathbb{B}\cap[0,\infty)$
$a\oplus b$	$a+b+\mu ab$, $(a,b\in\mathcal{R})$
$\ominus a$	$\frac{-a}{1+\mu a}$
$n\odot a$	$\frac{(\mu a+1)^{n}-1}{\mu}$

DETS	Dynamic Equation on Time Scales
EFO	Evolution Family of Operator
STS	Semigroup Time Scale
RNN	Stochastics Differential Equation
HNN	Hopfield neural networks
CNN	Cellular Neural Networks
DCNN	Delayed Cellular Neural Networks
DDE	Delay dynamic equation
NDDE	Neutral-type delay dynamic equation
SICNN	Shunting inhibitory cellular neural network

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Abstract

The study of differential and difference equations is fundamental to understanding the qualitative behavior of dynamical systems arising in science, engineering, biology, and economics. Although many results for differential equations extend naturally to difference equations, others differ significantly from their continuous counterparts. The theory of dynamic equations on time scales highlights these similarities and differences and provides a unified framework that eliminates the need to establish parallel results separately for continuous and discrete systems. In this approach, the domain is a time scale, defined as an arbitrary nonempty closed subset of the set of real numbers. This formulation enables the simultaneous treatment of continuous-time, discrete-time, and hybrid dynamical systems. Motivated by the need to model real-world phenomena that evolve continuously, discretely, or through a combination of both, Hilger introduced the theory of time scales in 1988, thereby unifying continuous and discrete analysis and facilitating the study of dynamical systems on arbitrary time domains.

Within the framework of time scales, the qualitative analysis of dynamic equations, such as periodicity, stability, and asymptotic behavior of solutions, has received significant attention. Periodic behavior is particularly important for modeling recurrent phenomena in population dynamics, neural networks, epidemiology, and control systems. However, classical periodicity is often too restrictive to describe more general repetitive behaviors arising from scaling effects or weighted repetitions. To address these limitations, several generalized notions of periodicity have been introduced. Among them, the concept of (ω, c)-periodicity has emerged as a natural extension of classical periodicity, characterizing functions whose values repeat up to a constant multiplicative factor after a fixed time shift. This generalized periodic structure allows for the modeling of phenomena where the system's amplitude evolves over time while preserving a repetitive pattern, making it particularly suitable for the analysis of nonlinear evolution equations.

An important aspect of (ω, c)-periodicity is that it is not solely a property of the function, but also depends fundamentally on the underlying time domain. Consequently, the structure of the time scale plays a decisive role in determining whether such oscillatory behavior can occur. This observation highlights the necessity of studying (ω, c)-periodic dynamics within a general time-scale framework rather than restricting

attention to purely continuous settings. The notion of (ω, c)-periodicity encompasses several well-known types of periodic behavior as special cases, including classical periodicity, antiperiodicity, Bloch-type periodicity, as well as bounded and unbounded scaled periodicity. As such, it provides a unified and robust framework for understanding a wide variety of repetitive phenomena across different time scales. It is worth noting that in practical applications, exact (ω, c)-periodicity may not always be observed, as real-world systems are often influenced by perturbations, uncertainties, or external disturbances. Nevertheless, the concept remains highly relevant, as it can be extended to describe approximate or generalized (ω, c)-periodic behavior, allowing for deviations within a controlled margin of error. This flexibility enables a more nuanced and realistic analysis of systems that exhibit nearly (ω, c) periodic dynamics. Motivated by these observations, this study focuses on (ω, c)-periodic functions and their generalizations on periodic time scales, with applications to the qualitative analysis of dynamic equations on time scales. Special attention is given to population models and neural networks, where such generalized periodic structures naturally arise and play a significant role in characterizing long-term system behavior. In realistic environments, however, system behavior is often affected by stochastic perturbations arising from environmental fluctuations, measurement noise, or intrinsic system randomness. Accordingly, this study further includes the analysis of (ω, c)-periodic stochastic processes, which capture recurrent dynamics in a statistical sense rather than in a strictly deterministic manner. The major contributions of the study can be summarized as follows.

Chapter 1 provides the necessary background, including an introduction to time-scale calculus, the motivation of the work, a review of related literature, and the mathematical preliminaries required for the subsequent chapters.

Bridging the gap between continuous and discrete systems, **Chapter 2** presents a detailed characterization of (ω, c)-periodic functions on periodic time scales and examines their fundamental properties. Several classes of delayed population models are analyzed to establish the existence of exponentially stable (ω, c)-periodic solutions, including Nicholson's model, Lasota–Ważewska model, and models incorporating a combination of both. The theoretical results are further illustrated and validated through numerical examples.

Chapter 3 investigates nonautonomous abstract dynamic equations on ω-periodic time scales, focusing on the existence of stable (ω, c)-periodic mild solutions. Sufficient conditions guaranteeing the existence, uniqueness and exponential stability of (ω, c)-periodic mild solutions for linear and semilinear abstract dynamic equations are derived using fixed point theory, Gronwall's inequality, semigroup theory, dichotomy theory, properties of (ω, c)-periodicity, and calculus on time scales. This study also delves into the continuous dependence of these exponentially stable (ω, c)-periodic mild solutions on the source term within a class of nonautonomous abstract semilinear dynamic equations.

Chapter 4 introduces the concept of (ω, c)-asymptotic periodicity on time scales, characterizing functions that do not exhibit (ω, c)-periodicity at early stages but eventually evolve into (ω, c)-periodic behavior. Recognizing the importance of qualitative analysis in time-delayed systems, this work investigates a class of delayed cellular neural networks (DCNNs). Under appropriate assumptions on the coefficients and activation functions, sufficient conditions for the existence of stable (ω, c)-asymptotically periodic solutions for DCNNs are derived. The analysis relies fundamentally on the Banach fixed point theorem, combined with time-scale inequalities and composition results for (ω, c)-asymptotically periodic functions on time scales. Numerical examples and simulations are used to validate the proposed results across various time scales.

Further generalizing the notion of (ω, c)-asymptotic periodicity, **Chapter 5** provides the concept weighted pseudo (ω, c)-periodicity on time scales. This chapter deals with a class of neutral-type evolution systems with piecewise impulsive effects on time scales, which can be used to model a general and more realistic physical phenomenon in several situations. This system is investigated for the existence of (doubly) weighted pseudo (ω, c)-periodic solutions. Using Lyapunov function techniques, the global exponential synchronization of weighted pseudo (ω, c)-periodic solutions is further obtained for this system. Numerical examples are used to demonstrate that the obtained results effectively unify continuous and discrete analysis.

Chapter 6 develops a theory for p-th mean S-asymptotically (ω, c)-periodic stochastic processes on periodic time scales and applies it to stochastic shunting-inhibitory cellular neural networks characterized by discrete time-varying delays and infinite distributed delays. In the presence of standard Lipschitz and growth conditions, as well as stochastic perturbations driven by a Wiener process, the existence and uniqueness of stable p-th mean S-asymptotically (ω, c)-periodic solutions are established. The analysis utilizes time scale calculus, a Banach space setup that is appropriate for the (ω, c)-weighted processes, fixed-point arguments, and estimates of the Burkholder–Davis–Gundy type inequality for the stochastic integrals on time scales. The results of the theoretical analysis are illustrated and validated through the use of numerical simulations on representative continuous, discontinuous, and nonuniform time scales.

Chapter 7 presents the conclusions of the study, outlines directions for future research, and is followed by the bibliography.

Keywords: Time Scales, (ω, c)-periodicity, asymptotic (ω, c)-periodicity, Weighted pseudo (ω, c)-periodicity, S-asymptotic (ω, c)-periodicity, Stochastic Process, Evolution System, Cellular Neural networks, Nicholson's Blowflies model, Lasota–Ważewska model, Abstract dynamic equation, Impulses, Existence and uniqueness, Exponential Stability, Synchronization, Fixed point theory, Lyapunov function, Semigroup theory, Dichotomy theory.

CHAPTER 1

Introduction, Literature Survey and Mathematical Preliminaries

1.1 Introduction

Periodic behavior is one of the most fundamental qualitative properties encountered in the study of dynamical systems. Many natural and engineered processes exhibit recurrent patterns over time, including oscillatory motion in mechanical systems, seasonal fluctuations in biological populations, cyclic phenomena in economic models, and signal propagation in neural and control systems. Mathematically, a function $f : \mathbb{R} \to \mathbb{R}^n$ is said to be periodic with period $\omega > 0$ if

$$f(t + \omega) = f(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

This definition plays a central role in the qualitative theory of dynamical systems. However, strict periodicity is often too restrictive for accurately modeling real-world systems, which are frequently influenced by random perturbations, noise, and other deviations from idealized behavior. These complexities motivate the development of generalized notions of periodicity that better capture realistic dynamics while retaining analytical tractability. In recent decades, several such generalizations have been introduced and extensively studied, including asymptotic periodicity, pseudo periodicity, weighted pseudo periodicity, almost periodicity, and automorphicity. These concepts relax the strict equality condition inherent in classical periodicity and provide a broader framework for analyzing recurrent behavior. Nevertheless, none of these notions accounts for simultaneous translation and scaling effects, which naturally arise in systems exhibiting amplification, attenuation, or proportional responses over time.

The classical definition of periodicity can be interpreted as a special case of a more general functional invariance under time translation. Extending this idea, consider a function $f : \mathbb{R} \to \mathbb{R}^n$ satisfying

$$f(t+\omega) = cf(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

where $\omega > 0$ and $c \in \mathbb{R} \setminus \{0\}$ are constants. When $c = 1$, the function is periodic, and when $c = -1$, it is anti-periodic. For general values of c, this relation defines a broader class of functions exhibiting both translational and scaling symmetries. This observation motivates the concept of (ω, c)-periodicity, which has emerged as a powerful tool in the analysis of dynamical systems with nonconservative or scaling behaviors. Functions exhibiting (ω, c)-periodicity naturally arise in models involving exponential growth or decay, feedback mechanisms, and adaptive responses. The primary objective of this thesis is to investigate (ω, c)-periodicity and its generalizations for dynamic equations on time scales, whose solutions take values in both finite and infinite dimensional spaces. Since real-world systems are often subject to random disturbances, we also study the behavior of (ω, c)-periodicity in stochastic dynamic equations. Furthermore, we analyze the impact of (ω, c)-periodicity on several models defined on time scales.

This section presents the fundamental concepts of time scale theory, dynamic equations on time scales, their abstract formulation, stochastic dynamic equations on time scales, as well as selected classical models from population dynamics and neural networks. In addition, we review classical (ω, c)-periodicity in continuous-time systems, which serves as essential background for the results developed in subsequent chapters.

1.1.1 Time scale theory

"A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both."

— E. T. BELL (1937)

"On becoming familiar with difference equations and their close relation to differential equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations."

— Hugh L. Turrittin (1973)

The study of differential and difference equations is fundamental to understanding the qualitative behavior of dynamical processes arising in science, engineering, biology, and economics. Although many results for differential equations extend naturally to difference equations, significant differences also arise between continuous and discrete systems. This creates the need for a unified theory that highlights both their similarities and distinctions, thereby eliminating the necessity of developing parallel results separately for continuous and discrete systems. Moreover, several real-world phenomena inherently involve both continuous and discrete temporal dynamics and therefore cannot be adequately described using only differential or difference equations. Periodical cicadas (Magicicada spp.) in the eastern United States provide a striking natural example of dynamics evolving on mixed continuous–discrete time domains. These insects spend most of their lives underground as juveniles, 13 years in southern regions (*Magicicada tredecim*, *M. tredecassini,* and *M. tredecula*) and 17 years in northern regions (*M. septendecim*, *M. cassini*, and *M. septendecula*), followed by a brief, synchronous adult emergence lasting approximately 4–6 weeks. Individuals emerging in the same year form a single-year class, known as a brood. During the brief adult phase, cicadas undergo mating and oviposition before dying. The eggs hatch shortly thereafter into nymphs, which develop underground for 13 or 17 years, producing discrete, nonoverlapping generations. Since this life cycle combines continuous activity with long discrete developmental intervals, it cannot be adequately modeled using only differential or difference equations.

Table 1.1: Some periodical cicadas (*Magicicada spp.*).

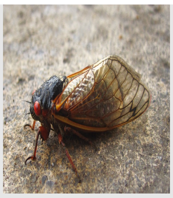
17-year Cicadas			**13-year Cicadas**		
Image	**Scientific name**	**Common name**	**Image**	**Scientific name**	**Common name**
	*Magicicada septendecim*	Pharaoh cicada		*Magicicada tredecim*	13-year cicada
	*Magicicada cassini*	Dwarf periodical cicada		*Magicicada tredecassini*	13-year cicada
	*Magicicada septendecula*	—		*Magicicada tredecula*	—

Table 1.2: Major periodical cicada broods, their life cycles, emergence years, and geographic distributions. (adapted from Wikipedia [1])

Brood	Common Name	Cycle (yrs)	Last Emergence	Next Emergence	Geographic Distribution
XXI	Floridian Brood	13	1870	–	Florida panhandle (extinct)
XI	–	17	1954	–	Connecticut, Massachusetts, Rhode Island (extinct; last seen in Ashford, CT)
I	Blue Ridge Brood	17	2012	2029	Western Virginia, West Virginia
II	East Coast Brood	17	2013	2030	Connecticut, Maryland, North Carolina, New Jersey, New York, Pennsylvania, Delaware, Virginia, District of Columbia
XXII	Baton Rouge Brood	13	2014	2027	Louisiana, Mississippi
III	Iowan Brood	17	2014	2031	Iowa
XXIII	Mississippi Valley Brood	13	2015	2028	Arkansas, Illinois, Indiana, Kentucky, Louisiana, Missouri, Mississippi, Tennessee
IV	Kansan Brood	17	2015	2032	Eastern Nebraska, southwestern Iowa, eastern Kansas, western Missouri, Oklahoma, north Texas
V	–	17	2016	2033	Eastern Ohio, western Maryland, southwestern Pennsylvania, northwestern Virginia, West Virginia, Suffolk County (NY)
VI	–	17	2017	2034	Northern Georgia, western North Carolina, northwestern South Carolina
VII	Onondaga Brood	17	2018	2035	Central New York (Onondaga, Cayuga, Seneca, Ontario, Yates counties)
VIII	–	17	2019	2036	Eastern Ohio, western Pennsylvania, northern West Virginia
IX	–	17	2020	2037	Southwestern Virginia, southern West Virginia, western North Carolina
X	Great Eastern Brood	17	2021	2038	New York, New Jersey, Pennsylvania, Delaware, Maryland, District of Columbia, Virginia, West Virginia, North Carolina, Georgia, Tennessee, Kentucky, Ohio, Indiana, Illinois, Michigan (A premature emergence occurred in 2017.)
XIX	Great Southern Brood	13	2024	2037	Alabama, Arkansas, Georgia, Indiana, Illinois, Kentucky, Louisiana, Maryland, Missouri, Mississippi, North Carolina, Oklahoma, South Carolina, Tennessee, Virginia
XIII	Northern Illinois Brood	17	2024	2041	Northern Illinois, parts of Iowa, Wisconsin, and Indiana (A premature emergence occurred in 2020.)
XIV	–	17	2025	2042	Southern Ohio, Kentucky, Tennessee, Massachusetts, Maryland, North Carolina, Pennsylvania, northern Georgia, southwestern Virginia and West Virginia, parts of New York and New Jersey

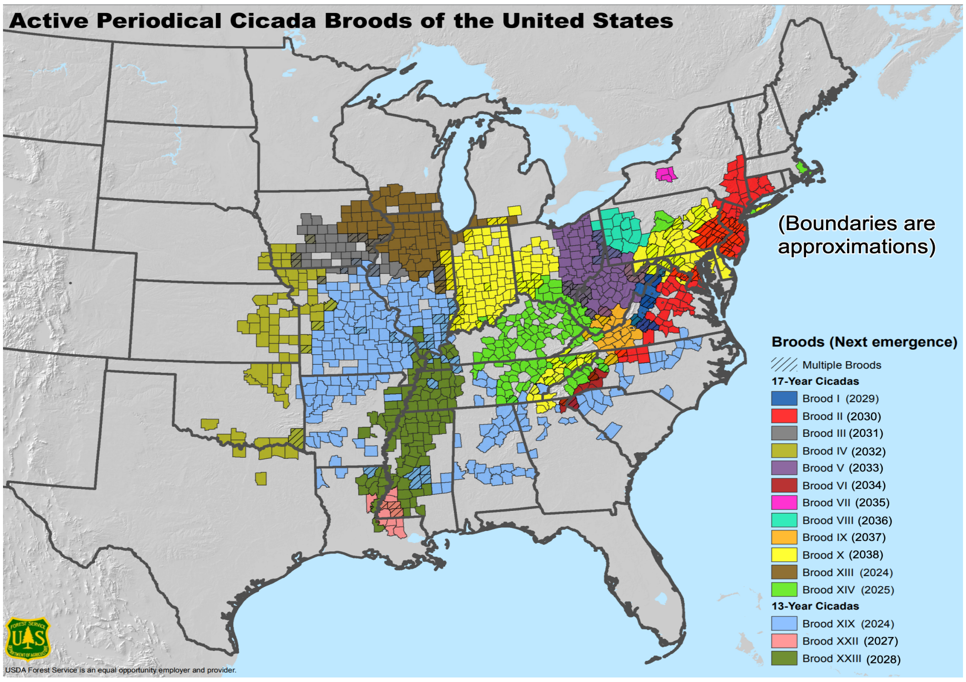


Figure 1.1: Active periodical Cicada broods of the United States [2].

Similar hybrid temporal structures arise in multidose pharmacokinetic models, which combine continuous processes such as absorption, metabolization, and elimination with discrete events corresponding to drug administration [3]. Another example arises in the analysis of an RLC circuit (capacitance C, inductance L and resistence R) where the capacitor is periodically discharged at each time instants, resulting in a time domain that is neither purely continuous nor completely discrete [4]. Moreover, large-scale models in DNA dynamics, gene mutation fixation, stock dynamics and option pricing in finance, as well as the analysis of market frequency and trade duration in economics, naturally involve continuous–discrete hybrid processes (see the references in [5, 6, 7]). These examples collectively illustrate the necessity of a mathematical framework capable of handling hybrid time domains.

In this context, Stefan Hilger introduced the theory of *measure chains* in his Ph.D. thesis in 1988 [8] with the aim of unifying continuous and discrete analysis within a single framework and extending it to hybrid domains. Shortly thereafter, the term *time scale* was adopted in the literature as a more intuitive and application-oriented designation. Bernd Aulbach, who supervised Hilger's doctoral work [8], identified three fundamental objectives of this calculus on measure chains (now known as time scale calculus), namely unification, extension, and discretization. Time scale theory generalizes and connects differential and difference equations by treating them as special cases of dynamic equations on time scales. This unified framework provides a powerful and flexible tool for modeling and analyzing systems exhibiting mixed continuous–discrete behavior, and it has found wide-ranging applications in physics, chemical technology, population dynamics, biotechnology, economics, neural networks, and social sciences.

A **time scale** is a nonempty closed subset of the set of real numbers ($\mathbb{R}$). It may take various forms, including closed intervals, the set of integers ($\mathbb{Z}$), the set of natural numbers ($\mathbb{N}$), the Cantor set, the set of harmonic numbers, the q-scale

$$\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z},\ q > 1\} \cup \{0\},$$

discrete time scale

$$h\mathbb{Z} := \{hk : k \in \mathbb{Z}\},\ h > 0,$$

a more general discrete time scale of the form

$$\{t_k : k \in \mathbb{Z}\}, \quad \text{where} \quad t_k \in \mathbb{R}, t_k < t_{k+1} \quad \text{for all} \quad k \in \mathbb{Z},$$

which is particularly useful for discretization purposes, and periodic time scales such as

$$\mathbb{P}_{a,b} := \bigcup_{k=1}^{\infty} [k(a+b), k(a+b)+b],\ a, b \in \mathbb{R},\ a, b \geq 0.$$

The periodic time scale $\mathbb{P}_{a,b}$ is of particular importance, since $\mathbb{P}_{0,1} \equiv \mathbb{R}$ and $\mathbb{P}_{1,0} \equiv \mathbb{Z}$. For nontrivial values of a and b, this mixed continuous–discrete domain naturally arises in the modeling of many real-world phenomena. For example, the population dynamics of periodic cicada populations, such as *Floridian brood* and *Pharaoh cicadas*, which are cicadas with larval stages lasting 13 and 17 years, respectively, can be modeled using the time scale $\mathbb{P}_{a,b}$ with $(a, b) = (12, 1)$ and $(16, 1)$, respectively. Similarly, the dynamics of an *RLC* circuit in which a capacitor is periodically discharged every unit time, with each discharge lasting for a duration $\delta > 0$, can be modeled on the time scale $\bigcup_{k=0}^{\infty}[k, k+1-\delta]$ [4].

Time scale calculus is also highly applicable in economics, a discipline in which many dynamic processes are traditionally modeled in either continuous or discrete time. In a discrete-time model, for example, a consumer receives income over a period and decides how much to consume and save during the same period, with all decisions assumed to occur at evenly spaced intervals. The time scales approach provides a more flexible and realistic framework, allowing income receipt, asset adjustments, and consumption decisions to occur at different points in time and with arbitrary, time-varying frequency. This enables the modeling of situations in which the timing of decisions directly affects outcomes, such as the impact of delayed consumption on savings or the effect of irregular investment intervals on portfolio growth, thereby offering clear advantages over traditional discrete or continuous models.

Furthermore, in many biological and ecological systems, particularly in the modeling of agricultural pest populations, data collection and system evolution are governed by environmental factors rather than uniform calendar time. For poikilothermic organisms such as insects, development depends on accumulated temperature, leading to sampling regimes that are inherently irregular when viewed in standard time units. Time-scale based models allow population dynamics to be formulated directly on such nonuniform sampling domains, enabling both time-based and temperature-based observations to be analyzed within a single framework. This makes it possible to compare alternative sampling strategies and assess their impact on parameter estimation accuracy using quantitative measures, thereby providing a realistic and flexible tool for modeling temperature-dependent population growth in applied biological systems [10].

Time scale calculus also plays an important role in quantum mathematics and quantum calculus, which have attracted considerable attention due to their wide-ranging applications in physics, including black holes and cosmic strings, conformal quantum mechanics, high-energy and nuclear physics, and the fractional quantum Hall effect. Quantum calculus (or q-calculus) generalizes classical calculus by replacing limits with finite q-differences and arises naturally when functions are defined on the q-scale $\overline{q^{\mathbb{Z}}}$. It has been successfully applied to study thermodynamic quantities such as entropy, pressure, internal energy, and specific heat. In particular, q-deformed thermostatistics is more accurately formulated using Jackson derivatives rather than classical derivatives. q-deformation significantly modifies the energy spectrum, enabling the modeling of complex systems or real gases, in contrast to classical thermodynamics, which is typically restricted to ideal gases. Consequently, the physical effects of q-deformation are more effectively captured by Jackson derivatives and the associated q-difference equations than by conventional differential equations.

1.1.2 Dynamic equations on time scales

In recent years, dynamical systems on time scales has been extensively studied in literature due to their wide range of applications in science and engineering, particularly in population dynamics, control theory, neural networks, and thermal physics. Let $\mathbb{T}$ be a time scale and $x : \mathbb{T} \to \mathbb{R}$ be a function defined on $\mathbb{T}$. The notion of the Δ-derivative x^{Δ} is introduced in such a way that it coincides with the classical derivative when $\mathbb{T} = \mathbb{R}$, and with the forward difference operator when $\mathbb{T} = \mathbb{Z}$. Within this framework, one can study dynamic equations involving higher-order delta derivatives of the form

$$f(t, x, x^{\Delta}, x^{\Delta^2}, \ldots, x^{\Delta^n}) = 0, \quad t \in \mathbb{T},$$

together with associated initial value or boundary value problems. These equations reduce to ordinary differential equations when $\mathbb{T} = \mathbb{R}$, difference equations when $\mathbb{T} = \mathbb{Z}$, and q-difference equations when $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

The study of dynamic equations on time scales provides a unified framework that replaces the separate analysis of models defined on different time domains. For example, the qualitative behavior of solutions of the logistic dynamic equation on time scales applies to both the classical logistic differential equation and its time-scale discretizations, which is distinct from the standard logistic map. This further illustrates that time scales offer alternative discretizations of continuous models that differ from commonly used discretizations and may exhibit significantly different dynamical behaviors.

Within the framework of time scales, several special classes of dynamic equations have attracted considerable attention due to their ability to model memory, abrupt changes, and hereditary effects. In particular, delay dynamic equations incorporate past states of the system and are essential in describing phenomena where the present evolution depends on historical information. Impulsive dynamic equations account for sudden changes in the state at specified moments, making them suitable for systems subject to instantaneous perturbations or external interventions. Furthermore, neutral dynamic equations involve delays not only in the state variables but also in their delta derivatives, allowing for a more accurate representation of systems with aftereffects in both the state and its rate of change. In addition, switched dynamic systems consist of multiple subsystems and a switching rule that governs transitions among them, enabling the modeling of systems whose structure or parameters change over time. Together, these classes of equations significantly extend the modeling capability of dynamic equations on time scales and arise naturally in a wide range of applications.

1.1.2.1 Delay dynamic equations

Delay dynamic equations (DDEs) play a fundamental role in modeling systems whose current state depends on past behavior, a feature that arises naturally in many physical, biological and engineered processes. Unlike ordinary dynamic equations, DDEs evolve on infinite-dimensional state spaces, which allows them to generate rich and realistic dynamics even in the presence of simple nonlinearities. This observation explains why classical models, such as the Lotka–Volterra predator–prey system, often fail to exhibit sustained oscillations, whereas delay-based models like Nicholson's blowflies equation successfully capture complex population cycles.

A DDE is a dynamic equation in which the Δ-derivative of the unknown function at a given time depends on the values of the functions at previous times. Such equations arise in mathematical models when the evolution of physical or biological system depends on its past states. With increasing demands for dynamic performance, engineers require models that more accurately reflect real processes. Many systems exhibit aftereffect phenomena, where the current state is influenced by historical events. Moreover, actuators, sensors, and communication networks in feedback control loops introduce unavoidable delays. In addition to actual delays, time lags are oftenly employed to simplify very high order models by representing complex dynamics through delayed responses. Consequently, DDEs have extensive applications in diverse fields, including age-structured population dynamics, control theory, signal transmission, transportation systems, and nuclear reactor control with time delays.

On a time scale $\mathbb{T}$, the general first-order DDE is written as

$$x^{\Delta}(t) = f(t, x(t), x_t), \quad t \in \mathbb{T},$$

where $x_t = \{x(\tau) : \tau \leq t\}$ is the history function of the solution, representing its past trajectory up to time t. Delays are commonly classified as discrete or distributed, depending on whether the influence of past states occurs at specific time instants or is spread over an interval of time. In DDEs with discrete delays, the evolution of the system at time t depends on the state of the system at one or more specific past instants. Such delays represent situations where aftereffects occur after fixed or time-varying lags. These models are suitable for processes with well-defined reaction or maturation times, including population dynamics, neural signal transmission, and feedback control systems. The general form is

$$x^{\Delta}(t) = f\big(t, x(t), x(t - \eta_1(t)), ..., x(t - \eta_n(t))\big), \quad \eta_1, \eta_2, ..., \eta_n \geq 0,$$

where it is assumed that $t - \eta_i(t) \in \mathbb{T}$ for all i.

In contrast, DDEs with distributed delays describe the cumulative influence of the system's history over an interval of past times, typically represented by integral terms with suitable kernel functions. This formulation captures memory effects in systems where the influence of the past is gradual rather than instantaneous, as in physiological processes, epidemiological models, and materials with hereditary properties. A general form is

$$x^{\Delta}(t) = f\Big(t, x(t), \int_0^{\infty} K(t, s) x(t - s) \ \Delta s\Big),$$

where the Δ-integral term represents an infinitely distributed delay, since the present state depends on the entire past history of the system. When the integration interval is restricted to a finite delay length $\delta > 0$, the model reduces to a finite distributed delay, in which only past states over a bounded interval contribute to the system's evolution. On general time scales, the integration is taken over those past points $s \geq 0$ for which $t - s$ lies in the time scale, ensuring that both the shifted solution $x(t - s)$ and the kernel $K(t, s)$ are well defined.

Both discrete and distributed delay dynamic equations play a crucial role in accurately modeling complex real-world phenomena, and in many applications, they appear together within a single dynamical system. Their combination allows for the realistic representation of systems exhibiting both instantaneous aftereffects and cumulative memory effects, providing a versatile framework for studying a wide range of biological, physical, and engineering processes.

1.1.2.2 Impulsive dynamic equations

In many practical systems, abrupt changes in state occur at specific moments due to sudden external or internal events such as harvesting, stocking, control actions, switching mechanisms, or shocks. These abrupt changes, commonly referred to as impulsive effects, can significantly alter the system's dynamics and must be incorporated into mathematical models to achieve accurate and realistic descriptions. Impulsive phenomena naturally arise in a wide range of applications, including population dynamics, where sudden harvesting or stocking events occur; neural networks, where instantaneous signal transmissions or resets may happen; epidemiological interventions, such as mass vaccinations or treatment campaigns; and engineering or control systems, where switching or corrective actions are applied at discrete times. To model such phenomena rigorously, impulsive dynamic equations on time scales offer a versatile framework that unifies continuous and discrete dynamics, enabling the system to capture both gradual evolution and instantaneous changes within a single formulation. Depending on their characteristics, impulses are generally classified into two main types:

1. Instantaneous impulses: These occur over a very short duration relative to the overall evolution of the system, such as shocks or sudden environmental disasters. Systems subject to such abrupt changes are typically modeled using instantaneous impulsive differential equations, which capture the immediate effect of the impulse at a specific moment.

2. Non-instantaneous impulses: These impulses begin abruptly at a certain time but continue to influence the system over a finite interval. For example, in the case of a hyperglycemic patient, the administration of insulin causes an immediate change in blood glucose levels, followed by a gradual process as the medication is absorbed. Such phenomena are modeled using non-instantaneous impulsive differential equations, where the effect of the impulse remains active over a finite time period.

Any dynamic equation that incorporates impulsive conditions is referred to as an impulsive dynamic equation. The theory of impulsive dynamic equations has become an active area of research due to its wide-ranging applications in biological systems, mechanical systems, electronics, economics, and other fields. These models are particularly relevant because they provide a natural framework for describing processes that experience sudden, rapid changes in state at specific moments, which cannot be adequately captured by ordinary dynamic equations alone. The general form of an impulsive dynamic equation with instantaneous impulses on time scales is given by

$$\begin{cases} x^{\Delta}(t) = F(t, x(t)),\ t \in \mathbb{T},\ t \neq t_k, \\ x\left(t_k^{+}\right) - x\left(t_k^{-}\right) = \mathcal{I}_k\left(x\left(t_k^{-}\right)\right),\ k = 1, 2, \ldots, \ell, \end{cases}$$

where $x(t)$ is the state function, $F : \mathbb{T} \times \mathbb{B} \to \mathbb{B}$ defines the rd-continuous dynamics, and $\mathcal{I}_k$ represents the impulsive effect at the moments t_k.

However, certain evolution processes, such as those arising in pharmacotherapy, cannot be adequately described using instantaneous impulses. To address such phenomena, theory of non-instantaneous impulsive systems have been developed. The general form of an impulsive dynamic equation with non-instantaneous impulses dynamic equation with non-instantaneous impulses on time scales is

$$\begin{cases} x^{\Delta}(t) = F(t, x(t)),\ t \in \bigcup_{k=0}^{\ell} (s_k, t_{k+1}]_{\mathbb{T}}\,, \\ x(t) = g_k\left(t, x\left(t_k^-\right)\right),\ t \in (t_k, s_k]_{\mathbb{T}}\,,\ k = 1, 2, \ldots, \ell, \end{cases}$$

where $x(t)$ is the state variable, $F : [0,T]_{\mathbb{T}} \times \mathbb{B} \to \mathbb{B}$ defines the rd-continuous dynamics, $g_k\left(t, x\left(t_k^-\right)\right)$ represents the non-instantaneous impulses acting over the intervals $(t_k, s_k]_{\mathbb{T}}$, with s_k, t_k satisfying

$$0 = s_0 = t_0 < t_1 < s_1 < t_2 < \ldots < t_\ell < s_\ell < t_{\ell+1} = T < \infty.$$

1.1.2.3 Functional dynamic equations

In functional dynamic equations on time scales, the state variable $x(t)$ depends not only on the current state but also on past or delayed states. These equations arise in models where the evolution of the system is explicitly influenced by historical events, allowing past behavior to directly affect future outcomes. In contrast to ordinary dynamic equations, in which the effect of the past is only implicit, functional dynamic equations provide a more accurate framework for processes with memory or delay. Delay dynamic equations form a specific subclass of functional dynamic equations, in which the system depends explicitly on its state at one or more discrete past times or over a past interval, which may be finite or infinite. Functional dynamic equations thus provide a general framework that encompasses more specific types of equations, such as neutral dynamic equations, integro-dynamic equations, and other delayed or memory-dependent systems. On general time scales, the general form of a first-order functional dynamic equation is

$$x^{\Delta}(t) = F(t, x(t), x(h(t))),\ t \in \mathbb{T},$$

where $x(t)$ is the state variable and $h(t)$ is a real-valued function of t, typically representing a delay or a functional argument of the past state.

1.1.2.4 Neutral dynamic equations

A neutral dynamic equation on time scales is a dynamic equation in which the highest-order Δ-derivative of an unknown function depends not only on the present state but also on one or more delayed states. This feature distinguishes neutral equations from standard delay dynamic equations, as the derivative itself incorporates after-effects or memory terms. Neutral dynamic equations arise in many real-world systems with after-effects and have abundant applications in control theory, oscillation theory, electrodynamics, biomathematics, economic modeling, and other areas of science and engineering. The general form of a first-order neutral dynamic equation on time scales can be written as

$$[x(t) - a_1 x(t - \eta_1(t))]^{\Delta} = a_2 x(t) + a_3 x(t - \eta_2(t)) + F(t), \ t \in \mathbb{T},$$

where a_1, a_2 and a_3 are given constants, $\eta_1, \eta_2 \geq 0$ represent delays, and $F(t)$ is a known function. The term $x(t) - a_1 x(t - \eta_1(t))$ is called the neutral term, representing the direct influence of past states on the derivative of the system, while the right-hand side may include additional delayed contributions and external inputs. This formulation provides a versatile framework for modeling a wide variety of systems in which the memory or after-effects may vary over time.

1.1.2.5 Integro-dynamic equations

An integro-dynamic equation on time scales is a dynamic equation in which the unknown function $x(t)$ contains an ordinary Δ-derivative of $x(t)$ appears both through its delta derivative and under an integral operator. Such equations naturally arise in many scientific and engineering applications, including chemical kinetics, electronics, biological models, and fluid dynamics. The general form of a first-order integro-dynamic equation on time scales is

$$x^{\Delta}(t) = F(t) + \lambda \int_{g(t)}^{h(t)} K(x, s) x(s) \Delta s, t \in \mathbb{T}$$

where $x(t)$ is a state variable, λ is a constant, $g(t)$ and $h(t)$ are the lower and upper limits of integration, F and K are are given functions defined on appropriate domains.

1.1.3 Abstract formulation on time scales

An abstract dynamic equation on time scales is a dynamic equation in which the unknown function and its Δ-derivatives take values in an abstract function space, typically a Banach or Hilbert space. Such equations provide a unified framework for modeling a wide variety of evolution problems, including partial dynamic equations, functional dynamic equations, age-structured population models, and delayed reaction-diffusion systems. In the classical continuous-time setting, several partial differential equations, such as the heat equation, wave equation, and delayed reaction-diffusion equations, can be reformulated as abstract differential equations. By formulating evolution problems in abstract spaces, one can apply functional-analytic techniques such as semigroup theory, operator theories, and spectral analysis to study qualitative behavior of solutions. The general form of an abstract dynamic equation on a time scale $\mathbb{T}$ is

$$x^{\Delta}(t) = \mathcal{A}(t) x(t) + f(t, x), \ t \in \mathbb{T}$$

where $x(t)$ takes values in a Banach or Hilbert space X, $\mathcal{A}(t) : \mathrm{D}(\mathcal{A}) \subset \mathrm{X} \to \mathrm{X}$ is an (typically linear, possibly unbounded) operator, and $f : \mathbb{T} \times X \to X$ represents an external input or forcing term.

Abstract dynamic equations can be classified as follows:

1. Autonomous abstract dynamic equations: When the operator $\mathcal{A}$ and the system do not explicitly depend on time, i.e.,
$$x^{\Delta}(t) = \mathcal{A}x(t) + f(x),\ t \in \mathbb{T},$$
the system is called autonomous. Autonomous systems are used to study time-invariant processes, such as classical heat or wave equations. Their long-term behavior can often be analyzed using semigroup theory.

2. Non-autonomous abstract dynamic equations: When the system explicitly depends on time, typically through a time-varying operator or forcing term, i.e.,
$$x^{\Delta}(t) = \mathcal{A}(t)x(t) + f(t,x),\ t \in \mathbb{T},$$
the system is called non-autonomous. Non-autonomous formulations are essential for modeling time-dependent processes, such as seasonal population models, time-varying control systems, and variable-coefficient partial dynamic equations. In this case, classical semigroup methods are no longer directly applicable. Instead, one can use evolution families, which generalize semigroups, to describe the evolution of solutions, and exponential dichotomy theory to study time-dependent stability. For nonlinear systems, variation-of-constants formulas together with fixed-point methods are commonly employed to establish existence, uniqueness, and qualitative properties of solutions.

By working in an abstract setting, only the invariant properties of a class of problems are emphasized, while many unnecessary details of a particular model are suppressed. This allows the rich theory of functional analysis to be applied, enabling the study of existence, uniqueness, asymptotic behavior, continuous dependence on initial data, and controllability of solutions in a systematic and unified manner.

1.1.4 Stochastic dynamic equations on time scales

Classical dynamical systems are typically developed under idealized assumptions of perfectly controlled environments, precise parameter values, and complete knowledge of system interactions. While these assumptions facilitate simplified mathematical analysis, they rarely reflect practical conditions encountered in real-world applications. Systems arising in physics, biology, engineering, and finance are inherently subject to random fluctuations, measurement uncertainties, environmental disturbances, and sudden structural variations. Such uncertainties may originate from thermal noise in physical processes, random switching in networked or hybrid systems, demographic variability in biological populations, or unpredictable market forces in financial systems. Neglecting these stochastic influences often limits the ability of deterministic models to capture essential qualitative and statistical features, including variability, instability, sensitivity to perturbations, and long-term probabilistic behavior. Consequently, the incorporation of stochastic effects into dynamical systems has become essential for developing mathematically rigorous frameworks that more accurately describe and predict the evolution of complex real-world systems.

A stochastic dynamic equation (SDE) on time scales describes a dynamic system whose evolution is influenced by both deterministic and random effects. Unlike deterministic dynamic equations, the state variables in SDEs evolve as random processes due to the presence of uncertainties and external disturbances. In classical continuous-time settings, stochastic differential equations provide a well-established framework for modeling such randomness through probabilistic noise terms. The theory of time scales, offers a natural and powerful framework for extending stochastic modeling to arbitrary time domains. Within this setting, SDEs unify continuous-time, discrete-time, and hybrid stochastic models under a single mathematical formulation. A general SDE on a time scale $\mathbb{T}$ can be be expressed as

$$\Delta x(t) = f(t, x(t))\Delta t + g(t, x(t))\Delta \mathcal{X}(t), \ t \in \mathbb{T},$$

where $f(t, x(t))$ represents the deterministic component of the system, commonly referred to as the drift term, and describes the average or systematic evolution of the state in the absence of randomness. The function $g(t, x(t))$ denotes the diffusion term, which characterizes the intensity of stochastic perturbations. The stochastic process $\mathcal{X}(t)$ models the source of randomness and may represent various stochastic inputs depending on the nature of the system under consideration.

SDEs establish a fundamental connection between probability theory and the theory of dynamic equations on time scales. This connection allows classical results from deterministic analysis to acquire new probabilistic interpretations and enables the extension of stability theory, boundedness analysis, and long-term behavior studies to stochastic settings. In applied fields such as artificial intelligence and data-driven modeling, SDEs play an important role in understanding uncertainty, learning dynamics, and probabilistic decision-making processes, particularly in systems that evolve over continuous or hybrid time domains. In continuous-time frameworks, stochastic perturbations are commonly modeled using processes that describe continuous random fluctuations, such as the Wiener process (Brownian motion). Nevertheless, many real-world systems exhibit discontinuities caused by abrupt environmental changes, impulsive effects, or switching mechanisms. Such non-smooth stochastic behavior can be effectively represented using jump processes, including Poisson noise, Lévy processes, and Markov switching models. For SDEs on time scales, the stochastic component adapts naturally to the structure of the underlying time domain through appropriate reformulation. When the time scale contains continuous intervals, stochastic disturbances resemble classical continuous-time noise and generate gradual random fluctuations, whereas at points preceded by a finite temporal gap, stochastic effects appear as random jumps represented through stochastic increments consistent with discrete-time transitions. Thus, the evolution of stochastic disturbances reflects the geometric structure of the time scale. This unified representation enables stochastic time scale models to describe systems in which uncertainty develops either either continuously in time or through abrupt transitions, thereby providing a comprehensive framework for continuous, discrete, and hybrid stochastic behavior. Mathematically, stochastic jumps at discrete time points are modeled as random increments added to the system state, where the post-jump value is expressed as the sum of the pre-jump state and a noise term scaled by an appropriate coefficient, allowing system dynamics to be analyzed probabilistically through statistical measures such as expectations, variances, and stability properties and thereby offering a realistic description of uncertainty in hybrid dynamical systems.

1.1.5 Classical population and neural network models

This subsection introduces classical population dynamics models and early neural network formulations, with emphasis on their mathematical structure and role in the development of modern nonlinear dynamical systems.

1.1.5.1 Nicholson's blowflies model

The Australian sheep blowfly, *Lucilia cuprina*, has been one of the most destructive pests affecting the Australian wool industry. Its larvae attack the skin of sheep, temporarily stopping wool growth and leaving weak breaks in the fibres, which significantly reduce the quality and commercial value of the fleece.

Figure 1.2: *Lucilia Cuprina*: Australian sheep blowfly.

Figure 1.3: Australian sheep.

Figure 1.4: Lucilia Cuprina on skin of a Australian sheep.

During controlled laboratory experiments conducted in the 1950s, Nicholson observed pronounced population oscillations of *Lucilia cuprina* even under constant environmental conditions. These large-amplitude fluctuations could not be explained by classical population models, such as the Malthusian (exponential) or logistic equations, which predict either unbounded growth or convergence to a steady state. Nicholson attributed this behavior to delayed density dependence, arising from the finite time required for eggs to develop into mature, reproductive adults. Subsequent delay logistic models incorporated maturation delays; however, they still failed to reproduce the multiple reproductive bursts observed in Nicholson's experimental data. This limitation motivated the development of a more realistic model. In 1980, *Gurney et al.* introduced the Nicholson blowflies model [11], which accounts explicitly for delayed reproduction and nonlinear density dependence. The classical form of this model is given by

$$x'(t) = -dx(t) + \beta x(t-\tau)e^{-\lambda x(t-\tau)}, \quad t \geq 0$$

where $x(t)$ denotes the population density at time t, d (> 0) is the per-capita mortality rate, β (> 0) represents the maximum egg production rate, λ^{-1} (> 0) is the inflection point of the of the nonlinear recruitment function $f(x) = xe^{-\lambda x}$, and τ (> 0) denotes the average maturation delay.

1.1.5.2 Lasota–Ważewska model

Introduced in 1976, the Lasota–Ważewska model captures the regulatory dynamics of red blood cell populations in the human circulatory system. Red blood cells are continuously produced in the bone marrow and removed from circulation due to aging and natural destruction. An important characteristic of this process is the time delay between the formation of precursor cells and their release as mature red blood cells into the bloodstream. To capture this delayed feedback mechanism, Lasota and Ważewska proposed a delay differential equation in which the rate of red blood cell production depends on the population level at an earlier time. The classical Lasota–Ważewska equation is given by [12]

$$x'(t) = -dx(t) + \beta e^{-\lambda x(t-\tau)},$$

where x denotes the concentration of circulating red blood cells, d (> 0) represents the rate of cell removal, while β and λ are parameters controlling the production dynamics, and τ denotes the time required for denotes the time required for cell maturationangle. The nonlinear production term $\beta e^{-\lambda x(t-\tau)}$ reflects an exponentially decreasing birth rate as the delayed population density increases. Biologically, this represents a regulatory mechanism in which high levels of circulating red blood cells suppress further production, whereas low levels promote increased production. This delayed negative feedback is essential for maintaining homeostasis but may also lead to complex dynamical behavior when the delay or feedback strength is sufficiently large.

1.1.5.3 Mackey–Glass model

The Mackey–Glass model, introduced in 1977, was developed to describe the regulation of blood cell production in the human body, a physiological process known as hematopoiesis. In this system, the production of mature blood cells in the bone marrow is governed by feedback mechanisms that involve significant time delays, reflecting the time required for stem cells to mature and enter the bloodstream. Mackey and Glass demonstrated that disruptions in this delayed feedback can give rise to complex and irregular oscillations in blood cell concentrations, phenomena that are commonly associated with pathological conditions such as periodic anemia and leukemia. To model the regulation of red blood cell formation, Mackey and Glass proposed the following delay differential equation [13]

$$x'(t) = -dx(t) + \frac{\beta\theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)},$$

where all parameters d, β, θ, n, and τ are all positive. In this model, x represents the density of red blood cells in the bloodstream. The parameter β denotes the maximum rate of red blood cell production that the body approaches when the circulating red blood cell level is very low, while parameter τ represents the maturation delay. The parameter θ acts as a shape parameter, n controls the degree of nonlinearity in the production response, and d denotes the rate at which red blood cells are removed from circulation.

1.1.5.4 Hopfield neural network

Artificial Neural Networks (ANNs) are computational models inspired by the structure and functioning of biological neural systems. One of the earliest and most widely cited definitions was given by Robert Hecht-Nielsen, who described neural networks as "computing systems composed of many simple, highly interconnected processing elements that process information through their dynamic responses to external inputs". The first artificial neural network, the perceptron, was introduced in 1958 by Frank Rosenblatt to model pattern recognition in visual perception. Since then, neural network research has advanced substantially, both theoretically and computationally, leading to modern architectures consisting of large collections of simple processing units (neurons) with continuously varying activation values governed by nonlinear and, in some cases, stochastic dynamics. These properties enable neural networks to effectively model complex, adaptive, and distributed nonlinear systems, making them an important research area in applied mathematics, engineering, and information theory.

Recurrent neural networks (RNNs) form an important class of neural network models characterized by feedback connections that allow the network to exhibit dynamic temporal behavior and memory effects. Among the earliest and most influential recurrent models are Hopfield Neural Networks (HNNs), which were introduced to describe associative memory and optimization problems within a dynamical systems framework. Inspired by neurobiological principles and concepts from statistical mechanics, HNNs are distinguished by symmetric synaptic connections and energy-based dynamics that guarantee convergence to stable equilibrium states. Structurally, an HNN consists of a set of n interconnected neurons, where each

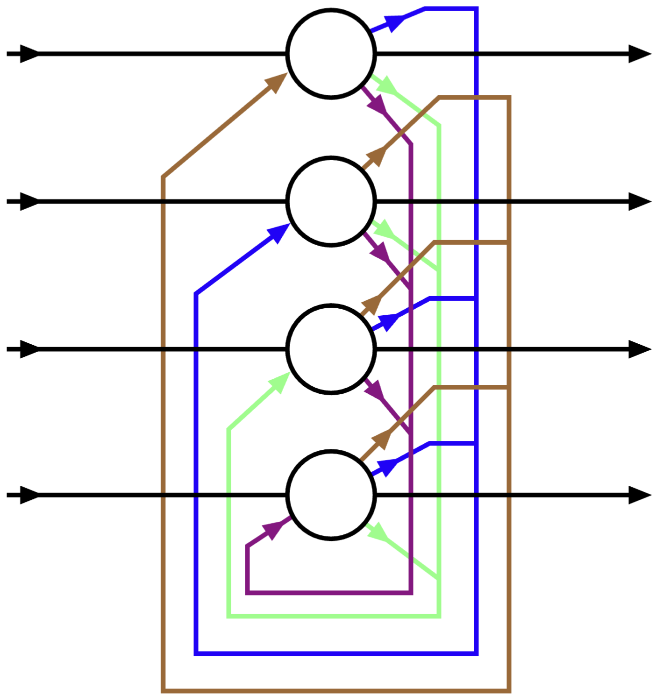


Figure 1.5: A four-unit Hopfield-type RNN.

neuron is connected to every other neuron except itself. The network is fully recurrent, allowing feedback interactions among neurons that govern the temporal evolution of the system. HNNs can be formulated in either discrete-time or continuous-time settings. In the continuous-time case, the network dynamics are described by the system of differential equations

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} A_{ij} f(x_j(t)) + u_i, \quad i = 1, 2, \ldots, n,$$

where $x_i(t)$ denotes the state of the ith neuron, a_i (> 0) represents the passive decay (leakage) rate, A_{ij} (≥ 0) is the synaptic weight from neuron j to neuron i, $f(\cdot)$ is a nonlinear activation function, and u_i denotes an external input or bias term. The synaptic weights satisfy $A_{ij} = A_{ji}$ and $A_{ii} = 0$, ensuring symmetric connectivity and the absence of self-feedback. Relaxing these classical assumptions on the synaptic weights leads to generalized Hopfield neural networks with more flexible and complex recurrent dynamics.

HNNs have been widely applied in associative memory, pattern recognition, and combinatorial optimization. Typical applications include image restoration, constraint satisfaction, and the traveling salesman problem. Despite inherent limitations such as finite storage capacity and the existence of spurious equilibrium states, HNNs remain a cornerstone of RNN theory. Their energy-based formulation and dynamical properties continue to influence the development of modern neural network architectures and energy-based learning models.

1.1.5.5 Cellular neural network

In 1988, Leon O. Chua and Lin Yang introduced the Cellular Neural Network (CNN) as a new class of artificial neural networks. CNNs are large-scale, nonlinear, and locally connected dynamical systems, originally conceived for real-time signal and image processing applications. A CNN is composed of a large number of identical processing units called cells, arranged in a regular lattice structure. Unlike traditional neural networks, where each neuron may be connected to many others, CNNs exhibit local connectivity, i.e., each cell interacts directly only with its neighboring cells. This structural property is similar to that of cellular automata, but CNNs differ in that their state variables are continuous and governed by differential equations. Each CNN cell consists of linear and nonlinear circuit components, including capacitors, resistors, controlled sources, and independent sources. From a system-theoretic perspective, each cell is characterized by an input u_{ij}, a state x_{ij}, and an output y_{ij}, where the indices (i,j) denote the position of the cell in the array. Cells can be arranged in one-dimensional, two-dimensional, or higher-dimensional configurations. The most common and widely studied architecture is the two-dimensional CNN, where cells are arranged on a rectangular grid. Each cell interacts only with the cells within its radius of neighborhood r. For the commonly used case $r = 1$, the neighborhood of a cell includes the cell itself and its eight nearest neighbors, forming a 3×3 local region (see Figure 1.6). This local connectivity enables massive parallelism, making CNNs particularly suitable for real-time information processing. As a result, CNNs have been successfully applied in numerous fields, including image processing (edge detection, noise removal, and segmentation), pattern recognition, signal processing, associative memory, optimization problems, and sensory data processing.

The dynamics of a CNN are described by a system of nonlinear ordinary differential equations. For the simplest case of first-order cell dynamics with linear interactions, the state equation of a cell located at position (i, j) is given by

$$\frac{dx_{ij}(t)}{dt} = -x_{ij}(t) + \sum_{(k,l)\in\mathcal{N}(i,j)} A_{ij}^{kl} y_{kl}(t) + \sum_{(k,l)\in\mathcal{N}(i,j)} B_{ij}^{kl} u_{kl}(t) + z_{ij},$$

where $\mathcal{N}(i, j)$ denotes the neighborhood of cell (i, j), A_{ij}^{kl}, represents the feedback template, defining the influence of neighboring cell outputs, B_{ij}^{kl} represents the control template, defining the influence of neighboring cell inputs, z_{ij} is a bias or threshold term. The output of each cell is typically defined by a

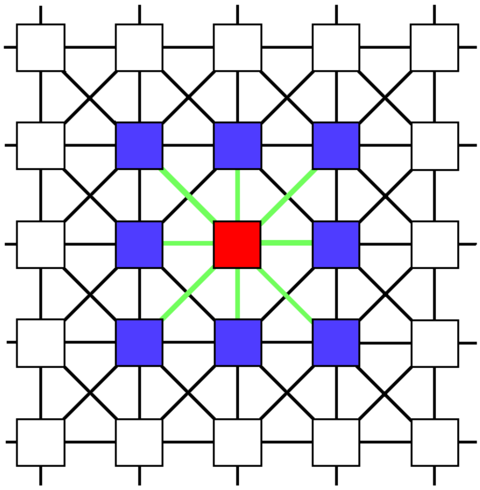


Figure 1.6: Two-dimensional CNN with neighborhood radius $r = 1$, showing the nine-cell neighborhood of the central (red) cell.

nonlinear function of the state, commonly a piecewise-linear saturation function:

$$y_{ij}(t) = f(x_{ij}(t)) = \frac{1}{2}\left(|x_{ij}(t) + 1| - |x_{ij}(t) - 1|\right).$$

This nonlinearity plays a crucial role in the pattern-forming and decision-making capabilities of CNNs.

In a two-dimensional space-invariant CNN with radius $r = 1$, feedback term A is a 3×3 matrix, control term B is a 3×3 matrix, and the bias z is a scalar. Thus, the entire network behavior is determined by only 19 real parameters, regardless of the network size. This compact parameterization is one of the most powerful features of CNNs. Since CNNs are finite arrays, the computation of cell states at the boundary requires the specification of boundary conditions. The choice of boundary conditions can significantly affect the network dynamics and the final steady-state solution. The dynamic behavior of CNNs depends on the interaction templates, initial conditions, and external inputs. By appropriately designing the cloning templates, CNNs can exhibit a wide range of behaviors, including pattern formation, wave propagation, stability, oscillations, and chaos.

1.1.5.6 Shunting inhibitory cellular neural network

Shunting Inhibitory Cellular Neural Networks (SICNNs) constitute an important class of cellular neural networks that explicitly incorporate biologically inspired shunting inhibition mechanisms. This inhibition allows neurons to dynamically control the strength of excitation and inhibition, thereby enhancing stability, contrast sensitivity, and adaptability. Motivated by observations from neurobiology, SICNNs were proposed as an extension of classical CNNs to model more realistic neural interactions and to improve performance in signal and image processing tasks.

Structurally, SICNNs preserve the locally connected lattice architecture of conventional CNNs, in which cells are arranged in one or two-dimensional arrays and interact only with their neighboring cells. However, unlike standard CNNs where interactions are typically additive, SICNNs employ multiplicative (shunting) inhibitory interactions. These interactions modulate the cell state in proportion to its current activity, enabling effective signal normalization and suppression of excessive excitation, which leads to more controlled and robust dynamics. For a two-dimensional SICNN, the state equation of a cell located at position (i,j) is given by

$$\frac{dx_{ij}(t)}{dt} = -a_{ij}x_{ij}(t) + \sum_{(k,l)\in\mathcal{N}(i,j)} A_{ij}^{kl}x_{ij}(t)f(x_{kl}(t)) + \sum_{(k,l)\in\mathcal{N}(i,j)} B_{ij}^{kl}u_{kl}(t) + z_{ij},$$

where a_{ij} (>0) denotes the passive decay (leakage) rate of the cell, A_{ij}^{kl} (≥ 0) represents the shunting inhibitory coupling strength, and f is a nonlinear activation function. All remaining parameters and variables have the same interpretation as in the conventional CNN model. The term $x_{ij}(t)f(x_{kl}(t))$ characterizes the shunting (multiplicative) inhibition that distinguishes SICNNs from conventional CNN models. The activation function $f(\cdot)$ is typically assumed to be bounded, continuous, and monotonically increasing, so that the inhibitory influence remains controlled and biologically plausible.

Due to the presence of multiplicative inhibitory interactions, SICNNs exhibit richer dynamical behaviors than standard CNNs, including enhanced stability properties, contrast enhancement, oscillatory responses, and complex spatiotemporal pattern formation. These features make SICNNs particularly suitable for applications in image processing such as edge detection, noise suppression, image enhancement, and segmentation, where shunting inhibition naturally performs contrast normalization. Additionally, SICNNs have been successfully applied to pattern recognition, associative memory, optimization problems, and real-time signal processing, demonstrating their effectiveness in handling sensory data with varying intensity levels.

1.1.6 (ω, c)-periodicity in continuous-time domain

Consider the one-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \quad x \in \mathbb{R}, \tag{1.1.1}$$

where $\psi(x)$ denotes the wave function, $V(x)$ is the potential energy, E is the energy eigenvalue, $\hbar$ is the reduced Planck constant, and m is the mass of the particle. We assume that the potential $V(x)$ is periodic with period $\omega > 0$, that is,

$$V(x+\omega) = V(x) \text{ for all } x \in \mathbb{R}. \tag{1.1.2}$$

Such periodic potentials naturally arise in the quantum mechanical description of electrons in crystalline solids and other structured media. By Bloch-Floquet theory, solutions of (1.1.1) with periodic potential (1.1.2) admit the representation

$$\psi_k(x) = e^{ikx} u_k(x), \tag{1.1.3}$$

where $k \in \mathbb{R}$ and $u_k(x)$ is an ω-periodic function. Consequently, the wave function satisfies the Bloch-periodicity (quasi-periodicity) condition.

Definition 1.1.1. *A function $f : \mathbb{R} \to \mathbb{C}$ is said to be* Bloch periodic *with period ω if there exists a real number k such that*

$$f(t+\omega) = e^{ik\omega} f(t) \text{ for all } t \in \mathbb{R}.$$

The complex constant $e^{ik\omega}$ is called the Bloch multiplier.

Bloch periodicity was introduced in 1928 by the Swiss physicist Felix Bloch in his study of electron motion in crystalline solids. It provides a fundamental framework for the spectral analysis of Schrödinger operators with periodic potentials and plays a central role in solid-state physics, photonic crystals, and other areas involving wave propagation in periodic media. Bloch periodicity generalizes classical periodicity by allowing a function to repeat up to a constant phase factor after each spatial period ω. In particular, when $k = 0$, the Bloch periodic function reduces to a conventional periodic function, while the choice $k = \pi/\omega$ leads to antiperiodic behavior. Since the quasi-momentum k is real in physical applications, the Bloch multiplier satisfies $|e^{ik\omega}| = 1$, ensuring that the magnitude of the function remains unchanged under translations by ω.

In the real world, there exist phenomena whose magnitudes scale by an approximately constant factor after each fixed time interval, rather than repeating exactly. A classical example is population growth.

Table 1.3: Official censuses of the USA population

Year	Population	Growth Ratio
1790	3,929,827	–
1800	5,305,925	1.350
1810	7,239,814	1.364
1820	9,638,131	1.331
1830	12,866,020	1.335
1840	17,062,566	1.326

Consider the population of the United States as recorded by official censuses over successive decades, as shown in Table 1.3. The ratios of the population between consecutive census years remain relatively stable, taking values close to $c \approx 1.34$. This observation suggests an approximate multiplicative relation of the form $P(t+\omega) \approx cP(t)$, where $P(t)$ denotes the population at time t and $\omega = 10$ years represents the census interval. Over longer time spans, this scaling accumulates multiplicatively; for example, over 25 years the population increases by a factor of approximately $c^{2.5} \approx 2.08$. Such behavior cannot be described by classical periodicity, which requires exact repetition after each period. Instead, it motivates a generalized notion of periodicity in which a function reproduces itself up to a constant scaling factor after each interval. This perspective naturally extends the concept of Bloch periodicity beyond oscillatory phase modulation to encompass a broader class of growth and decay phenomena.

Consider a second-order linear differential equation with periodic coefficients, given by

$$\frac{d^2y(x)}{dx^2} + \big(a - 2q\cos(2x)\big)y(x) = 0, \tag{1.1.4}$$

where $a, q \in \mathbb{R}$ are parameters. Equation (1.1.4), known as Mathieu equation, is a special case of Hill's equation and arises in diverse applications, including seasonally forced population dynamics and stability analysis of railway tracks under moving loads [14]. By Floquet theory, (1.1.4) admits a fundamental set of solutions of the form

$$y(x) = e^{\mu x}p(x),$$

where $\mu \in \mathbb{C}$ is the Floquet exponent and $p(x)$ is a π-periodic function [15]. The value of μ depends on the parameters a and q, and determines the qualitative behavior and stability of the solutions. In particular, if $\mathrm{Re}(\mu) = 0$, the solutions are bounded, whereas $\mathrm{Re}(\mu) \neq 0$ leads to exponential growth or decay and hence instability. For general values of a, called characteristic values, solutions satisfy the relation

$$y(x+\pi) = e^{\pi\mu}y(x),$$

whereas only for certain special values of the parameter a, equation (1.1.4) admits periodic or anti-periodic solutions.

The solution structure of the Mathieu equation motivates the notion of (ω, c)-periodicity. In 2018, Álvarez et al. [14] introduced the class of (ω, c)-periodic functions, defined by

$$f(t+\omega) = cf(t), \quad t \in \mathbb{R},$$

to formalize this Floquet-type behavior. This framework extends classical periodic and anti-periodic functions and naturally includes Bloch functions, as well as a wide range of bounded and unbounded functions with prescribed growth characteristics. Since its introduction, several generalizations of this class have been studied, together with their theoretical properties and applications in various fields.

1.2 Literature survey

The study of periodic solutions of dynamical systems has a long history, originating from classical results for ordinary differential and difference equations. However, conventional periodicity is a restrictive assumption and may not adequately describe many real-world models in which the underlying dynamics are nearly, but not exactly, periodic, as observed in applications such as signal processing and astrophysics [21, 22, 23]. To address this limitation, Bohr [24, 25] introduced the notion of almost periodicity as a natural generalization of classical periodicity, capable of capturing recurrent yet nonexactly periodic behavior. This theory was subsequently developed by Besicovitch [26] and Bochner [27], and later extended to include automorphic functions [28], asymptotically periodic functions [29, 30], pseudo almost periodic functions [31], S-asymptotically periodic functions [34], weighted pseudo almost periodic functions [32] and others [35]. The existence of such generalized periodic-type solutions has become a central topic in the qualitative theory of differential equations, owing to their wide-ranging applications in physics, mathematical biology, neural networks, control theory, and related areas [36, 37, 38, 39, 40, 41]. These concepts have been extensively studied in the context of functional differential equations, partial differential equations, fractional integro differential equations, stochastic differential equations and control systems, highlighting their effectiveness in modeling complex dynamical processes in applied sciences [42, 43, 44, 45, 46, 47]. There have also been significant studies on generalized anti-periodic and Bloch-periodic solutions of differential equations in recent years [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58].

Since the introduction of time scales theory in 1988 by Stefan Hilger [8, 9], research in this area has grown steadily. Dynamic equations on time scales have attracted considerable attention due to their unifying framework, which bridges discrete and continuous analysis and offers powerful tools for applications in economics [59], population dynamics [60, 61], quantum physics, and other applied fields by allowing the study of hybrid and nonclassical time domains. Two monographs, *Dynamic equations on time scales: an introduction with applications* [16] and *Advances in dynamic equations on time scales* [17], provide a comprehensive and systematic treatment of the theory. Moreover, numerous authors have extended Hilger's original work in various directions; for further related contributions, see [62, 63, 64, 65]. Over time, a substantial literature has emerged on periodicity, anti-periodicity, Bloch-periodicity, and their generalized forms on time scales, with applications analyzed for a wide range of models described by dynamic equations on time scales. The concept of almost periodic functions on time scales was introduced in 2011 [66, 67]. Since then, various extensions of this concept, including almost automorphic functions [68], pseudo almost periodic functions [69], weighted pseudo almost periodic functions [70], etc., have been studied to establish the existence of such solutions for dynamic equations on time scales in finite as well as infinite dimentional spaces, with notable applications in areas such as population dynamics and neural networks [71, 72, 73, 74, 75, 76, 77, 78]. Several analytical approaches, primarily based on time scale inequalities, operator theory, and fixed point methods, have been employed to establish the existence and uniqueness of these solutions. In addition to existence results, qualitative properties such as positivity, convergence, and stability of solutions [79, 80, 81, 82, 83, 84], as well as synchronization of systems on time scales possessing such solutions [85, 86, 87, 88, 89], have also been extensively investigated using different analytical techniques. Time scale calculus has been further advanced by incorporating stochastic perturbations into dynamic equations on time scales, thereby accounting for unavoidable and ubiquitous noise present in real-world systems. Within this stochastic framework, generalized concepts of periodicity, anti-periodicity, and Bloch-periodicity have been extended to stochastic processes on time scales. Consequently, the existence and stability of such solutions to stochastic dynamic equations on time scales have also been actively studied, with numerous applications in various applied fields [90, 91, 92, 93, 94, 95, 96, 97].

Recently, the classical notion of Bloch periodicity has been extended to describe unbounded periodic oscillations, leading to the concept of (ω, c)-periodicity, which has been formally introduced by Álvarez et al. in 2018 [14]. Discrete analogues of these functions has been subsequently developed in the form of (ω, c)-periodic sequences, commonly referred to as (N, λ)-periodic sequences; see [98, 99, 100]. Motivated by the need to capture more general recurrent behaviors, several refinements of this framework have been proposed, including asymptotically (ω, c)-periodic functions [101, 102], pseudo (ω, c)-periodic functions [103, 104], (ω, c)-pseudo almost periodicity [105], (ω, c)-pseudo almost automorphicity [105] and pseudo S-asymptotically (ω, c)-periodic functions [106, 107]. In parallel, measure-theoretic extensions have been introduced, such as measure (ω, c)-pseudo almost periodic functions [108], pseudo (S-asymptotically) (ω, c)-periodic sequences [104, 109], and measure pseudo S-asymptotically (ω, c)-periodic functions [113]. Along with these theoretical developments, increasing attention has been devoted to the study of differential and difference equations admitting solutions with such generalized periodicity. In [14], fundamental properties of (ω, c)-periodic functions were established, and the existence of solutions for abstract fractional integro-differential equations was obtained using semigroup theory and fixed point arguments. Discrete models have been investigated via Volterra and fractional difference equations in [98, 99] and convolution-type abstract difference equations in [100], where existence and uniqueness results have been obtained using operator-theoretic techniques. Fractional difference equations have been examined in [99] for (ω, c)-periodic solutions using fractional discrete operators and fixed point methods, while convolution-type abstract difference equations are investigated in [100], where the existence of such solutions is derived via discrete convolution operators and resolvent approaches. Applications to biological and population dynamics models have been explored in [101], where asymptotically (ω, c)-periodic behavior is analyzed for first-order Cauchy problems and the Lasota–Ważewska model with unbounded oscillating coefficients. Fractional Cauchy problems have been studied in [102], where the existence of (ω, c)-periodic and asymptotically (ω, c)-periodic mild solutions is established via fractional semigroup theory. Evolution equations in Banach spaces are considered in [103, 55], where pseudo and pseudo S-asymptotically (ω, c)-periodic solutions are obtained using decomposition techniques, operator semigroups, and suitable fixed point theorems. Semilinear difference equations are investigated in [109, 104], where discrete operator frameworks and fixed point principles are employed to analyze recurrent dynamics. Measure-theoretic approaches have been adopted in [108, 113] to analyze delay differential equations and Lasota–Ważewska-type models for the existence of generalized (ω, c)-periodic solutions, using measures of noncompactness and operator-theoretic methods to handle systems with delays and noncompact solution operators. Moreover, increasing attention has been devoted to fractional-order models with delays within this generalized periodic framework. In [110], pseudo S-asymptotically (ω, c)-periodic solutions are investigated for fractional differential equations with delay terms, yielding existence and uniqueness results. Delay differential equations involving mixed delays are studied in [111], where the existence of pseudo S-asymptotically (ω, c)-periodic mild solutions is established using semigroup theory and fixed point methods. Furthermore, neutral fractional models with delay terms are considered in [112], where the existence and uniqueness of pseudo S-asymptotically (ω, c)-periodic solutions are proved via fixed point techniques and differential inequalities.

Most existing works in literature have primarily focused on the existence and uniqueness of solutions within the generalized (ω, c)-periodic framework. By contrast, only a limited number of studies have addressed additional qualitative properties, such as asymptotic behavior and stability; see, for example, [102, 108, 113]. This observation highlights a clear research gap, as systematic investigations of qualitative properties for generalized (ω, c)-periodic solutions remain relatively scarce. Furthermore, the analysis of dynamical systems on time scales generally presents greater challenges than that of purely continuous or discrete systems, owing to the richer and more intricate behaviors that may arise. Although (ω, c)-periodic dynamical systems in continuous time have been extensively studied, research on (ω, c)-periodic dynamical systems on time scales is still limited. To the best of our knowledge, the study of (ω, c)-periodic solutions in time-scale dynamical systems was first initiated in [114], where the notion of (ω, c)-periodicity on time scales was introduced with results of a preliminary nature.

Building on the identified research gaps and the intrinsic significance of dynamic equations on time scales, our recent research has focused on the analysis of qualitative properties of dynamical systems exhibiting this generalized periodic behavior within the unified framework of time scales. In [115], we established a Banach space framework for (ω, c)-periodic functions on time scales and proved the unique existence and exponential stability of such solutions for population models. Furthermore, in [116], existence and stability results were obtained for (ω, c)-asymptotically periodic solutions of neural networks on time scales. In [117], we investigated the synchronization of neutral-type evolution systems with weighted pseudo (ω, c)-periodic solutions, and in [118], existence and stability of p-th mean S-asymptotically (ω, c)-periodic solutions for stochastic shunting-inhibitory cellular neural networks were investigated.

1.3 Motivation and research objectives

Periodicity is a fundamental notion in analysis and dynamical systems, and its classical interpretations are well understood in continuous, discrete, and time scale settings. Nevertheless, periodicity is not determined solely by the functional form but also depends crucially on the underlying domain on which the function is defined. A simple illustration of this dependence is provided by the function $f(t) = \sin(t)$. While this function is periodic on $\mathbb{R}$, its restriction to the set of integers $\mathbb{Z}$ is no longer periodic, since the classical period does not belong to the domain. This example highlights that periodic behavior may be lost when transitioning between continuous and discrete frameworks, even when the function itself remains unchanged. Since classical periodicity is a particular case of (ω, c)-periodicity, this domain dependence is naturally inherited by this broader class of functions. This necessitates the analysis of the properties of such functions in different domains. Traditionally, these properties are analyzed separately within the theories of differential equations and difference equations. Such a separation often leads to parallel results obtained through different techniques and obscures the intrinsic connections between continuous and discrete periodic behaviors. In the existing literature, most studies on (ω, c)-periodicity have been carried out in the continuous setting, while only a limited number of works address the discrete case. In contrast, classical periodicity and its well-known generalizations, such as almost periodicity, asymptotic periodicity, pseudo periodicity, and automorphicity, have been extensively studied within the framework of time scales, providing unified treatments that encompass both continuous and discrete domains. Despite these advances, a systematic extension of (ω, c)-periodicity to time scales remains largely unexplored. Motivated by this gap, the primary objective of this thesis is to develop a comprehensive and unified theory of (ω, c)-periodicity and its extensions within the framework of time scales. This approach not only recovers existing continuous and discrete results as special cases but also provides a systematic methodology for analyzing these phenomena on hybrid domains.

There are many phenomena in the real world that often, or under certain conditions, exhibit (ω, c)-periodic behaviour. In population dynamics, such behavior naturally arises in models that combine seasonal environmental effects with long-term growth or decline. Periodic factors such as food availability, temperature variation, and breeding cycles often induce oscillatory population patterns, while processes such as migration, harvesting, climate change, or pest-control interventions may systematically alter population levels over successive seasons. In particular, insect populations exposed to regular pest-control measures may experience recurring reductions, followed by partial recovery due to evolutionary adaptation or behavioral resistance, so that a similar temporal pattern is repeated while the population size is scaled by a fixed proportion after each cycle. In these situations, populations do not return to identical levels after each season, and (ω, c)-periodicity provides a more realistic modeling framework than classical periodicity by capturing the coexistence of periodic forcing and non-conservative population trends commonly observed in ecological systems. Here, (ω, c)-periodicity should be interpreted as a structural symmetry of the governing equations or dominant solution modes, rather than as an exact empirical law. In models describing the dynamics of red blood cell (RBC) concentrations, (ω, c)-periodicity can be used as a mathematical abstraction rather than a description of exact physiological behavior. Although RBC counts are typically regulated around homeostasis, recurring influences such as periodic medical treatments, chronic blood loss, or cyclic hypoxic exposure may introduce approximately periodic forcing with proportional changes in cell levels. In such cases, RBC dynamics may repeat a similar temporal pattern while exhibiting gradual amplification or attenuation across cycles. The concept of (ω, c)-periodicity provides a biologically reasonable and mathematically consistent framework for modeling these cyclic effects without neglecting regulatory feedback mechanisms. Similarly, in neural network models, (ω, c)-periodic solutions can be useful for describing oscillatory behaviors under adaptive or evolving conditions. Neural systems are known to respond to periodic or quasi-periodic stimuli, such as rhythmic sensory inputs or oscillatory control signals. However, mechanisms such as synaptic plasticity, learning, and neuromodulation may cause the strength of neural responses to change gradually over time. As a result, activity patterns may recur in a similar temporal form while exhibiting systematic amplification or attenuation across successive cycles. In this sense, (ω, c)-periodicity serves as a mathematically convenient and rigorous framework for modeling oscillatory neural dynamics with slowly varying amplitude, rather than as a claim of exact periodic behavior in biological systems.

The dynamics described above may evolve on hybrid temporal domains, where continuous-time evolution is coupled with discrete, recurrent interventions or adaptations, making purely continuous or purely discrete modeling frameworks inadequate. In addition, random environmental fluctuations, physiological variability, and synaptic noise play a significant role in shaping system behavior, making stochastic effects essential for realistic modeling and analysis. These observations provide the motivation for the research carried out in this thesis, in which we consider the following five major problems.

P-1: Existence of non-negative and stable (ω, c)-periodic solution to population models on time scales.

We investigate the conditions for existence of nonnegative and stable (ω, c)-periodic solutions for the following classes of population models on time scales, which incorporate time-dependent variable delays and nonlinear feedback mechanisms:

$$x^{\Delta}(t) = -d(t)x(t) + \sum_{k=1}^{m} F_k(t, x(t - \tau_k(t))) + H(t, x(t)),\ t \in \mathbb{T}_\omega,$$

and

$$x^{\Delta}(t) = -d(t)x^{\sigma}(t) + \sum_{k=1}^{m} F_k(t, x(t - \tau_k(t))) + H(t, x(t)),\ t \in \mathbb{T}_\omega.$$

Furthermore we take particular forms of birth function that arises in the Nicholson and Lasota–Ważewska type population models and carried out a detailed analytical investigation.

P-2: Existence and stability of (ω, c)-periodic mild solutions for nonautonomous abstract dynamic equation on time scales.

We analyze existence and stability of (ω, c)-periodic mild solutions for a class of linear and semilinear abstract dynamic equation , given as

$$x^{\Delta}(t) = \mathcal{A}x(t) + f(t), \quad t \geq t_0,\ t \in \mathbb{T}_{\omega},$$

and

$$x^{\Delta}(t) = \mathcal{A}x(t) + f(t, x(t)), \quad t \geq t_0,\ t \in \mathbb{T}_{\omega},$$

where $\mathcal{A} \in \mathscr{L}(\mathbb{B})$, space of linear operators on abstract Banach space $\mathbb{B}$. We analyze two distinct cases: when $\mathcal{A}$ generates a C_0-semigrroup and when $\mathcal{A}$ generates an evolution familly of operators.

P-3: (ω, c)-asymptotically periodic oscillation in delayed cellular neural networks on time scales.

We establish results on the existence and stability of (ω, c)-asymptotically periodic solutions for a system of delayed cellular neural networks (DCNNs) incorporating leakage and multiple mixed time-varying delays, given by

$$\begin{aligned} x_i^{\Delta}(t) = -a_i(t)x_i(t - \eta_i(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} e_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) \\ + \sum_{j=1}^{n} d_{ij}(t) \int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))\Delta s + U_i(t), \quad t \in \mathbb{T}_{\omega}^{0+}. \end{aligned}$$

P-4: Exponential synchronization of weighted pseudo (ω, c)-periodic solutions for neutral-type impulsive evolution systems on time scales.

We examine sufficient criteria for the existence of unique weighted pseudo (ω, c)-periodic solution for a neutral-type evolution system subject to instantaneous impulses, given by

$$\begin{aligned} (x(t) + \mathtt{b}(t)x(t{-}\gamma(t)))^{\Delta} &= -\mathtt{a}(t)x(t) + f(t, x(t{-}\tau(t))) + \mathtt{e}(t), \quad t \geq \mathtt{t}_0,\ t \neq t_j,\ t \in \mathbb{T}_{\omega}, \\ x(t_j{}^{-}) - x(t_j{}^{+}) &= I_j(x(t_j)), \quad t = t_j,\ j = 1, 2, \ldots, q, \end{aligned}$$

and establish results guaranteeing the synchronization between the system and its corresponding response system.

P-5: S-asymptotically (ω, c)-periodic solutions for shunting inhibitory stochastic delayed cellular neural networks on time scales.

We establish results on the existence, uniqueness and stability (in the p-th mean sense) of S-asymptotically (ω, c)-periodic solution or the following class of stochastic shunting inhibitory cellular neural networks driven by the Wiener process, given by

$$\begin{aligned}\Delta x_{ij}(t) \quad = \quad & \Bigg[- a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t-\eta_{kl}(t)))x_{ij}(t) \\ & - \sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t)\int_0^\infty \mathcal{K}_{ij}(u)g(x_{kl}(t-u))\Delta u x_{ij}(t) + L_{ij}(t)\Bigg]\Delta t \\ & + \sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(t)\delta_{ij}(t, x_{ij}(t))\Delta w_{ij}(t), \quad t \geq 0,\ t \in \mathbb{T}_\omega.\end{aligned}$$

1.4 Mathematical preliminaries

This section presents some basic definitions and results required for the analysis in the subsequent chapters. We generally state these results without proof, and appropriate references are provided for completeness.

1.4.1 Time scale calculus

This subsection is devoted to basic concepts and lemmas in time scales calculus. We review some fundamental notions and auxiliary results from the time scale calculus that will be used throughout this paper. A detailed analysis can be found in [16, 17].

Definition 1.4.1 (See [16]). *A time scale $\mathbb{T}$ is an arbitrary non empty closed subset of set of real numbers $\mathbb{R}$.*

Example 1.4.2. *The following are some examples of time scales:*

- *Set of real numbers $\mathbb{R}$, Set of integers $\mathbb{Z}$, Set of natural numbers $\mathbb{N}$.*
- *Closed inervals.*
- *Set of harmonic numbers, Cantor set.*
- $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \{0, 2\} \bigcup \{2 - \frac{1}{n} : n \in \mathbb{N}\}$.
- $\mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$.
- $\{q^n : n \in \mathbb{N},\ 1 < q \in \mathbb{Q}\} \cup \{0\}$,
- $\mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, *etc.*

Definition 1.4.3. *A closed interval on a time scale $\mathbb{T}$ is defined by*

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Similarly, the intervals $(a, b)_{\mathbb{T}}$, $[a, b)_{\mathbb{T}}$, and $(a, b]_{\mathbb{T}}$ are defined in the usual way.

Definition 1.4.4 (See [16])**.** *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is define by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

with the assumptions $\sup \mathbb{T} = \inf \emptyset$ *and* $\inf \mathbb{T} = \sup \emptyset$.

Note that, by definition, for each $t \in \mathbb{T}$, both $\sigma(t)$ and $\rho(t)$ belong to $\mathbb{T}$, since $\mathbb{T}$ is closed subset of $\mathbb{R}$. In particular, $\sigma(t)$ denotes the point in $\mathbb{T}$ mmediately following t, while $\rho(t)$ denotes the point in $\mathbb{T}$ immediately preceding t with respect to the natural ordering.

Definition 1.4.5 (Se [16])**.** *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) := \sigma(t) - t, \quad \forall\, t \in \mathbb{T}.$$

Example 1.4.6. *Operators in different time scales:*

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}}$	qt	t/q	$(q-1)t$
$\mathbb{N}_0^2$	$2\sqrt{t}+1$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$

Definition 1.4.7 (See [16])**.** *A point $t \in \mathbb{T}$ is said to be*

- *right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$;*
- *right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$;*
- *isolated if $\rho(t) < t < \sigma(t)$;*
- *dense if $\rho(t) = t = \sigma(t)$.*

Example 1.4.8. .

- *In* $\mathbb{R}$*, every point is dense point.*
- *In* $\mathbb{Z}, h\mathbb{Z}, q^{\mathbb{N}}, 2^{\mathbb{N}}$*, every point is isolated point.*

We seek the largest possible subset of $\mathbb{T}$ whose elements admit neighborhoods in $\mathbb{T}$. To this end, we define the set $\mathbb{T}^{\kappa}$ derived from $\mathbb{T}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

For example, if $\mathbb{T} = \{1, 3, 5, 7\}$, then $\mathbb{T}^{\kappa} = \{1, 3, 5\}$; and if $\mathbb{T} = \mathbb{Z}$, then $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 1.4.9 (See [16]). *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function and let* $t \in \mathbb{T}^{k}$*. We say that* f *is* Δ*-differential at* t *if there exists a real number* $f^{\Delta}(t)$ *(provided it exists) such that for each* $\varepsilon > 0$ *there is a neighborhood* $\mathcal{N} \subset \mathbb{T}$ *of* t *satisfying*

$$\big|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big| \leq \varepsilon|\sigma(t) - s|, \ \forall \ s \in \mathcal{N}.$$

In this case, $f^{\Delta}(t)$ *is called the* Δ*-derivative of* f *at* t*.*

Remark 1.4.10. *If* t *is right-dense point, the above definition reduces to the classical derivative. If* t *is right-scattered point, then* $f^{\Delta}(t) = \dfrac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$.

Remark 1.4.11. *If* f *is differentiable at* t*, then* $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

Example 1.4.12. *The* Δ*-derivatives of different functions on a general time scale* $\mathbb{T}$*:*

$f(t), \ t \in \mathbb{T}$	$f^{\Delta}(t), \ t \in \mathbb{T}^{\kappa}$
t^2	$t + \sigma(t)$
$\sqrt{t}$ *for* $t \geq 0$	$\frac{1}{\sqrt{t}+\sqrt{\sigma(t)}}$
$\frac{1}{t}$ *for* $t \neq 0$	$-\frac{1}{t\sigma(t)}$
For $a > 0, \ a^t$	$\frac{(a^{\mu(t)}-1)}{\mu(t)}a^t$
e^{at}	$\frac{(e^{a\mu(t)}-1)}{\mu(t)}e^{at}$
$\sin(t)$	$\dfrac{sin(\mu(t))cos(t) + \sin(t)(\cos(\mu(t)) - 1)}{\mu(t)}$
$sinh(at)$	$\dfrac{(e^{a(t+\mu(t))} - e^{at}) - (e^{-a(t+\mu(t))} - e^{-at})}{2\mu(t)}$
$\ln t^n$	$n\ln\left(1 + \frac{\mu(t)}{t}\right)\Big/\mu(t)$

Example 1.4.13. *The Δ-derivative of $f(t) = t^2$ on different time scales:*

$\mathbb{T}$	$f^{\Delta}(t)$
$\mathbb{R}$	$2t$
$h\mathbb{Z}$	$2t+h$
$\{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}$	$\frac{2t-t^2}{1-t}$
$\{q^n : n \in \mathbb{Z}, q > 1\} \bigcup \{0\}$	$(q+1)t$
$\bigcup_{k=0}^{\infty} [2k, 2k+1]$	$\begin{cases} 2t, & t \in \bigcup_{k=0}^{\infty} [2k, 2k+1), \\ 2t+1, & t \in \bigcup_{k=0}^{\infty} \{2k+1\} \end{cases}$

Example 1.4.14. *The Δ-derivative of $f(t) = a^t,\ a > 0$ on different time scales:*

$\mathbb{T}$	$f^{\Delta}(t)$
$\mathbb{R}$	$a^t \log(a)$
$h\mathbb{Z}$	$a^t \frac{(a^h-1)}{h}$
$\{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}$	$\left(\frac{a^{\frac{t^2}{(1-t)}} - 1}{t^2} \right) a^t (1-t)$
$\{q^n : n \in \mathbb{Z}, q > 1\} \bigcup \{0\}$	$a^t \frac{(a^{(q-1)t}-1)}{(q-1)t}$
$\bigcup_{k=0}^{\infty} [2k, 2k+1]$	$\begin{cases} a^t \log(a), & t \in \bigcup_{k=0}^{\infty} [2k, 2k+1), \\ a^t(a-1), & t \in \bigcup_{k=0}^{\infty} \{2k+1\} \end{cases}$

Theorem 1.4.15 (See [16])**.** *Let f and g are Δ-differentiable at $t \in \mathbb{T}^{\kappa}$, then the following hold:*

1. *The sum and difference $f \pm g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at t with*

$$(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t).$$

2. *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

4. If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Definition 1.4.16 (See [16]). *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists (finitely) at left- dense points in $\mathbb{T}$. We denote the collection of such functions by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.*

Theorem 1.4.17. *Assume $f : \mathbb{T} \to \mathbb{R}$.*

1. *If f is differentiable, then f is continuous.*
2. *If f is continuous, then f is rd-continuous.*
3. *The jump operator σ is rd-continuous.*

Definition 1.4.18 (See[16]). *Let f be an rd-continuous function. If $F^{\Delta}(t) = f(t)$, then delta integral is defined by*

$$\int_s^r f(t)\Delta t = F(r) - F(s), \quad r, s \in \mathbb{T}.$$

Lemma 1.4.19 (See [17]). *If f is rd-continuous, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

Lemma 1.4.20 (See [17]). *If f and f^{Δ} are continuous, then for $t, s, a \in \mathbb{T}$ we have*

$$\left(\int_a^t f(t,s)\Delta s\right)^{\Delta} = f(\sigma(t), t) + \int_a^t f^{\Delta_t}(t,s)\Delta s.$$

Example 1.4.21. *The Δ-integral of $f(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ on different time scales:*

$\mathbb{T}$	$\int_a^b f(t)\Delta t \ \ (a < b,\ a, b \in \mathbb{R})$
$\mathbb{R}$	$\int_a^b f(t)dt$ *(Riemann integral from calculus)*
$[a, b]$ *consists of only isolated points*	$\sum_{t\in[a,b)} \mu(t)f(t)$
$h\mathbb{Z} = \{hk : k \in \mathbb{Z},\ h > 0\}$	$\sum_{k=a/h}^{b/h-1} f(kh)h$
$\mathbb{Z}$	$\sum_{t=a}^{b-1} f(t)$

Definition 1.4.22 (See [16]). *A function* $p : \mathbb{T} \to \mathbb{R}$ *is said to be regressive if*

$$1 + \mu(t)p(t) \neq 0, \ \forall \ t \in \mathbb{T}.$$

The set of all regressive and rd-continuous function is denoted by $\mathcal{R}$*. If* $1 + \mu(t)p(t) > 0, \ \forall \ t \in \mathbb{T}$*, then* p *is said to be positively regressive. The set of all positively regressive, rd-continuous function is denoted by* $\mathcal{R}^+$.

Definition 1.4.23 (See [17]). *Let* $p \in \mathcal{R}$*. The generalized exponential function is defined as*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad t, s \in \mathbb{T},$$

with

$$\xi_h(z) = \begin{cases} \frac{1}{h} Log(1 + zh), & \text{if } h \neq 0 \\ z, & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 1.4.24 (See [17]). *Let* $p, q \in \mathcal{R}$ *and* $n \in \mathbb{N}$*. We define*

$$p \oplus q = p + q + \mu pq, \quad \ominus p = \frac{-p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q), \ \text{and } n \odot p := p \oplus p \oplus \ldots \oplus p \ (n \text{ times}) = \frac{(zh + 1)^n - 1}{h}.$$

Theorem 1.4.25 (See [17]). *Assume that* $p, q \in \mathcal{R}$*, then for* $s, t \in \mathbb{T}$

1. $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p)e_p(t, s)$;
3. $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
4. $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
5. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
6. $(e_{\ominus p}(t, s))^\Delta = -p(t)e_{\ominus p}(\sigma(t), s)$.

Theorem 1.4.26 (See [17]). *Let* $a, b, c \in \mathbb{T}$ *and* $p \in \mathcal{R}$*, then*

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Lemma 1.4.27 (See [16]). *If* $p \in \mathcal{R}^+$ *and* $t_0 \in \mathbb{T}$*, then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$.

Lemma 1.4.28 (See [18]). *Let* $p \in \mathcal{R}$*. If* $p > 0$*, then* $e_{\ominus p}(t, s) \leq 1$*, for all* $t,s \in \mathbb{T}$ *where* $t \geq s$.

Lemma 1.4.29 (See [19]). *Let* $\alpha > 0$ *be a constant such that* $-\alpha \in \mathcal{R}^+$*. Then*

$$0 < e_{-\alpha}(t, s) \leq e^{-\alpha(t-s)} \leq e_{\ominus\alpha}(t, s) \ \text{for all } s \leq t, \ s, t \in \mathbb{T}.$$

Theorem 1.4.30 (Variation of Constants [16]). *Suppose $f, p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $p \in \mathcal{R}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then the unique solution to the initial value problem (IVP)*

$$x^{\Delta} = -p(t)x^{\sigma} + f(t), \quad x(t_0) = x_0, \tag{1.4.1}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\Delta s. \tag{1.4.2}$$

Theorem 1.4.31 (Variation of Constants [16]). *Suppose $f, p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $p \in \mathcal{R}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then the unique solution to the IVP*

$$x^{\Delta} = p(t)x + f(t), \quad x(t_0) = x_0, \tag{1.4.3}$$

is given by

$$x(t) = e_p(t, t_0)x_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s. \tag{1.4.4}$$

Definition 1.4.32 (See [18]). *A time scale $\mathbb{T}$ is said to be invariant under translations if*

$$\Pi_{\mathbb{T}} = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T} \text{ for all } t \in \mathbb{T}\} \neq \{0\}.$$

Example 1.4.33. a) *The time scales $\mathbb{Z}$ and $\mathbb{R}$ are invariant under translations.*

b) $\mathbb{T} = h\mathbb{Z},\ h \in \mathbb{N}$; $\mathbb{T} = \frac{1}{n}\mathbb{Z},\ n \in \mathbb{N}$; $\mathbb{P}_{a,b} = \bigcup_{k=-\infty}^{\infty} [k(a+b), k(a+b)+b],\ a, b > 0$ *are time scales invariant under translations.*

c) *The time scales $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\},\ q > 1$; $\mathbb{T} = [a, b],\ a, b \in \mathbb{R}$; $\mathbb{T} = {\mathbb{N}_0}^2$; and $\mathbb{T} = 2^{\mathbb{N}}$ are not invariant under translations.*

Remark 1.4.34. *If the time scale $\mathbb{T}$ is invariant under translations, then the graininess function $\mu(\cdot)$ will be bounded. Furthermore, translation invariance implies that $\mathbb{T}$ is unbounded in both directions, that is, $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = \infty$*

Definition 1.4.35. *A time scale $\mathbb{T}$ that is invariant under translations is called ω-periodic if $\omega \in \Pi_{\mathbb{T}}^{+} := \{\tau \in \Pi_{\mathbb{T}} : \tau > 0\}$ and $t + \omega \in \mathbb{T}$ for all $t \in \mathbb{T}$. We denote such a time scale by $\mathbb{T}_{\omega}$.*

1.4.2 Semigroup theory, evolution system, and general dichotomy on time scales

We begin with some results regarding the semigroup of linear operators on time scales.

Definition 1.4.36 (See [119]). *A time scale $\mathbb{T}$ is called a semigroup time scale (STS) if, for all $t_1, t_2 \in \mathbb{T}$, $t_1 - t_2 \in \mathbb{T}$.*

Definition 1.4.37 (See [120]). *Let $\mathbb{T}$ be a STS containing zero and $\mathscr{L}(\mathbb{B})$ is the space of all bounded linear operators on a Banach space $\mathbb{B}$. A one parameter family $\mathscr{F} = \{\mathscr{F}(t) : t \in \mathbb{T}\} \subset \mathscr{L}(\mathbb{B})$ is said to be a* C_0*-semigroup (strongly continuous semigroups) if*

(i) $\mathscr{F}(t+s) = \mathscr{F}(t)\mathscr{F}(s)$ *for all* $t, s \in \mathbb{T}$.

(ii) $\mathscr{F}(0) = \mathscr{I}$, *where* $\mathscr{I}$ *is the identity operator on* $\mathbb{B}$.

(iii) $\lim\limits_{t\to 0^+} \mathscr{F}(t)x = x$ *for each* $x \in \mathbb{B}$.

Definition 1.4.38. *If* $\lim\limits_{t\to 0^+} \|\mathscr{F}(t) - \mathscr{I}\| = 0$, *then* $\mathscr{F}$ *is called a uniformly continuous semigroup on time scales.*

Definition 1.4.39 (See [120]). *A linear operator $\mathcal{A}$ is said to be the generator of a* C_0*-semigroup $\mathscr{F}$ if*

$$\mathcal{A}x = \lim_{s\to 0^+,\ s\in\mathbb{T}} \frac{\mathscr{F}(\mu(t))x - \mathscr{F}(s)x}{\mu(t) - s}, \quad x \in \mathfrak{D}(\mathcal{A}),$$

where $\mathfrak{D}(\mathcal{A})$ is the set of all $x \in \mathbb{B}$ for which the above limit exists uniformly in $t \in \mathbb{T}$.

Remark 1.4.40. *When $\mathbb{T} = \mathbb{R}_{\geq 0}$, we have $\mu(t) = 0$ for all $t \in \mathbb{T}$. Consequently,*

$$\mathcal{A}x = \lim_{s\to 0^+} \frac{\mathscr{F}(s)x - x}{s},$$

which coincides with the classical definition of the generator of a C_0*-semigroup.*

Theorem 1.4.41 (See [119]). *Let $\mathscr{F}$ be a* C_0*-semigroup generated by the operator $\mathcal{A}$. Then, for any $x \in \mathfrak{D}(\mathcal{A})$ and $t \in \mathbb{T}$, the system*

$$x^{\Delta}(t) = \mathcal{A}x(t) \quad \text{with} \quad x(0) = x_0 \in \mathfrak{D}(\mathcal{A}),$$

has a unique solution given by $x(t) = \mathscr{F}(t)x_0$.

Definition 1.4.42 (See [121]). *A* C_0*-semigroup $\mathscr{F}$ is said to be exponentially stable if there exist constants $K \geq 1$ and $\alpha > 0$ such that*

$$\|\mathscr{F}(t-s)\|_{\mathbb{B}} \leq K e_{\ominus\alpha}(t,s) \quad \text{for all} \quad t, s \in \mathbb{T}, \quad t > s.$$

Definition 1.4.43 (See [121]). *Let $\mathcal{A}$ be the generator of a* C_0*-semigroup $\mathscr{F}$. We say that the function $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{B})$ is a mild solution of*

$$x^{\Delta}(t) = \mathcal{A}x(t) + f(t, x(t)), \quad t \geq t_0, \quad t, t_0 \in \mathbb{T}, \tag{1.4.5}$$

if it satisfies

$$x(t) = \mathscr{F}(t - t_0)x(t_0) + \int_{t_0}^{t} \mathscr{F}(t - \sigma(s)) f(s, x(s)) \Delta s. \tag{1.4.6}$$

Remark 1.4.44. *Every solution of* (1.4.5) *satisfies the integral equation* (1.4.6)*, but the converse is not always true, because a solution of* (1.4.6) *is not necessarily* Δ*-differentiable.*

Definition 1.4.45 (See [122])**.** *A two-parameter family* $\{\mathscr{F}(t,s)\}_{t\geq s;s,t\in\mathbb{T}} \subset \mathscr{L}(\mathbb{B})$ *defined on* $\mathbb{T}\times\mathbb{T}$ *is said to be an evolution family of operator (EFO), if*

(i) $\mathscr{F}(t,r)\mathscr{F}(r,s) = \mathscr{F}(t,s)$ *and* $\mathscr{F}(t,t) = \mathscr{I}$ *for all* $t,r,s \in \mathbb{T}$ *with* $t \geq r \geq s$*.*

(ii) For $t \geq s$*, the map* $(t,s) \mapsto \mathscr{F}(t,s)x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{B})$ *for each* $x \in \mathbb{B}$*.*

Definition 1.4.46 (See [122])**.** *We call an EFO* $\{\mathscr{F}(t,s)\}$ *exponentially stable, if there exist some constants* $K \geq 1$ *and* $\alpha > 0$ *such that*

$$\|\mathscr{F}(t,s)\|_{\mathbb{B}} \leq K e_{\ominus\alpha}(t,s) \quad \text{for all} \quad s,t \in \mathbb{T},\ t > s.$$

Theorem 1.4.47. *Let* $\mathscr{F}$ *be an EFO generated by* $\mathcal{A}$*. Then, a function* $x \in C_{rd}(\mathbb{T},\mathbb{B})$ *is called a mild solution of* (1.4.5) *on* $\mathbb{T}$*, if it satisfies*

$$x(t) = \mathscr{F}(t,t_0)x(t_0) + \int_{t_0}^{t} \mathscr{F}(t,\sigma(s))f(s,x(s))\Delta s, \tag{1.4.7}$$

for all $t \geq t_0$*,* $t,s \in \mathbb{T}$*.*

Next, we recall the concept of general dichotomy on time scales.

Definition 1.4.48 (See [123])**.** *A system*

$$x^{\Delta}(t) = \mathcal{A}x(t), \quad \mathcal{A} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathscr{L}(\mathbb{B})),$$

is said to have a general exponential dichotomy on $\mathbb{T}$ *if projections* $\mathscr{P}(\cdot)$ *can be defined along with constant* $K > 0$*,* $\ell_1,\ell_2 : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^+$ *and bounded functions* $\eta,\zeta \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$ *satisfying* $\underline{\eta} = \inf_{t\in\mathbb{T}} \eta(t) > 0$ *and* $\underline{\zeta} = \inf_{t\in\mathbb{T}} \zeta(t) > 0$*, so that*

$$\mathscr{P}(t)\mathscr{F}(t,s) = \mathscr{F}(t,s)\mathscr{P}(s),$$

and, for $s > k > t_0$

$$\begin{aligned} \|\mathscr{F}(t,s)\mathscr{P}(s)\|_{\mathbb{B}} &\leq K\ell_1(s,k)e_{\ominus\eta}(t,s), \quad s \leq t, \\ \|\mathscr{F}(t,s)\mathscr{Q}(s)\|_{\mathbb{B}} &\leq K\ell_2(s,k)e_{\ominus\zeta}(t,s), \quad t < s \end{aligned}$$

hold, where $\mathscr{Q} = \mathscr{I} - \mathscr{P}$ *denotes the projections complementary to* $\mathscr{P}$*, and* $\mathscr{F}(t,s)$ *is the evolution operator that satisfies* $\mathscr{F}(t,s)x(s) = x(t)$ *for* $t,s \in \mathbb{T}$ *and any solution* x *of the given system.*

Examples of general exponential dichotomies on different time scales can be found in [123].

1.4.3 Δ-measure on time scales

Here, we recall the concept of the Δ-measure, which extends the classical Lebesgue measure to arbitrary time scales, ensuring that standard results like the Dominated Convergence Theorem remain valid.

Definition 1.4.49. *On a set $\mathcal{A} = \{[\alpha,\beta)_{\mathbb{T}} = [\alpha,\beta) \cap \mathbb{T} : \alpha,\beta \in \mathbb{T}, \alpha \leq \beta\}$, we define a countably additive measure $\boldsymbol{m}$ that assigns to every interval in $\mathcal{A}$ its length. The outer measure $\boldsymbol{m}^*$, induced on the power set of $\mathbb{T}$ using $\boldsymbol{m}$, is defined as*

$$\boldsymbol{m}^*(\mathcal{B}) := \begin{cases} \inf\limits_{\mathcal{E}} \sum\limits_{i \in I_{\mathcal{E}}} \boldsymbol{m}([\alpha_i,\beta_i)_{\mathbb{T}}) \in \mathbb{R}^+, & \beta \in \mathbb{T}\backslash\mathcal{B}, \\ \infty, & \beta \in \mathcal{B}, \end{cases}$$

with

$$\mathcal{E} = \left\{ \{[\alpha_i,\beta_i)_{\mathbb{T}} \in \mathcal{B}\}_{i \in I_{\mathcal{E}}} : I_{\mathcal{E}} \subset \mathbb{N}, \mathcal{B} \subset \bigcup_{i \in I_{\mathcal{E}}} [\alpha_i,\beta_i)_{\mathbb{T}} \right\}.$$

A set $\mathcal{C} \subset \mathbb{T}$ is said to be Δ-measurable if for every $\mathcal{B} \subset \mathbb{T}$,

$$\boldsymbol{m}^*(\mathcal{B}) = \boldsymbol{m}^*(\mathcal{B} \cap \mathcal{C}) + \boldsymbol{m}^*(\mathcal{B} \cap (\mathbb{T}\backslash\mathcal{C})),$$

where $\boldsymbol{m}^*$ denotes the outer measure induced by assigning to each interval $[\alpha,\beta)_{\mathbb{T}} = [\alpha,\beta) \cap \mathbb{T}$ its length. The Lebesgue Δ-measure ($\boldsymbol{\mu}_\Delta$) is the restriction of $\boldsymbol{m}^*$ to the family of Δ-measurable sets.

If $\mathscr{M}(\boldsymbol{m}^*) := \{\mathcal{B} \subset \mathbb{T} : \mathcal{B} \text{ is } \Delta - \text{measurable}\}$, then the Lebesgue Δ-measure ($\boldsymbol{\mu}_\Delta$) is the restriction of $\boldsymbol{m}^*$ to $\mathscr{M}(\boldsymbol{m}^*)$.

Definition 1.4.50. *A function $f : \mathbb{T} \to \mathbb{R}$ is Δ-measurable iff $f^{-1}([-\infty, a)) := \{t \in \mathbb{T} : f(t) < \mathtt{a}\}$ is Δ-measurable for all $a \in \mathbb{R}$.*

Lemma 1.4.51. *Let $\mathcal{B} \subset \mathbb{T}$. Then $\mathcal{B}$ is Δ-measurable iff $\mathcal{B}$ is Lebesgue measurable.*

Corollary 1.4.52. *If $\mathcal{B} \subset \mathbb{R}$ is closed in $\mathbb{R}$, then $\mathcal{B} \cap \mathbb{T}$ is Δ-measurable.*

Corollary 1.4.53. *If $f : \mathbb{T} \to \mathbb{R}$ is continuous, then f is Δ-measurable.*

Theorem 1.4.54. *Consider $\mathcal{B} \subset \mathbb{T}$ to be Δ-measurable. If $(f_k)_{k \in \mathbb{N}}$ is a sequence of Δ-measurable functions and g is Δ-integrable function over $\mathcal{B}$ such that*

(a) $\|f_k(t)\| \leq g(t)$ *on $\mathcal{B}$ for almost all $\tau \in \mathcal{B}$, and*

(b) $\lim\limits_{k \to \infty} f_k(t) = f(t)$,

then $\lim\limits_{k \to \infty} \int_{\mathcal{B}} f_k(t)\Delta t = \int_{\mathcal{B}} f(t)\Delta t$ *and f is Δ-measurable.*

1.4.4 Stochastic analysis in the time scale setting

In this subsection, we give a brief introduction of the general probability theory and stochastic process on a time scale $\mathbb{T}$.

Definition 1.4.55. *Let Ω be a given set. A σ-*algebra *on Ω is a collection $\mathcal{F}$ of subsets of Ω satisfying the following properties:*

(i) $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$.

(ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$, where $E^c = \Omega \setminus E$ denotes the complement of E in Ω.

(iii) If $\{E_k\}_{k=1}^{\infty} \subset \mathcal{F}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$.

Definition 1.4.56. *Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A function $\mathbb{P} : \mathcal{F} \to [0,1]$ is called a* probability measure *if the following conditions are satisfied:*

(i) $\mathbb{P}(\Omega) = 1$.

(ii) For any sequence $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$, $\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$.

Remark 1.4.57. *From the above definition, it follows immediately that $\mathbb{P}(\emptyset) = 0$. Moreover, $\mathbb{P}$ is countably subadditive, that is, for any sequence $\{E_k\}_{k=1}^{\infty} \subset \mathcal{F}$, $\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mathbb{P}(E_k)$.*

Definition 1.4.58. *A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a* probability space *if Ω is a nonempty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$.*

Remark 1.4.59. *A set $E \in \mathcal{F}$ is said to be an* event*, the elements $\omega \in \Omega$ are called* sample points*, and $\mathbb{P}(E)$ denotes the probability of the event E.*

Remark 1.4.60. *A property is said to hold* almost surely *(a.s.) if it holds except on an event of probability zero.*

Definition 1.4.61. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A mapping $X : \Omega \to \mathbb{R}^n$ is called an n-dimensional random variable if it is $\mathcal{F}$-measurable, that is,*

$$X^{-1}(B) \in \mathcal{F} \quad \textit{for every } B \in \mathcal{B}(\mathbb{R}^n),$$

where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra on $\mathbb{R}^n$.

Definition 1.4.62. *A stochastic process on $\mathbb{T}$ is a family*

$$\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+} := \mathbb{T} \cap [0, \infty)\}$$

of $\mathbb{R}^n$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For each $\zeta \in \Omega$, the mapping

$$t \mapsto X(t, \zeta), \quad t \in \mathbb{T}^{0+},$$

is called a sample path (or simply a path) of the stochastic process $\mathcal{X}$.

Definition 1.4.63. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration on the time scale $\mathbb{T}$ is a nondecreasing family $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ of σ-algebras on Ω such that*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \quad \text{for all } s, t \in \mathbb{T}^{0+} \text{ with } s \leq t.$$

Definition 1.4.64 (Filtration satisfying usual conditions)**.** *A filtration $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ is said to satisfy the* usual conditions *if:*

(i) $\mathcal{F}_t$ is increasing and rd-continuous on $\mathbb{T}^{0+}$.

(ii) $\mathcal{F}_{t_0}$ contains all $\mathbb{P}$-null sets, where $t_0 = \inf \mathbb{T}^{0+}$.

Definition 1.4.65. *Let $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is said to be adapted to the filtration if for every $t \in \mathbb{T}^{0+}$, the random variable $X(t)$ is $\mathcal{F}_t$-measurable.*

Definition 1.4.66. *Let $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is said to be* progressively measurable *with respect to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ if, for every $t \in \mathbb{T}^{0+}$, the mapping*

$$(s, \zeta) \mapsto X(s, \zeta), \quad s \in [0, t]_{\mathbb{T}},$$

is measurable with respect to the product σ-algebra $\mathcal{B}([0, t]_{\mathbb{T}}) \otimes \mathcal{F}_t$.

Remark 1.4.67. *Every progressively measurable stochastic process is adapted, but the converse is not true in general. However, if an adapted stochastic process on the time scale $\mathbb{T}$ has rd-continuous sample paths, then it is progressively measurable. Progressive measurability is the natural measurability assumption required for the definition of stochastic integrals on time scales.*

Definition 1.4.68. *A Wiener process (or Brownian motion) indexed by a time scale $\mathbb{T}$ is an adapted stochastic process $W = \{W(t) : t \in \mathbb{T}^{0+}\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$ and satisfying the following properties:*

(i) $W(t_0) = 0$ almost surely, where $t_0 = \inf \mathbb{T}^{0+}$;

(ii) If $t_0 \leq s < t$ with $s, t \in \mathbb{T}^{0+}$, then the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance $t - s$.

(iii) The sample paths $t \mapsto W(t, \zeta)$ are rd-continuous on $\mathbb{T}^{0+}$ for almost all $\zeta \in \Omega$.

Remark 1.4.69. *A Wiener process indexed by a time scale $\mathbb{T}$ can be regarded as the restriction of a standard Wiener process defined on $\mathbb{R}$ to the time scale $\mathbb{T}$. All properties, including independent Gaussian increments and rd-continuous paths, are preserved under this restriction.*

Definition 1.4.70. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}^{0+}}$. A stochastic process $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is said to belong to the class $\mathcal{L}^p(\mathcal{I}_\mathbb{T})$, $p \geq 1$, if the following conditions hold:*

(i) $X(t)$ is adapted, that is, for each $t \in \mathcal{I}_\mathbb{T}$, the mapping $X(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable;

(ii) $X(t)$ is p-integrable on $\mathcal{I}_\mathbb{T}$ in the sense that

$$\mathbb{P}\left(\int_{\mathcal{I}_\mathbb{T}} |X(t,\zeta)|^p \, \Delta t < \infty\right) = 1,$$

where $\mathcal{I}_\mathbb{T} \subseteq \mathbb{T}^{0+}$ is a given measurable subset of the time scale $\mathbb{T}$.

Definition 1.4.71. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ be an integrable random variable, that is, $\mathbb{E}|X| < \infty$. The* expectation *(or mean) of X is defined by*

$$\mathbb{E}[X] := \int_\Omega X(\zeta) \, d\mathbb{P}(\zeta).$$

Remark 1.4.72. *If $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is a stochastic process such that $X(t)$ is integrable for each $t \in \mathbb{T}^{0+}$, then the function*

$$t \mapsto \mathbb{E}[X(t)]$$

is called the mean function *of the process.*

Definition 1.4.73. *Let X be a random variable such that $\mathbb{E}|X|^p < \infty$ for some $p \geq 1$. The L^p-norm of X is defined by*

$$\|X\|_{L^p(\Omega)} := \big(\mathbb{E}|X|^p\big)^{1/p}.$$

Remark 1.4.74. *If $X(t, \omega)$ is a measurable stochastic process such that $\int_{\mathcal{I}_\mathbb{T}} \mathbb{E}|X(t)| \, \Delta t < \infty$, then Fubini's theorem allows the interchange of expectation and the delta integral:*

$$\mathbb{E}\left[\int_{\mathcal{I}_\mathbb{T}} X(t) \, \Delta t\right] = \int_{\mathcal{I}_\mathbb{T}} \mathbb{E}[X(t)] \, \Delta t.$$

Definition 1.4.75. *A stochastic process $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is said to be* stochastically continuous *(or continuous in the p-th mean) if, for every $s \in \mathbb{T}^{0+}$,*

$$\lim_{t\to s} \mathbb{E}[|X(t) - X(s)|^p] = 0.$$

Definition 1.4.76. *A stochastic process $\mathcal{X} = \{X(t) : t \in \mathbb{T}^{0+}\}$ is said to be* stochastically bounded in the p-th moment *if there exists a constant $M > 0$ such that*

$$\mathbb{E}[|X(t)|^p] \leq M \quad \textit{for all } t \in \mathbb{T}^{0+}.$$

Remark 1.4.77. *If $\mathcal{I}_{\mathbb{T}} \subset \mathbb{T}^{0+}$ is a bounded time-scale interval and $\mathcal{X} \subset \mathcal{L}^p(\mathcal{I}_{\mathbb{T}})$ is stochastically bounded, then $\mathbb{E}|X(t)|^p$ is uniformly bounded on $\mathcal{I}_{\mathbb{T}}$.*

Lemma 1.4.78 (Hölder's inequality)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $X \in \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$ and $Y \in \mathcal{L}^q(\Omega, \mathbb{H}_{\mathbb{F}})$, then*

$$\mathbb{E}\big[\|XY\|\big] \leq \big(\mathbb{E}\|X\|^p\big)^{1/p}\big(\mathbb{E}\|Y\|^q\big)^{1/q}.$$

Theorem 1.4.79 (Tonelli's theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $f : \mathbb{T} \times \Omega \to [0, \infty]$ be jointly measurable. Then*

$$\mathbb{E}\left[\int_{\mathbb{T}} f(t, \zeta)\, \Delta t\right] = \int_{\mathbb{T}} \mathbb{E}\big[f(t, \zeta)\big]\, \Delta t,$$

where both sides may be equal to $+\infty$.

1.4.5 Some important results

In this subsection, we state some essential definitions, lemmas, and theorems which has been used in subsequent chapters.

1.4.5.1 Banach fixed point theorem

Definition 1.4.80. *Let $\mathbb{B}$ be a Banach space and let $\Upsilon : \mathbb{B} \to \mathbb{B}$ be an operator. A point $x \in \mathbb{B}$ is called a* fixed point *of Υ if $\Upsilon(x) = x$.*

Theorem 1.4.81 (Banach fixed point theorem [124]). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach space and let S be a nonempty closed subset of $\mathbb{B}$. Suppose that $\Upsilon : S \to S$ is a contraction mapping, that is, there exists a constant γ with $0 < \gamma < 1$ such that*

$$\|\Upsilon(x_1) - \Upsilon(x_2)\| \leq \gamma\|x_1 - x_2\| \textit{ for all } x_1, x_2 \in S.$$

Then Υ has a unique fixed point in S.

1.4.5.2 Some important inequalities on time scales

Lemma 1.4.82 (See [125])**.** *Assume that $a > 0$ and $-a \in \mathcal{R}^+$.*

(i) *If* $x^{\Delta}(t) \leq b - ax(t)$, *then* $\limsup_{t\to\infty} x(t) \leq \frac{b}{a}$.

(ii) *If* $x^{\Delta}(t) \geq b - ax(t)$, *then* $\liminf_{t\to\infty} x(t) \geq \frac{b}{a}$.

Lemma 1.4.83 (Halanay inequality on time scales [126]). *Let* $x(\cdot)$ *be a nonnegative function satisfying*

$$x^{\Delta}(t) \leq -a(t)x(t) + b(t) \sup_{s\in[t-\tau,t]} x(s) + c(t)x(t), \quad t > t_0$$

$$x(s) = |\varphi(s)| \text{ for all } s \in [t_0 - \tau^*, t_0]_{\mathbb{T}},$$

where $\varphi(t)$ *defined for* $t \in [-\infty, t_0]_{\mathbb{T}}$ *is bounded, rd-continuous and* $a(\cdot), b(\cdot), c(\cdot), \tau(\cdot)$ *are nonnegative, rd-continuous and bounded functions defined on* $\mathbb{T}$ *with* $\sup_{t\in\mathbb{T}}\tau(t) = \tau^*$ *and* $\sup_{t\in(-\infty,t_0]} x(t) = N > 0$. *Suppose* $a(t) - b(t) - c(t) \geq \inf_{t\in\mathbb{T}}\{a(t) - b(t) - c(t)\} > 0,\ t \in \mathbb{T}$, *Then there exists a positive constant* M *such that*

$$x(t) \leq \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}}} x(s) \right) e_{\ominus M}(t, t_0), \quad t \in (t_0, \infty)_{\mathbb{T}}.$$

Lemma 1.4.84. *Let* $u(t)$ *be an rd-continuous, non-negative function for* $t \in \mathbb{T}_0^+$, *and* α, γ *be some positive constants. If*

$$u(t) \leq \alpha e_{\ominus\beta}(t, 0) + \gamma \int_0^t e_{\ominus\beta}(t, s)u(s)\Delta s, \quad t \in \mathbb{T}_0^+ \tag{1.4.8}$$

then

$$u(t) \leq \alpha e_{\ominus(\beta\ominus\gamma)}(t, 0). \tag{1.4.9}$$

Proof. For $t \in \mathbb{T}_0^+$, define a function as $v(t) = e_{\beta}(t, 0)u(t)$. Multiplying (1.4.8) by $e_{\beta}(t, 0)$, we can have

$$v(t) \leq \alpha + \gamma \int_0^t v(s)\Delta s,$$

which by Gronwall's dynamic inequality on time scales implies that

$$v(t) \leq \alpha e_{\gamma}(t, 0).$$

Hence, (1.4.9) which can be obtained directly by multiplying both sides of the above by $e_{\ominus\beta}(t, 0)$. □

1.4.5.3 Stability of dynamic systems on time scales

Stability theory studies how solutions of dynamic systems respond to small perturbations in initial conditions or external disturbances. It provides a framework to understand whether a system maintains its qualitative behavior over time when subject to such perturbations. In particular, stability analysis concerns the behavior of a perturbed solution $\tilde{x}(t)$ relative to a reference solution $x(t)$ of the system. The primary questions addressed are:

(i) Does the perturbed solution remain close to the reference solution over time?

(ii) Does the deviation between the perturbed and reference solutions decay to zero as time evolves?

(iii) Can the rate of decay of this deviation be quantified, e.g., exponentially?

These questions lead to several standard notions of stability, including stability in the sense of Lyapunov, asymptotic stability, and exponential stability. Consider the dynamic equation on a time scale

$$x^{\Delta}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n, \tag{1.4.10}$$

with reference solution $x(t)$ and perturbed solution $\tilde{x}(t)$.

Definition 1.4.85 (Stability of a reference solution)**.** *The reference solution $x(t)$ of (1.4.10) is said to be*

(i) stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|\tilde{x}(t_0) - x(t_0)\| < \delta \implies \|\tilde{x}(t) - x(t)\| < \varepsilon, \quad \forall t \geq t_0, \quad t_0, t \in \mathbb{T}.$$

(ii) asymptotically stable *if it is stable and, in addition,*

$$\lim_{t\to\infty} \|\tilde{x}(t) - x(t)\| = 0, \quad t \in \mathbb{T}.$$

(iii) exponentially stable *if there exist constants $L > 0$ and $\lambda > 0$ such that*

$$\|\tilde{x}(t) - x(t)\| \leq L\|\tilde{x}(t_0) - x(t_0)\| e_{\ominus\lambda}(t, t_0), \quad t \geq t_0, \quad t_0, t \in \mathbb{T}.$$

1.4.5.4 Lyapunov functions on time scales

Lyapunov functions provide a systematic way to study the stability of a system without explicitly solving it. Intuitively, a Lyapunov function can be thought of as an "energy" that measures the deviation of a perturbed solution from a reference solution.

Definition 1.4.86. *A scalar function $V : \Omega \to \mathbb{R}$ is*

(i) positive definite *if $V(0) = 0$ and $V(y) > 0$ for all $y \neq 0$,*

(ii) negative definite *if $V(0) = 0$ and $V(y) < 0$ for all $y \neq 0$.*

Definition 1.4.87 (Lyapunov function for a reference solution)**.** *Let V be positive definite. For the deviation $e(t) := \tilde{x}(t) - x(t)$, define*

$$V^{\Delta_t}(e) := \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} e_i^{\Delta}(t).$$

If V^{Δ_t} is negative semidefinite, V is called a Lyapunov function for the deviation $e(t)$.

Theorem 1.4.88 (Local stability via Lyapunov function)**.** *Let* $V \in \mathrm{C}^1_{\mathrm{rd}}(\Omega)$ *be positive definite. If*

$$V^{\Delta_t}(e) \leq 0 \text{ for all } e \in \Omega,$$

then the reference solution $x(t)$ *of* (1.4.10) *is locally stable. If* $V^{\Delta_t}(e) < 0$ *for all* $e \neq 0$*, then* $x(t)$ *is locally asymptotically stable.*

Theorem 1.4.89 (Global asymptotic stability via Lyapunov function)**.** *If* $V \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{R}^{\mathrm{n}})$ *is positive definite, radially unbounded (that is,* $V(x) \to \infty$ *as* $\|x\| \to \infty$*), and*

$$V^{\Delta_t}(e) < 0, \quad \text{for all} \;\; e \neq 0,$$

then the reference solution $x(t)$ *is globally asymptotically stable.*

1.5 Thesis outlines

This thesis is organized into seven chapters, with Chapter 1 serving as an introduction. The rest of the thesis has the following structure.

In **Chapter 2**, we establish fundamental properties of (ω, c)-periodic functions on ω-periodic time scales. We also examine delayed population models for the existence of exponentially stable (ω, c)-periodic solutions. A separate analysis is provided for particular classes of population models, namely Nicholson's model, the Lasota–Ważewska model, and models incorporating a combination of both. To illustrate the theoretical results graphically, some numerical examples from various time scales are presented. The results of this chapter have been published in *Discrete and Continuous Dynamical Systems - Series S* [115].

In **Chapter 3**, we consider linear and semilinear nonautonomous abstract dynamic equations on ω-periodic time scales and establish the conditions for existence of stable (ω, c)-periodic mild solutions. To achieve this, we employ Banach fixed point theorem, along with semigroup theory and exponential dichotomy theory. We also analyze the continuous dependence of these exponentially stable (ω, c)-periodic mild solutions on the source term for the considered class of nonautonomous semilinear equations. Finally, a few finite-dimensional Banach space models, including the Hopfield neural network, the Lasota–Ważewska model, cellular neural networks, and a model from economics, serve as simple illustrations of the abstract results.

In **Chapter 4**, we introduce the concept of (ω, c)-asymptotically periodic functions on ω-periodic time scales, to include functions that do not exhibit (ω, c)-periodicity initially but eventually attain this behavior. We then investigate a class of delayed cellular neural networks on time scales for the existence of stable (ω, c)-asymptotically periodic solutions. The analysis is primarily based on the Banach fixed point theorem, combined with time-scale inequalities and composition results for (ω, c)-asymptotically periodic functions. Numerical examples and simulations are provided to illustrate and validate the theoretical results across various types of time scales. The results of this chapter have been published in *Neural Networks* [116].

In **Chapter 5**, we further generalize (ω, c)-asymptotic periodicity and introduce the concept of weighted pseudo (ω, c)-periodicity on ω-periodic time scales. We then consider a class of neutral-type evolution systems with instantaneous impulsive effects on time scales and investigate the existence of (doubly) weighted pseudo (ω, c)-periodic solutions. Using Lyapunov function techniques and by introducing a suitable control term, we also establish the global exponential synchronization between the system with weighted pseudo (ω, c)-periodic solutions and its identical response system. Numerical examples and simulations are provided to illustrate how the obtained results unify continuous and discrete analyses. The results of this chapter have been published in *Communications in Nonlinear Science and Numerical Simulation* [117].

In **Chapter 6**, we develop a theory for p-th mean S-asymptotically (ω, c)-periodic stochastic processes on ω-periodic time scales and apply it to delayed stochastic shunting-inhibitory cellular neural networks driven by a Wiener process. We establish the existence, uniqueness, and stability of p-th mean S-asymptotically (ω, c)-periodic solutions. The analysis is carried out using time scale calculus, an appropriate Banach space framework, fixed point theorem, and stochastic integral inequalities on time scales. The theoretical results are illustrated and validated through numerical simulations on different time scales. The results of this chapter have been published in *Applied Mathematics and Computation* [118].

Chapter 7 presents an overall summary of the key findings from the previous chapters and offers concluding remarks. It also outlines several directions for future research.

CHAPTER 2

Existence of Non-negative and Stable (ω, c)-Periodic Solutions to Population Models on Time Scales [1]

This chapter is devoted to the study of (ω, c)-periodic functions on time scales and their applications to dynamic systems. We first investigate several fundamental properties of this class of functions on ω-periodic time scales, establishing a rigorous theoretical foundation. As a significant application of this framework, we analyze the existence, uniqueness, and exponential stability of (ω, c)-periodic solutions for certain families of generalized dynamic equations on time scales incorporating feedback mechanisms and time-varying delays arising in population dynamics. In particular, we derive sufficient conditions for the existence of non-negative (ω, c)-periodic solutions for several well-known population models, including the Nicholson model, the Lasota–Ważewska model, and their mixed formulations on time scales.

[1] The content of this chapter is published with title "*Dynamics beyond bounds: (ω, c)-periodic functions and their impact on delayed population models on time scales*" in Discrete and Continuous Dynamical Systems- Series S, 18(9), 2586–2621 (2025). https://www.aimsciences.org/article/doi/10.3934/dcdss.2025049.

2.1 Introduction

Over the past few decades, considerable attention has been devoted to periodic and generalized periodic types of solutions of differential and difference equations due to their broad applicability in the natural and social sciences. A pivotal advancement in this direction occurred in 2018, when Alvarez et al. [14] introduced the concept of (ω, c)-periodic functions. In applied contexts, the existence of (ω, c)-periodic solutions has been investigated for several classes of fractional integro-differential equations [14, 128] and impulsive differential equations [127]. In addition, the notion of (ω, Q)-periodicity has been introduced in [129, 130], where the scalar parameter c is replaced by a linear isomorphism Q on a Banach space $\mathbb{B}$. Affine periodic solutions, corresponding to the case $Q = GL(\mathbb{R}^n)$ have also been explored in the framework of discrete dynamical systems [131]. More recently, (N, λ)-periodic solutions for certain difference equations have been studied [98, 99], extending the concept of (ω, c)-periodicity to discrete-time systems. In the context of time scales, Wang and Li [132] explored affine periodic solutions with $Q = GL_n(\mathbb{R})$, while Federson et al. [133] introduced the concept of affine-periodicity with $Q = GL_n(\mathbb{C})$. Notably, the concept of (ω, c)-periodicity can be viewed as a special case of affine-periodicity with $Q = cI$; however apart from the study in [114], the study of (ω, c)-periodicity on time scales remains relatively limited. Motivated by this observation, this chapter aims to bridge this gap by investigating previously unexplored properties of (ω, c)-periodic functions on time scales and applying them to models of species population growth.

A comprehensive and compelling approach to modeling the growth dynamics of single or multiple species requires the incorporation of seasonal environmental variations, cyclical behavioral patterns, and the significant influence of time delays. Two particularly notable models in this context are the Nicholson's blowflies model and the Lasota–Ważewska model. Nicholson's model [134, 135], deeply rooted in biological systems, has attracted substantial attention due to its realism, detailed structure, and natural discrete analogs, making it an effective representation of population dynamics under fluctuating environmental conditions. Over the past four decades, extensive research on this model and its variants has significantly advanced the understanding of biological systems, including the existence of positive solutions, periodic and oscillatory behaviors, and stability properties; comprehensive discussions can be found in [36, 93, 136, 137, 138] and the references therein. While continuous-time formulations are valuable, discrete-time models governed by difference equations are often more appropriate in situations involving large population sizes or nonoverlapping generations, which has motivated in-depth studies of discrete Nicholson's blowflies models [139, 140]. Similarly, the Lasota–Ważewska model [12] represents a fundamental and influential contribution to ecological modeling. Originally introduced to describe the survival of red blood cells in animals, it has since inspired extensive investigations into its dynamic behavior. Numerous studies have addressed the existence, characterization, and stability of periodic solutions [141, 142], as well as oscillatory behavior and global attractivity [143], with further extensions and generalizations explored in [144, 145, 146, 147]. Despite the significance of these models, relatively few studies have explored their dynamics on general time scales, leaving an important avenue for further investigation. Notable contributions on Nicholson's blowfly models formulated on time scales can be found in [148, 149, 150] and the references therein. For analyses of generalized Lasota–Ważewska models on time scales with multiple time-varying delays, the reader is referred to [151, 152, 153] and the associated literature. However, as noted earlier, for a general constant c, there are currently no results addressing the (ω, c)-periodicity of these dynamic models on time scales.

Motivated by this gap in the existing literature, the aim of this chapter is to analyze the concept of (ω, c)-periodicity on time scales and to investigate the fundamental properties of (ω, c)-periodic functions within this framework. Our study focuses on (ω, c)-periodic dynamic equations defined on ω-periodic time scales. Employing concepts from time scale calculus and related mathematical techniques, we establish sufficient conditions for the unique existence and exponential stability of (ω, c)-periodic solutions for the following classes of dynamic equations on time scales incorporating nonlinear feedback mechanisms and time-dependent delays:

$$x^{\Delta}(t) = -d(t)x(t) + \sum_{k=1}^{m} F_k(t, x(t - \tau_k(t))) + H(t, x(t)) \tag{2.1.1}$$

and

$$x^{\Delta}(t) = -d(t)x^{\sigma}(t) + \sum_{k=1}^{m} F_k(t, x(t - \tau_k(t))) + H(t, x(t)). \tag{2.1.2}$$

In (2.1.1) and (2.1.2), the functions $d(\cdot) : \mathbb{T}_{\omega}^{+} \to \mathbb{R}^{+}$ and $\tau_k(\cdot) : \mathbb{T}_{\omega}^{+} \to \Pi_{\mathbb{T}_{\omega}} \cap [0, \infty)$ are assumed to be ω-periodic on $\mathbb{T}_{\omega}$. The (ω, c)-periodic functions $F_k(\cdot, x(\cdot)) : \mathbb{T}_{\omega}^{+} \times \mathbb{B}_0^{+} \to \mathbb{B}_0^{+}$ and $H(\cdot, x(\cdot)) : \mathbb{T}_{\omega}^{+} \times \mathbb{R}_0^{+} \to \mathbb{B}_0^{+}$ are assumed to be locally Lipschitz in the second variable. Moreover, the function $d(\cdot)$ is assumed to be regressive.

Equations (2.1.1) and (2.1.2) arise naturally in modeling the population growth of a species, where $x(t)$ denotes the population density, $d(t)$ represents the instantaneous loss (death) rate, $\sum_{k=1}^{m} F_k(t)(t, x(t - \tau_k(t)))$ describes the birth process incorporating time-dependent delays $\tau_k(\cdot)$, and $H(t, x(t))$ accounts for harvesting or immigration, depending on whether its contribution to the system is negative or positive, respectively. In particular, choosing $F_k(t, x(t)) = \beta_k(t)x(t)e^{\lambda_k(t)x(t)}$, (2.1.1) reduces to Nicholson-type models on time scales. Likewise, by taking $F_k(t, x(t)) = \beta_k(t)e^{\lambda_k(t)x(t)}$, the equation yields Lasota–Ważewska-type models on the time scale. More generally, various hybrid models on time scales can be obtained by combining Nicholson and Lasota–Ważewska type nonlinearities in (2.1.1). The assumption that the birth and harvesting functions are (ω, c)-periodic allows the model to incorporate environments in which seasonal patterns recur while their overall impact on the population is gradually amplified or attenuated across successive cycles. This framework reflects populations experiencing recurring ecological conditions that do not reproduce identical outcomes each season, due to cumulative effects such as changing climate conditions, adaptive responses, or evolving management strategies. Consequently, the model is well suited to describe species whose population trajectories exhibit scaled oscillatory behavior rather than strictly periodic repetition, a phenomenon frequently observed in real ecological systems. The results established in this chapter offer a unified and systematic framework for analyzing (ω, c)-periodic dynamics in these biologically motivated systems.

The organization of this chapter is as follows. In Section 2.2, we discuss the concept of (ω, c)-periodicity on periodic time scales and present its basic properties, including a composition theorem. The main results concerning the existence of such solutions for (2.1.1) and (2.1.2) are presented in Section 2.3. In the subsequent section, the exponential stablity of these solution is analyzed. Section 2.4 focuses on applications of the main results and provides illustrative numerical examples. The chapter concludes with a summary of the main findings.

2.2 (ω, c)-periodic functions on ω-periodic time scales

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a real (or complex) Banach space with scalar field $\mathbb{F}_{\mathbb{B}} = \mathbb{R}$ (or $\mathbb{C}$). We begin with the definition of an (ω, c)-periodic $\mathbb{B}$-valued function defined on an ω-periodic time scale $\mathbb{T}_\omega$.

Definition 2.2.1. *For $c \in \mathbb{F}_{\mathbb{B}} \setminus \{0\}$, a function $f \in \mathrm{C_{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is said to be (ω, c)-periodic on $\mathbb{T}_\omega$ if*

$$f(t+\omega) = cf(t) \text{ for all } t \in \mathbb{T}_\omega.$$

Here, ω is called the c-period of f, and the collection of all functions with the same c-period ω is denoted by $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.

Remark 2.2.2. *The ω-periodicity of the time scale $\mathbb{T}_\omega$ plays a crucial role in the definition 2.2.1, as it ensures that $\mathbb{T}_\omega$ is invariant under translations by ω (> 0), thereby making the definition well defined.*

Example 2.2.3. *The function $f(t) = \sin(\pi t/k)$ is a $(k, -1)$-periodic function on the time scale $\mathbb{Z}$, where $k \in \mathbb{Z}^+$.*

Example 2.2.4. *Let $\mathbb{T}$ be a time scale invariant under translation, satisfying $s+t \in \mathbb{T}$ for all $s, t \in \mathbb{T}$. For a positive $\omega \in \mathbb{T}$, consider $u : \mathbb{T} \to \mathbb{B}$ to be an ω-periodic function. Let $f : \mathbb{T} \to \mathbb{C}$ be a function such that $f(t+s) = f(s)f(t)$ for all $s, t \in \mathbb{T}$ and $f(\omega) \neq 0$. Then $v(t) = f(t)u(t)$ is an $(\omega, f(\omega))$-periodic function.*

Example 2.2.5. *Consider the heat equation on $\mathbb{T}_\omega = \mathbb{R}$, given as*

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \quad t > 0, \ x \in \mathbb{R}, \\ u(x,0) &= f(x), \quad f \in P_{\omega,c}(\mathbb{R}, \mathbb{R}). \end{aligned} \tag{2.2.1}$$

It is well-known that the regular solution $u(x,t)$ of (2.2.1) is given as

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}} f(s)ds.$$

For a fixed $t_0 > 0$, we have

$$u(x,t_0) = \frac{1}{2\sqrt{\pi t_0}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t_0}} f(s)ds.$$

Then,

$$\begin{aligned} u(x+\omega,t_0) &= \frac{1}{2\sqrt{\pi t_0}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t_0}} f(s)ds \\ &= \frac{1}{2\sqrt{\pi t_0}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t_0}} f(s+\omega)ds \\ &= c\frac{1}{2\sqrt{\pi t_0}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t_0}} f(s)ds \\ &= cu(x,t_0), \end{aligned}$$

hence $u(x,t)$ is (ω,c)-periodic with respect to x.

Example 2.2.6. *Consider the heat equation on $\mathbb{T}_1 \times \mathbb{T}_2$ (where the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ are invariant under translations) in one (spatial) dimension, given as*

$$\begin{aligned} &u^{\Delta_t}(x,t) = \alpha^2 u^{\Delta_x^2}(x,t),\ \ \alpha \in \mathbb{R} \text{ is a constant} \\ &u(x,0) = \cos x, \end{aligned} \tag{2.2.2}$$

where u^{Δ_t} denotes the partial delta derivative of u with respect to variable t, and $u^{\Delta_x^2}$ indicates that partial derivative of u with respect to x are taken twice. Then, the particular solution to IVP (2.2.2), given as

$$u_p(x,t) = e_{-\alpha^2 q^2}(t,0)\cos x,$$

is $(2\pi,1)$-periodic with respect to x.

Theorem 2.2.7. *If $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$ is Δ-differentiable and $f^\Delta \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega,\mathbb{B})$, then $f^\Delta \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$.*

Proof. As the time scale is ω-periodic, $\sigma(t+\omega) = \sigma(t)+\omega$, and $\mu(t+\omega) = \mu(t)$ for all $t \in \mathbb{T}_\omega$. Now, for $t \in \mathbb{T}_\omega^\kappa$ with $\mu(t) \neq 0$, $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$ implies

$$\begin{aligned} f^\Delta(t+\omega) &= \frac{f(\sigma(t+\omega)) - f(t+\omega)}{\mu(t+\omega)} \\ &= \frac{f(\sigma(t)+\omega) - f(t+\omega)}{\mu(t)} = \frac{cf(\sigma(t)) - cf(t)}{\mu(t)} = cf^\Delta(t). \end{aligned}$$

When $\mu(t) = 0$ at $t \in \mathbb{T}_\omega^\kappa$, then $f^\Delta(t) = \lim\limits_{s\to t, s\in\mathbb{T}_\omega} \dfrac{f(t)-f(s)}{t-s} = \lim\limits_{\tau\to 0,\tau\in\Pi_{\hat{\mathbb{T}}_\omega}} \dfrac{f_\tau(t)-f(t)}{\tau}$ with $f_\tau(t) = f(t+\tau)$ and $\hat{\mathbb{T}}_\omega = \{t \in \mathbb{T}_\omega |\ \mu(t) = 0\}$. This implies

$$f^\Delta(t+\omega) = \lim_{\tau\to 0,\tau\in\Pi_{\hat{\mathbb{T}}_\omega}} \frac{f_\tau(t+\omega) - f(t+\omega)}{\tau} = \lim_{\tau\to 0,\tau\in\Pi_{\hat{\mathbb{T}}_\omega}} \frac{cf_\tau(t) - cf(t)}{\tau} = cf^\Delta(t).$$

The proof is complete. $\square$

Example 2.2.8. *Consider the linear delay dynamic equation*

$$x^{\Delta}(t) + \eta x(t-r) = 0 \quad \textit{with} \quad \eta, r > 0, \ r \in \Pi_{\mathbb{T}}, \tag{2.2.3}$$

on a $\frac{2\pi}{y_0}$*-periodic time scale* $\mathbb{T}$*. Then, for some* $t_0 \in \mathbb{T}$*,* $x(t) = e_{z_0}(t, t_0)$ *with* $z_0 = x_0 \oplus i_\mu y_0$*, where* $x_0 = \mathrm{Re}_\mu(z_0)$ *and* $y_0 = \mathrm{Im}_\mu(z_0) > 0$ *is a* $\left(\frac{2\pi}{y_0}, e_{x_0}\left(\frac{2\pi}{y_0}, 0\right)\right)$*-periodic solution for* (2.2.3) *provided* $z_0 + \eta e_{\ominus z_0}(t, t-r) = 0$*. Here* $\mathrm{Re}_\mu(z_0) = \frac{|z_0\mu+1|-1}{\mu}$ *and* $\mathrm{Im}_\mu(z_0) = \frac{\mathrm{Arg}(z_0\mu+1)}{\mu}$ *respectively are the Hilger real and imaginary parts of* z_0*, where* $\mathrm{Arg(z)}$ *denotes the principal argument of* z*. Also,* $i_\mu y_0 = \frac{e^{iy_0\mu}-1}{\mu}$ *denotes the Hilger purely imaginary number.*

Remark 2.2.9. *The definition of* (ω, c)*-periodic functions on* $\mathbb{T}_\omega$ *suggests that* $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *is a translation invariant subspace of* $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$*.*

Remark 2.2.10. *The definition of the class of* (ω, c)*-periodic functions on* $\mathbb{T}_\omega$ *clearly indicates that this class can include both bounded and unbounded functions, depending on the magnitude of* c*. Specifically, if* $|c| = 1$*, then all functions in the class will be bounded, whereas, for* $|c| \neq 1$*, there will be unbounded functions within the class. If* $|c| < 1$*, any* $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *converges to zero as* t *approaches* ∞*, and it is unbounded if* t *approaches* $-\infty$*. This situation is reversed if* $|c| > 1$*. In other words, the magnitude of* c *determines the rate at which the functions in the class* $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *grow or decay at different points in* $\mathbb{T}_\omega$*, which ultimately determines whether they are bounded or unbounded.*

We now state a proposition that is a time scale generalization of [14, Proposition 2.2] and can be found very useful to characterize (ω, c)-periodic functions on $\mathbb{T}_\omega$.

Proposition 2.2.11. *A function* $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ *is an* (ω, c)*-periodic function if and only if there exist* $p \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ *such that*

$$f(t) = c^{t/\omega} p(t) \ \textit{for all} \ t \in \mathbb{T}_\omega,$$

where $c^{t/\omega} = exp(\frac{t}{\omega}\log(c))$*.*

Proof. We start by considering f to be in $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. If we write $p(t) = c^{-t/\omega} f(t)$, then

$$p(t+\omega) = c^{-(t+\omega)/\omega} f(t+\omega) = c^{-t/\omega} f(t) = p(t),$$

i.e., $p \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$. Conversely, if p is in $\mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, then

$$f(t+\omega) = c^{(t+\omega)/\omega} p(t+\omega) = c(c^{t/\omega} p(t)) = cf(t),$$

implies $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. □

Remark 2.2.12. *Considering Proposition 2.2.11, for any $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, we say that $\mathrm{c}^{t/\omega}p(t)$ is the c-factorization of f. We call $p(t)$ the periodic part of f on $\mathbb{T}_\omega$. Note that using this convention, an anti-periodic function f on $\mathbb{T}_\omega$ can be written as $f(t) = (-1)^{t/\omega}p(t)$, where ω is the anti-period of f on $\mathbb{T}_\omega$.*

Example 2.2.13. *The decomposition of the anti-periodic function $f(t) = \sin(\pi t/k)$ on $\mathbb{Z}$ with anti-period k can be expressed as*

$$f(t) = (-1)^{t/k}p(t) = [\cos(\pi t/k) + i\sin(\pi t/k)]p(t),$$

where $p(t) = \sin(\pi t/k)[\cos(\pi t/k) + \sin(\pi t/k)]$.

Remark 2.2.14. *In view of Proposition 2.2.11, it should be noted that the nature of a function f on $\mathbb{T}_\omega$ can vary based on the selected value of ω. Specifically, f may exhibit periodicity, anti-periodicity, or behave as a Bloch wave on $\mathbb{T}_\omega$ simultaneously, depending on the chosen ω. For example, consider an $(\omega, e^{i\pi/2})$-periodic function f on $\mathbb{T}_\omega$. It is worth noting that this function can exhibit different properties depending on different values of ω:*

- *For $t \in \mathbb{T}_\omega$, f can behave as a Bloch wave, satisfying $f(t+\omega) = e^{i\pi/2}f(t)$.*
- *For $t \in \mathbb{T}_\omega$, f can also be an anti-periodic function with anti-period 2ω, meaning that $f(t+2\omega) = -f(t)$.*
- *Additionally, for $t \in \mathbb{T}_\omega$, f can be a 4ω-periodic function, such that $f(t+4\omega) = f(t)$.*

Now we have the formal definition of (ω, c)-periodicity on $\mathbb{T}_\omega$, let us look at some basic properties of (ω, c)-periodic functions on the time scale $\mathbb{T}_\omega$.

Theorem 2.2.15. *If $f, g \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, then*

1. *$f + g \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$,*
2. *$\alpha f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ for each $\alpha \in \mathbb{F}_B$,*
3. *$f_\tau(t) := f(t+\tau) \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ for each fixed $\tau \in \Pi_\mathbb{T}$.*

Proof. The proof is immediate. □

It is worth noting that the first two properties of Theorem 2.2.15 imply that the collection of $\mathbb{B}$-valued (ω, c)-periodic functions on $\mathbb{T}_\omega$ form a vector space. To give the space of (ω, c)-periodic functions on $\mathbb{T}_\omega$ a Banach structure, we need to define a suitable norm. This norm should satisfy the properties of a norm, such as nonnegativity, homogeneity, and the triangle inequality. Once we have a norm, we can define a Banach space as a complete normed vector space, which will allow us to study the properties of this space and its functions in a rigorous manner. Note that two important characteristics of the elements in this space are their nonboundedness and periodicity. This indicates that the norm that is assigned to this space will need to address these characteristics in some way.

It is well-known that the space $\mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, equipped with the norm

$$\| \cdot \|_\omega := \sup_{[t_0, t_0+\omega]_{\mathbb{T}_\omega}} \| \cdot \|_{\mathbb{B}},$$

is a Banach space for any fixed $t_0 \in \mathbb{T}_\omega$ and Euclidean norm $\| \cdot \|_{\mathbb{B}}$ on $\mathbb{B}$. Is the space of (ω, c)-periodic functions on $\mathbb{T}_\omega$ along with the same norm a Banach space? It is important to note that the sup-norm is always a natural way to measure the length of periodic functions. But the (ω, c)-periodic functions on $\mathbb{T}_\omega$ are unbounded when $|c| \neq 1$. Thus, using a sup-norm for the case of $|c| > 1$ will be problematic as it leads to all the elements of $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ having an infinite norm. Also for $|c| < 1$, this norm assigns different values for $(2\pi, e^{-2\pi})$ periodic functions $e^{-t}\cos t$ and $e^{-t}\sin t$, although their periodic components differ only by a phase shift. To address these challenges, an alternative norm must be defined that considers both the periodicity and the unboundedness of the functions.

Theorem 2.2.16. $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *is a Banach space with the norm* $\sup_{t \in [t_0, t_0+\omega]_{\mathbb{T}}} \|f\|_{\omega,c}$, *where*

$$\|f\|_{\omega,c} := \|c^{-t/\omega} f(t)\|_{\mathbb{B}}, \tag{2.2.4}$$

for some fixed $t_0 \in \mathbb{T}_\omega$.

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be an Cauchy sequence in $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. Using the characterization of (ω, c)-periodic functions, $a_n(t)$ can be written as $c^{t/\omega} p_n(t)$, for some $p_n \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$. Now,

$$\|p_n - p_m\|_\omega = \sup_{t\in[t_0,t_0+\omega]} \|p_n(t) - p_m(t)\|_{\mathbb{B}} = \sup_{t\in[t_0,t_0+\omega]} \|c^{-t/\omega} a_n(t) - c^{-t/\omega} a_m(t)\|_{\mathbb{B}} = \|a_n - a_m\|_{\omega,c}$$

implies that $\{p_n\}_{n\in\mathbb{N}} \subset \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ is also a Cauchy sequence. As $\mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ is a Banach space equipped with the norm $\| \cdot \|_\omega$, there exist a uniform limit $p \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ of p_n in $[t_0, t_0 + \omega]$. This further implies that $a_n(t) \to a(t) = c^{t/\omega} p(t) \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ with respect to the norm $\| \cdot \|_{\omega,c}$. □

We now examine the invariance of the space $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ under the action of the operator $\mathcal{F}(\cdot, \varphi(\cdot))$, where $\mathcal{F}$ operates on $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$ and φ is a function in $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. Our objective is to identify conditions under which applying the operator $\mathcal{F}(\cdot, \varphi(\cdot))$ to a function in $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ yields another function belonging to $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$. In simpler terms, we are looking for a class of operators that, when applied, keep the space $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$ invariant.

Theorem 2.2.17. *Let* $\mathcal{F} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$. *Then the following are equivalent:*

1. $\mathcal{F}(\cdot, \varphi) \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$ *whenever* $\varphi \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.

2. $\mathcal{F}$ is ω-periodic in the first variable and homogeneous in the second variable.

Proof. Suppose that the first assertion holds true. For the second assertion to be true, we need to show that $\mathcal{F}(t+\omega, cx) = c\mathcal{F}(t,x)$ for all $(t,x) \in \mathbb{T}_\omega \times \mathbb{B}$. For this, it is enough to show that for every $(t,x) \in \mathbb{T}_\omega \times \mathbb{B}$, there exists a function $\varphi \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ such that $\varphi(t) = x$. Therefore, for such φ, we have

$$c\mathcal{F}(t,x) = c\mathcal{F}(t,\varphi(t)) = \mathcal{F}(t+\omega, \varphi(t+\omega)) = \mathcal{F}(t+\omega, c\varphi(t)).$$

Now, to prove our claim, it is enough to consider $\varphi(s) = c^{(s-t)/\omega}x$, which clearly is in $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ with $\varphi(t) = x$. For the converse part, assume that the second statement is true. Then for all $t \in \mathbb{T}_\omega$ and $\varphi \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, we have

$$\mathcal{F}(t+\omega, \varphi(t+\omega)) = \mathcal{F}(t+\omega, c\varphi(t)) = c\mathcal{F}(t,\varphi(t)),$$

proving that the first statement holds. $\square$

Remark 2.2.18. *We call the operator $\mathcal{T}(\varphi)(\cdot) := \mathcal{F}(\cdot, \varphi(\cdot))$ the Nemytskii dynamic composition operator. The operator $\mathcal{N}$ is called translation-invariant on $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ if either of the statements of Theorem 2.2.17 holds true.*

2.3 Existence of (ω, c)-periodic solutions for delayed population models on time scales

In this section, we present a set of sufficient conditions for the existence of (ω, c)-periodic solutions for dynamic equations with time-dependent discrete variable delays and nonlinear feedback on ω-periodic time scales. As we are working on a population model, we restrict the domain to $\mathbb{T}_\omega^{0+} = \mathbb{T}_\omega \cap [0, \infty)$. Let (2.1.1) and (2.1.2) be supplemented with an initial value given by

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau^*, t_0]_{\mathbb{T}_\omega} := [t_0 - \tau^*, t_0] \cap \mathbb{T}_\omega, \tag{2.3.1}$$

where $\tau^* = \max_{1 \le k \le n} \sup_{t \in \mathbb{T}_\omega} \tau_k(t)$, and $\varphi \in \mathcal{C}_{\omega,c} := (\mathrm{C_{rd}}([t_0 - \tau^*, t_0]_{\mathbb{T}_\omega}, \mathbb{R}), \|\cdot\|_{\omega,c})$, with t_0 is the smallest nonnegative element of $\mathbb{T}_\omega$. It is important to note that in a population model, only the nonnegative solutions are meaningful. Therefore, it is useful to consider $\phi(t_0) > 0$.

For practical importance of the population model, throughout this section we will consider $\mathbb{B}$ to be a real-valued Banach space. Before we proceed further, let us recall the necessary assumptions that are considered in the coefficients of (2.1.1) and (2.1.2):

H1 There exists an $c \in \mathbb{R}^+$ such that $F_k(t+\omega, cx) = cF_k(t,x), k = 1, 2, ..., m$ and $H(t+\omega, cx) = cH(t,x)$ for all $t \in \mathbb{T}_\omega^{0+}$ and $x \in \mathbb{B}_0^+$. Also, $d \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$ and $\tau_k \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \Pi_{\mathbb{T}_\omega}^{0+})$.

H2 There exist positive real-valued functions L_{F_k} and L_H corresponding to nonnegative functions F_k and H such that $\|F_k(t,x) - F_k(t,y)\|_{\mathbb{B}} \leq L_{F_k}(t)\|x-y\|_{\mathbb{B}}$ for all $1 \leq k \leq m$, and $\|H(t,x) - H(t,y)\|_{\mathbb{B}} \leq L_H(t)\|x-y\|_{\mathbb{B}}$ for all $x, y \in \mathbb{B}_0^+$ and $t \in \mathbb{T}_\omega^{0+}$.

Theorem 2.3.1. *Consider the following condition:*

H* *There exists a positive constant $\mathcal{L}_H$ such that $-\mathcal{L}_H x \leq H(t,x)$ for all $t \in \mathbb{T}_\omega^{0+}$ and $x \in \mathbb{B}_0^+$, if H is an immigration function (or $H(t,x) \leq \mathcal{L}_H x$, if H is a harvesting term), and $-(\overline{d} + \mathcal{L}_H) \in \mathcal{R}^+$, where $\overline{d} = \sup_{t \in \mathbb{T}_\omega^{0+}} d(t)$.*

Then any solution x of (2.1.1) satisfies $x(t) \geq 0$.

Proof. Let x be a solution of (2.1.1). Then

$$\begin{aligned} x^\Delta(t) &= -d(t)x(t) + \sum_{k=1}^{m} F_k(t, x(t - \tau_k(t))) + H(t, x(t)) \\ &\geq -(\overline{d} + \mathcal{L}_H)x(t) + \sum_{k=1}^{m} \underline{F_k}, \end{aligned}$$

where $\underline{F_k}$ is the infimum of $F_k(t,x)$ in the whole of its domain. Thus, by Lemma 1.4.82, we have $\liminf_{t\to\infty} x(t) \geq \dfrac{\sum_{k=1}^m \underline{F_k}}{\overline{d} + \mathcal{L}_H}$. □

Remark 2.3.2. *For $c \geq 1$, (2.1.1) under the condition **H*** is always uniformly persistent, i.e., there exists $x_* > 0$ such that $\liminf_{t\to\infty} x(t) \geq x_*$.*

Remark 2.3.3. *Consider that we have a positive constant $\mathcal{L}_H$ such that $H(t,x) \leq \mathcal{L}_H x$ for all $t \in \mathbb{T}_\omega^{0+}$ and $x \in \mathbb{B}_0^+$, if H denotes the immigration of the population (or $-\mathcal{L}_H x \leq H(t,x)$, if H is a harvesting function). Also, $\underline{d} > \mathcal{L}_H$ and $\mathcal{L}_H - \underline{d} \in \mathcal{R}^+$, where $\underline{d} = \inf_{t \in \mathbb{T}_\omega^{0+}} d(t)$. Then (2.1.1) is dissipative if $c \leq 1$, i.e., $0 \leq \limsup_{t\to\infty} x(t) \leq x^*(> 0)$.*

Now let us have a look at the following lemmas:

Lemma 2.3.4. *Let $f \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ and $t_1, t_2 \in \mathbb{T}_\omega$. Then*

$$\sigma(t+\omega) = \sigma(t) + \omega, \quad \mu(t+\omega) = \mu(t), \text{ and } \int_{t_1+\omega}^{t_2+\omega} f(\tau)\Delta\tau = \int_{t_1}^{t_2} f(\tau)\Delta\tau.$$

Lemma 2.3.5. *Consider a regressive function $\alpha \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{R})$. Then, $F(t) = \int_{-\infty}^{t} e_\alpha(t, \sigma(s)) f(s)\Delta s \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ provided $f \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.*

Proof. As $\mathbb{T}_\omega$ is ω-periodic, $\sigma(t+\omega) = \sigma(t) + \omega$ for all $t \in \mathbb{T}_\omega$. Also, for an ω-periodic regressive function α, $e_\alpha(t_2 + \omega, t_1 + \omega) = e_\alpha(t_2, t_1)$ for all $t_1, t_2 \in \mathbb{T}_\omega$, which can be easily shown using the representation of the exponential function in terms of the cylinder transform. Now,

$$\begin{aligned} F(t+\omega) &= \int_{-\infty}^{t+\omega} e_\alpha(t+\omega, \sigma(s)) f(s)\Delta s = \int_{-\infty}^{t} e_\alpha(t+\omega, \sigma(s+\omega)) f(s+\omega)\Delta s \\ &= c\int_{-\infty}^{t} e_\alpha(t, \sigma(s)) f(s)\Delta s = cF(t). \end{aligned}$$

Hence, the proof is complete. □

Lemma 2.3.6. *For a regressive function $\alpha \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{R})$, the function $F(t) := \int_{-\infty}^{t} e_{\ominus\alpha}(t,s) f(s)\Delta s \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ provided $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.*

Proof. As $\mathbb{T}_\omega$ is ω-periodic, $\mu(t+\omega) = \mu(t)$ for all $t \in \mathbb{T}_\omega$. Therefore, $e_\alpha(t_2+\omega, t_1+\omega) = e_\alpha(t_2, t_1)$ for all $t_1, t_2 \in \mathbb{T}$. Now,

$$\begin{aligned} F(t+\omega) &= \int_{-\infty}^{t+\omega} e_{\ominus\alpha}(t+\omega, s) f(s)\Delta s \\ &= \int_{-\infty}^{t} e_{\ominus\alpha}(t+\omega, s+\omega) f(s+\omega)\Delta s \\ &= c\int_{-\infty}^{t} e_\alpha(s+\omega, t+\omega) f(s)\Delta s \\ &= c\int_{-\infty}^{t} e_\alpha(s, t) f(s)\Delta s \\ &= c\int_{-\infty}^{t} e_{\ominus\alpha}(t, s) f(s)\Delta s \\ &= cF(t), \end{aligned}$$

completing the proof. □

Now, let us proceed to the main results of this section.

Theorem 2.3.7. *Assuming **H***, **H1** and **H2**, also consider the following conditions:*

H3 *There exists a positive constant $\underline{d}$ such that $-\underline{d} \in \mathcal{R}^+$, and the regressive function d satisfies $0 < \underline{d} \leq d(t)$ for all $t \in \mathbb{T}_\omega$, $\frac{\ln|c|}{\omega} + \underline{d} > 0$, and $|c| < e^{\frac{\omega}{\overline{\mu}}(1-\underline{d}\overline{\mu})}$.*

H4 $\mathtt{C}_1 := \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} \frac{e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(t,s)}{1-\underline{d}\mu(s)} \left(\sum_{k=1}^{m} |c|^{-\tau_k(s)/\omega} L_{F_k}(s) + L_H(s) \right) \Delta s < 1.$

Under these conditions, for any $\varphi \in \mathcal{C}_{\omega,c}$, (2.1.1) has a unique nonnegative (ω, c)-periodic solution.

Proof. To prove this result, first we define an operator $\mathcal{S} : \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega^{0+}, \mathbb{B}) \to \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ as

$$(\mathcal{S}x)(t) = \int_{-\infty}^{t} e_{-d}(t, \sigma(s))(\mathcal{T}x)(s)\Delta s, \tag{2.3.2}$$

where $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $\mathcal{T} : \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B}) \to \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is again an operator defined by

$$(\mathcal{T}x)(t) = \sum_{k=1}^{m} F_k(t, x(t-\tau_k(t))) + H(t, x(t)). \tag{2.3.3}$$

Note that $x(t-\tau_k(t)) = \varphi(t-\tau_k(t))$ for $t \in [t_0, t_0+\tau^*]_{\mathbb{T}_\omega}$, and since $\tau_k \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \Pi_{\mathbb{T}_\omega}^{0+})$, for an (ω, c)-periodic function $x : \mathbb{T}_\omega \to \mathbb{B}$, the functions $x(t-\tau_k(t))$ will also be in $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Also, being a Nemytskii dynamic composition operator, for all $1 \le k \le m$, $F_k(t, x(t-\tau_k(t))$ and $H(t, x(t))$ will be in the space $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ whenever $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Thus, by the first property of Theorem 2.2.15, it is evident that $(\mathcal{T}x)(t)$ belongs to $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ for any $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Finally, by Lemma 2.3.5, it follows that $\mathcal{S}$ maps the space $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ into itself. Now for any $x, y \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ with $x \neq y$, we obtain

$$\begin{aligned}
\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|\mathcal{S}(x) - \mathcal{S}(y)\|_{\omega,c} &= \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{-d}(t, \sigma(s)) \left((\mathcal{T}x)(s) - (\mathcal{T}y)(s)\right) \Delta s \right\|_{\mathbb{B}} \\
&\le \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-t/\omega} e_{-d}(t, \sigma(s)) \Bigg(\sum_{k=1}^{m} \|F_k(s, x(s-\tau_k(s))) \\
&\quad - F_k(s, y(s-\tau_k(s)))\|_{\mathbb{B}} + \|H(s, x(s)) - H(s, y(s))\|_{\mathbb{B}} \Bigg) \Delta s \\
&\le \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-(t-s)/\omega} e_{-\underline{d}}(t, \sigma(s)) \Bigg(\sum_{k=1}^{m} |c|^{-s/\omega} L_{F_k}(s) \|x(s-\tau_k(s)) \\
&\quad - y(s-\tau_k(s))\|_{\mathbb{B}} + L_H(s) |c|^{-s/\omega} \|x(s) - y(s)\|_{\mathbb{B}} \Bigg) \Delta s \\
&\le \mathtt{C}_1 \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x - y\|_{\omega,c} \\
&< \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x - y\|_{\omega,c},
\end{aligned}$$

which implies that the map $\mathcal{S}$ is a contraction map. Thus, by the Banach fixed-point theorem, $\mathcal{S}$ has a unique fixed point in the space $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$. It is also important to recall that (2.3.2) along with (2.3.3) constitute a solution of dynamic equation (2.1.1). Hence, under assumption **H***, the conclusion follows. It is worth noting that the condition $|c| < e^{\omega/\underline{\mu}}$ implies $-\frac{\ln|c|}{\omega} \in \mathcal{R}^+$, which allowed us to apply Lemma 1.4.29 and obtain the aforementioned result. In the proof, we have also used the fact that $-d(t) \le -\underline{d}$ $(\in \mathcal{R}^+)$ implies $e_{-d}(s, t) \le e_{-\underline{d}}(s, t)$. $\square$

Our focus in this paper is on the search for (ω, c)-periodic solutions to the delayed population model on a time scale with a particular type of birth term that arises in the Nicholson and Lasota–Ważewska type models. Specifically, we are interested in (ω, c)-periodic solutions of (2.1.2), where the birth term has the form

$$F_k(t, x(t-\tau_k(t)) := \beta_k(t) f_k(\lambda_k(t) x(t-\tau_k(t))).$$

This form of the birth term is of particular interest due to its relevance in modeling population dynamics and ecological systems. Our goal is to establish conditions under which (ω, c)-periodic solutions exist for this type of birth term. To this end, we present an important result that provides sufficient conditions for the existence of (ω, c)-periodic solutions of a particular form of (2.1.1). By establishing these conditions, we hope to contribute to a better understanding of the dynamics of Nicholson and Lasota–Ważewska type models on a time scale and to provide insights that may be useful in a variety of applied contexts.

Before we state the result, note that since we have considered $\mathbb{T}_\omega$ to be a periodic time scale, the graininess function μ must be bounded. For any function α that has its domain in $\mathbb{T}_\omega$, let us use the notations $\overline{\alpha}$ to represent $\sup\limits_{t\in\mathbb{T}_\omega} \alpha(t)$ and $\underline{\alpha}$ to denote $\inf\limits_{t\in\mathbb{T}_\omega} \alpha(t)$, where t_0 is the smallest element of $\mathbb{T}_\omega^{0+}$.

Theorem 2.3.8. *Assume the following conditions:*

A1 *There exists* $c \in \mathbb{R}^+$ *such that* $d \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$, $\gamma \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{R}_0^+)$, $\tau_k \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \Pi_{\mathbb{T}_\omega}^{0+})$, $\beta \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$ *and* $\lambda \in P_{(\omega,1/c)}(\mathbb{T}_\omega^{0+}, \mathbb{R}_0^+)$.

A2 $h(cx) = ch(x)$ *for all* $x \in \mathbb{B}$.

A3 *The nonnegative functions* $f_k, h : \mathbb{B}_0^+ \to \mathbb{B}_0^+$ *are Lipschitz, i.e., there exist positive constants* l_{f_k} *and* l_h *such that* $\|f_k(x) - f_k(y)\|_{\mathbb{B}} \leq l_{f_k}\|x-y\|_{\mathbb{B}}$ *for all* $1 \leq k \leq m$ *and* $\|h(x)-h(y)\|_{\mathbb{B}} \leq l_h\|x-y\|_{\mathbb{B}}$ *for all* $x, y \in \mathbb{B}_0^+$.

A4 *There exists a positive constant* $\underline{d}$ *such that* $-\underline{d} \in \mathcal{R}^+$, *and the regressive function* d *satisfies* $0 < \underline{d} \leq d(t)$ *for all* $t \in \mathbb{T}_\omega$, $\frac{\ln|c|}{\omega} + \underline{d} > 0$, *and* $|c| < e^{\frac{\omega}{\overline{\mu}}(1-\underline{d}\overline{\mu})}$.

A5 $$\mathtt{C}_2 := \frac{1+\overline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}\left(\sum_{k=1}^{m} l_{f_k}\ |c|^{-\tau_k^*/\omega}|\overline{\beta_k\lambda_k}| + l_h|\overline{\gamma}|\right) < 1,$$

$$\textit{where } \tau_k^* = \begin{cases} \overline{\tau}_k, & \textit{when } |c| \leq 1, \\ \underline{\tau_k} & \textit{when } |c| \geq 1. \end{cases}$$

Then, for any initial function $\varphi \in \mathcal{C}_{\omega,c}$,

$$x^\Delta(t) = -d(t)x(t) + \sum_{k=1}^{m} \beta_k(t) f_k(\lambda_k(t)x(t-\tau_k(t))) - \gamma(t)h(x(t)) \tag{2.3.4}$$

has a unique (ω, c)*-periodic solution.*

Proof. Similarly to the proof of Theorem 2.3.7, we define an operator $\mathcal{S} : \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B}) \to \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ given by (2.3.2). However, in this case, the operator $\mathcal{T} : \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B}) \to \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is defined as

$$(\mathcal{T}x)(t) = \sum_{k=1}^{m} \beta_k(t) f_k(\lambda_k(t)x(t-\tau_k(t))) - \gamma(t)h(x(t)) \tag{2.3.5}$$

To show that $\mathcal{S}(\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+},\mathbb{B})) \subset \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+},\mathbb{B})$, it is sufficient to demonstrate that $(\mathcal{T}x)(t) \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ whenever $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. This is evident because $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ is a vector space, and the remaining requirements are satisfied by the hypothesis on the coefficients of (2.3.4), as stated in the first and second statements. Now for any $x, y \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+},\mathbb{B})$ with $x \neq y$, we have

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|\mathcal{S}(x)-\mathcal{S}(y)\|_{\omega,c}\\
&\quad= \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \left\| c^{-t/\omega}\int_{-\infty}^{t} e_{-d}(t,\sigma(s))\left((\mathcal{T}x)(s)-(\mathcal{T}y)(s)\right)\Delta s\right\|_{\mathbb{B}}\\
&\quad\le \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-(t-s)/\omega}\frac{e_{-\underline{d}}(t,s)}{1-\underline{d}\mu(s)}\Bigg(\sum_{k=1}^{m}|c|^{-s/\omega}|\beta_k(s)\lambda_k(s)|l_{f_k}\|x(s-\tau_k(s))\\
&\qquad\quad -y(s-\tau_k(s))\|_{\mathbb{B}}+l_h|c|^{-s/\omega}|\gamma(s)|\ \|x(s)-y(s)\|_{\mathbb{B}}\Bigg)\Delta s\\
&\quad\le \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\|x-y\|_{\omega,c}\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\int_{-\infty}^{t}\frac{e_{\ominus(\frac{\ln|c|}{\omega}+\underline{d})}(t,s)}{1-\underline{d}\mu(s)}\Bigg(\sum_{k=1}^{m}|c|^{-\tau_k(s)/\omega}\\
&\qquad\quad l_{f_k}|(\overline{\beta_k\lambda_k})|+l_h|\gamma(s)|\Bigg)\Delta s\\
&\quad\le \frac{\sum_{k=1}^{m}|c|^{-\tau_k^*/\omega}l_{f_k}|\overline{\beta_k\lambda_k}|+l_h|\overline{\gamma}|}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)}\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\left(\int_{-\infty}^{t}e_{\ominus(\frac{\ln|c|}{\omega}+\underline{d})}(t,\sigma(s))\Delta s\right)\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\|x-y\|_{\omega,c}\\
&\quad\le \mathtt{C}_2\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\|x-y\|_{\omega,c}\\
&\quad< \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}}\|x-y\|_{\omega,c},
\end{aligned}$$

which suggests that $\mathcal{S}$ is a contraction map. Thus, applying the Banach fixed point theorem, $\mathcal{S}$ has a unique fixed point in $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+},\mathbb{B})$. Since (2.3.2) and (2.3.5) form a solution to the dynamic equation (2.3.4), the conclusion follows. □

Remark 2.3.9. *It will be appropriate to take $c > 0$ for the nonnegative population. Considering $|c| = 1$ is particularly valuable when studying the population dynamics of a species that exhibits oscillatory behavior. With a suitable initial function, this case guarantees that the model will retain the natural tendency of such species to exist and survive over time, with a maximum limit beyond which their population cannot increase. However, the other two cases are equally important. When investigating the population of a species in a specific location that needs to gradually decrease over time, such as insects harmful to plants, it is appropriate to consider the case where $|c| < 1$. On the other hand, if we wish to study the population that increases gradually over time, such as the production of animal red cells in special cases, the case $|c| > 1$ becomes relevant.*

Theorem 2.3.10. *Along with assumptions **A1**–**A5**, consider the following:*

A* *There exists a positive constant $\mathcal{L}_h$ such that $h(x) \leq \mathcal{L}_h x$ for all $x \in \mathbb{B}_0^+$, if h is a harvesting term (or $-\mathcal{L}_h x \leq h(x)$, if h is an immigration function), and $-(\overline{d} + \overline{\gamma}\mathcal{L}_H) \in \mathcal{R}^+$.*

Then the solution x of (2.3.4) satisfies $x(t) \geq 0$ for all $t \in \mathbb{T}_\omega^{0+}$.

Proof. Being a solution of (2.3.4), x satisfies

$$x^\Delta(t) \geq -(\overline{d} + \overline{\gamma}\mathcal{L}_h)x(t) + \sum_{k=1}^{m} \underline{\beta_k f_k}.$$

Thus, by Lemma 1.4.82 we have $\liminf_{t\to\infty} x(t) \geq \dfrac{\sum_{k=1}^m \underline{\beta_k}\ \underline{f_k}}{\overline{d} + \overline{\gamma}\mathcal{L}_h}$. Note that here $\underline{\beta_k} = \inf_{t\in[t_0,\infty)_\mathbb{T}} \beta_k(t)$. $\square$

Remark 2.3.11. *For $c \geq 1$, (2.3.4) under condition **A*** is uniformly persistent.*

Theorem 2.3.12. *Along with assumptions **H1** and **H2**, let us assume the following conditions:*

H5 *$|c| < e^{\omega/\overline{\mu}}$, where $\overline{\mu}$ is the least upper bound of $\mu(t)$.*

H6 *There exists a positive constant $\underline{d}$ such that the regressive function d satisfies $0 < \underline{d} \leq d(t)$ for all $t \in \mathbb{T}_\omega$.*

H7 $\mathtt{C}_3 := \sup_{t\in[t_0,t_0+\omega]_\mathbb{T}} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega}\oplus\underline{d}\right)}(t,s)\left(\sum_{k=1}^{m} |c|^{-\tau_k(s)/\omega} L_{F_k}(s) + L_H(s)\right)\Delta s < 1.$

Then, for any initial function $\varphi \in \mathcal{C}_{\omega,c}$, (2.1.2) has a unique (ω, c)-periodic solution.

Proof. We define an operator $\mathcal{S}^* : P_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B}) \to P_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ as

$$(\mathcal{S}^* x)(t) = \int_{-\infty}^{t} e_{\ominus d}(t,s)(\mathcal{T}x)(s)\Delta s, \tag{2.3.6}$$

where $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $\mathcal{T}$ is given by (2.3.3). From Lemma 2.3.6 it follows that $\mathcal{S}^*(\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})) \subset \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$. It is also important to recall that (2.3.6) along with (2.3.3) constitutes a solution of the dynamic equation (2.1.2). Note that for a positive constant $\underline{d} \in \mathcal{R}^+$, $e^{-\underline{d}(t-s)} \leq e_{\ominus\underline{d}}(t,s)$ for all $t \geq s$; $t, s \in \mathbb{T}_\omega$. As $|c| < e^{\omega/\overline{\mu}}$ implies $1 - \frac{\ln|c|}{\omega}\mu(t) > 0$ for all $t \in \mathbb{T}_\omega$, we have $|c|^{-(t-s)} = e^{-\frac{\ln|c|}{\omega}(t-s)} \leq e_{\ominus\frac{\ln|c|}{\omega}}(t,s)$.

Thus, for any $x, y \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ with $x \neq y$, we obtain

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|\mathcal{S}^*(x) - \mathcal{S}^*(y)\|_{\omega,c} \\
&= \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{\ominus d}(t,s) \left((\mathcal{T}x)(s) - (\mathcal{T}y)(s) \right) \Delta s \right\|_{\mathbb{B}} \\
&\leq \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-t/\omega} e_{\ominus \underline{d}}(t,s) \Bigg(\sum_{k=1}^{m} L_{F_k}(s) \|x(s-\tau_k(s)) \\
&\qquad - y(s-\tau_k(s))\|_{\mathbb{B}} + L_H(s)\|x(s) - y(s)\|_{\mathbb{B}} \Bigg) \Delta s \\
&\leq \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c} \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-(t-s)/\omega} e_{\ominus \underline{d}}(t,s) \\
&\qquad \left(\sum_{k=1}^{m} |c|^{-\tau_k(s)/\omega} L_{F_k}(s) + L_H(s) \right) \Delta s \\
&\leq \mathtt{C}_3 \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c} \\
&< \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c},
\end{aligned}$$

which implies that the map $\mathcal{S}$ is a contraction map. The conclusion follows from the Banach fixed-point theorem. $\square$

Theorem 2.3.13. *Consider that along with the assumptions of Theorem 2.3.12, assumption **H*** also holds. Then, the solution x of (2.1.2) satisfies $x(t) \geq 0$.*

Proof. Since x is a solution of (2.1.2), we have

$$\begin{aligned}
x^{\Delta}(t) &= -d(t)x^{\sigma}(t) + \sum_{k=1}^{m} F_k(t, x(t-\tau_k(t))) + H(t, x(t)) \\
&= -d(t)(x(t) + \mu x^{\Delta}(t)) + \sum_{k=1}^{m} F_k(t, x(t-\tau_k(t))) + H(t, x(t)) \\
&= -\frac{d(t)}{1+\mu d(t)} x(t) + \frac{1}{1+\mu d(t)} \sum_{k=1}^{m} F_k(t, x(t-\tau_k(t))) + \frac{H(t,x(t))}{1+\mu d(t)} \\
&\geq -\frac{\overline{d} + \mathcal{L}_H}{1+\underline{\mu}\underline{d}} x(t) + \frac{1}{1+\underline{\mu}\underline{d}} \sum_{k=1}^{m} \underline{F_k}.
\end{aligned}$$

Thus, by Lemma 1.4.82, we conclude that $\liminf_{t\to\infty} x(t) \geq \dfrac{\sum_{k=1}^{m} \underline{F_k}}{\overline{d} + \mathcal{L}_H}$. $\square$

Theorem 2.3.14. *Along with the first three assumptions of Theorem 2.3.8, let us assume the following conditions:*

A6 *There exists a positive constant $\underline{d}$ such that the regressive function d satisfies $0 < \underline{d} \leq d(t)$ for all $t \in \mathbb{T}_\omega$.*

A7 $|c| < e^{\omega/\overline{\mu}}$.

A8 $0 < \frac{1}{\frac{\ln|c|}{\omega} \oplus \underline{d}} \left(\sum_{k=1}^{m} |c|^{-\tau_k^*/\omega} |\overline{\beta_k \lambda_k}| l_{f_k} + l_h |\overline{\gamma}| \right) < 1$ *for all* $t \in \mathbb{T}_\omega$.

Then,

$$x^\Delta(t) = -d(t)x^\sigma(t) + \sum_{k=1}^{m} \beta_k(t) f_k(\lambda_k(t)x(t-\tau_k(t))) - \gamma(t)h(x(t)) \tag{2.3.7}$$

has an (ω, c)*-periodic solution that exists uniquely.*

Proof. To prove this result, let us define an operator $\mathcal{S}^*$ as given in (2.3.6), where the operator $\mathcal{T}$ is defined by (2.3.5). It is clear that $\mathcal{S}^*(\mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})) \subset \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$. Now for any $x, y \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{B})$ with $x \neq y$, we have

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|\mathcal{S}^*(x) - \mathcal{S}^*(y)\|_{\omega,c} \\
&= \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{\ominus d}(t,s) \left((\mathcal{T}x)(s) - (\mathcal{T}y)(s) \right) \Delta s \right\|_{\mathbb{B}} \\
&\leq \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} |c|^{-(t-s)/\omega} e_{\ominus \underline{d}}(t,s) \left(\sum_{k=1}^{m} |c|^{-s/\omega} |\beta_k(s)\lambda_k(s)| l_{f_k} \right. \\
&\qquad \left. \|x(s-\tau_k(s)) - y(s-\tau_k(s))\|_{\mathbb{B}} + l_h |\gamma(s)|\, |c|^{-s/\omega} \|x(s) - y(s)\|_{\mathbb{B}} \right) \Delta s \\
&\leq \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c} \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} \oplus \underline{d}\right)}(t,s) \left(\sum_{k=1}^{m} |c|^{-\tau_k(s)/\omega} \right. \\
&\qquad \left. |\beta_k \lambda_k(s)| l_{f_k} + l_h |\gamma(s)| \right) \Delta s \\
&\leq \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c} \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \int_{-\infty}^{t} \frac{e_{\ominus\left(\frac{\ln|c|}{\omega} \oplus \underline{d}\right)}(t,\sigma(s))}{1+\mu(s)\left(\frac{\ln|c|}{\omega} \oplus \underline{d}\right)} \left(\sum_{k=1}^{m} |c|^{-\tau_k^*/\omega} \right. \\
&\qquad \left. |\overline{\beta_k \lambda_k}| l_{f_k} + l_h |\overline{\gamma}| \right) \Delta s \\
&\leq \frac{\sum_{k=1}^{m} |c|^{-\tau_k^*/\omega} |\overline{\beta_k \lambda_k}| l_{f_k} + l_h |\overline{\gamma}|}{\inf_{\mathbb{T}_\omega} \left(\frac{\ln|c|}{\omega} \oplus \underline{d} \right)} \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c} \\
&< \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \|x-y\|_{\omega,c}.
\end{aligned}$$

Since (2.3.6) and (2.3.5) form a solution to the dynamic equation (2.3.7), the conclusion follows from the Banach fixed-point theorem. □

Theorem 2.3.15. *Suppose that along with the assumptions of Theorem 2.3.14, Assumption* ***A**** *also holds. Then, the solution* x *of* (2.3.7) *satisfies* $x(t) \geq 0$.

2.4 Exponential stability of the (ω, c)-periodic solutions on time scales

In Section 2.3, we established the existence and uniqueness of an (ω, c)-periodic solution for the families of delayed population models on the time scale, as given by (2.1.1) and (2.1.2), under certain conditions. In this section, we investigate whether the solution is stable, if not, identify sufficient conditions on the model coefficients that ensure its stability.

We first define the notion of exponential stability of the (ω, c)-periodic solution of (2.1.1) (or (2.1.2)), coupled with the initial function φ defined in (2.3.1), with respect to the (w, c)-norm defined in (2.2.4).

Definition 2.4.1. *The (ω, c)-periodic solution x of (2.1.1) (or (2.1.2)) with initial function φ is said to be exponentially stable if there exist positive constants M and K such that*

$$\|x(t) - x^*(t)\|_{\omega,c} \leq K e_{\ominus M}(t, t_0) \sup_{s \in [t_0 - \tau^*, t_0]_{\mathbb{T}_\omega}} \|\varphi(s) - \varphi^*(s)\|_{\omega,c} \text{ for all } t \in \mathbb{T}_\omega, \quad t \geq t_0,$$

where x^* is a solution of (2.1.1) (or (2.1.2)) corresponding to an arbitrary initial function $\varphi^* \in \mathcal{C}_{\omega,c}$, and $\tau^* = \max_{1 \leq k \leq n} \sup_{t \in \mathbb{T}_\omega} \tau_k(t)$.

Theorem 2.4.2. *Suppose that all the assumptions of Theorem 2.3.7 are satisfied. Then the (ω, c)-periodic solution of (2.1.1) is exponentially stable, provided*

$$\frac{\ln|c|}{\omega} + \underline{d} > \frac{1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}{1 - \overline{\mu}\underline{d}} \left(\sum_{k=1}^{m} |c|^{-\tau_k(t)} L_{F_k}(t) + L_H(t)\right) \quad \text{for all } \ t > t_0.$$

Proof. Let x be the (ω, c)-periodic solution of the delayed dynamic equation (2.1.1) corresponding to the initial function φ. For any initial function $\varphi^* \in \mathcal{C}_{\omega,c}$, let x^* denote the corresponding solution of (2.1.1). We now define

$$X(t) := x(t) - x^*(t).$$

Then, being a solution, X satisfies (2.1.1), and we have

$$X^{\Delta}(t) = -d(t)X(t) + \sum_{k=1}^{m}\{F_k(t, x(t - \tau_k(t))) - F_k(t, x^*(t - \tau_k(t)))\} + \{H(t, x(t)) - H(t, x^*(t))\}.$$

In consequence, for $t \geq t_0$, we have

$$X(t) = X(t_0)e_{-d}(t, t_0) + \int_{t_0}^{t} e_{-d}(t, \sigma(s)) \Bigg[\sum_{k=1}^{m} \big\{F_k(s, x(s - \tau_k(s))) - F_k(s, x^*(s - \tau_k(s)))\big\} + \big\{H(s, x(s)) - H(s, x^*(s))\big\} \Bigg] \Delta s. \quad (2.4.1)$$

As $-d \in \mathcal{R}^+$, $e_{-d}(t,t_0) > 0$ for all $t \in \mathbb{T}_\omega$. Let the functions X^* be defined as

$$X^*(t) = \|X(t)\|_{\omega,c} \quad \text{for } t \in (-\infty, t_0]_{\mathbb{T}},$$

and for $t > t_0$

$$X^*(t) = \|X(t_0)\|_{\omega,c} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(t,t_0) + \frac{1}{1-\overline{\mu}\underline{d}} \int_{t_0}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(t,s)\mathcal{F}(s)\Delta s, \tag{2.4.2}$$

where

$$\mathcal{F}(s) = \left[\sum_{k=1}^{m} \|F_k(s, x(s-\tau_k(s))) - F_k(s, x^*(s-\tau_k(s)))\|_{\omega,c} + \|H(s,x(s)) - H(s,x^*(s))\|_{\omega,c}\right]. \tag{2.4.3}$$

Then, we have $\|X(t)\|_{\omega,c} \le X^*(t)$ for all $t \in \mathbb{T}_\omega$. Taking Δ-derivative with respect to t, (2.4.2) yields

$$\begin{aligned}
X^{*\Delta}(t) &= \frac{-\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)} \left(\|X(t_0)\|_{\omega,c} + \frac{1}{1-\overline{\mu}\underline{d}}\int_{t_0}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(t_0,s)\mathcal{F}(s)\Delta s\right) e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(t,t_0) \\
&\quad + \frac{\mathcal{F}(t)}{1-\overline{\mu}\underline{d}} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}(\sigma(t),t) \\
&\le \frac{-\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)} X^*(t) + \frac{\mathcal{F}(t)}{(1-\overline{\mu}\underline{d})\left(1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)} \\
&\le \frac{-\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)} X^*(t) + \frac{1}{1-\overline{\mu}\underline{d}} \sum_{k=1}^{m} \|F_k(t,x(t-\tau_k(t))) - F_k(t,x^*(t-\tau_k(t)))\|_{\omega,c} \\
&\quad + \frac{1}{1-\overline{\mu}\underline{d}} \|H(t,x(t)) - H(t,x^*(t))\|_{\omega,c} \\
&\le \frac{-\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)} X^*(t) + \frac{1}{1-\overline{\mu}\underline{d}} \sum_{k=1}^{m} L_{F_k}(t)|c|^{-\tau_k(t)/\omega} \|X(t-\tau_k(t))\|_{\omega,c} + \frac{L_H(t)}{1-\overline{\mu}\underline{d}}\|X(t)\|_{\omega,c} \\
&\le \frac{-\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{1+\overline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)} X^*(t) + \frac{1}{1-\overline{\mu}\underline{d}} \sum_{k=1}^{m} \overline{L}_{F_k}|c|^{-\tau_k^*/\omega} \sup_{s\in[t-\tau_k(t),t]_{\mathbb{T}_\omega}} X^*(s) + \frac{\overline{L}_H}{1-\overline{\mu}\underline{d}} X^*(t),
\end{aligned}$$

where $\overline{L}_{F_k} = \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} L_{F_k}(t)$, $\overline{L}_H = \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} L_H(t)$, $\tau_k^* = \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \tau_k(t)$ when $|c| \le 1$, and $\tau_k^* = \inf_{t\in[t_0,t_0+\omega]_{\mathbb{T}_\omega}} \tau_k(t)$ when $|c| \ge 1$. Note that $1 \le 1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{d}\right) < \infty$ for all $t \in \mathbb{T}_\omega$. Thus, by the Halanay inequality (Lemma 1.4.83), there exists a positive constant M such that

$$X^*(t) \le \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} X^*(s)\right) e_{\ominus M}(t,t_0), \quad t \in (t_0,\infty)_{\mathbb{T}_\omega},$$

which implies

$$\|x(t) - x^*(t)\|_{\omega,c} \le \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} \|x(s)-x^*(s)\|_{\omega,c}\right) e_{\ominus M}(t,t_0), \quad t \in (t_0,\infty)_{\mathbb{T}_\omega}.$$

Hence, x is exponentially stable. □

Theorem 2.4.3. *Suppose that all the assumptions of Theorem 2.3.8 are satisfied. Then the (ω, c)-periodic solution of (2.3.4) is exponentially stable.*

Proof. Let x be the (ω, c)-periodic solution of (2.3.4) with the initial function φ, and x^* denote a solution of (2.3.4) corresponding to an arbitrary initial function $\varphi^* \in \mathcal{C}_{\omega,c}$. We define

$$X(t) := x(t) - x^*(t).$$

Then, being a solution, X satisfies (2.3.4), and we have

$$X^\Delta(t) = -d(t)X(t) + \sum_{k=1}^{m} \beta_k(t) \left\{ f_k(\lambda_k(t)x(t-\tau_k(t))) - f_k(\lambda_k(t)x^*(t-\tau_k(t))) \right\} - \gamma(t)\{h(x(t) - h(x^*(t))\}.$$

Consequently, for $t \geq t_0$, we have

$$\begin{aligned} X(t) = X(t_0)e_{-d}(t, t_0) + \int_{t_0}^{t} e_{-d}(t, \sigma(s)) \Bigg[\sum_{k=1}^{m} \beta_k(s) \Big\{ & f_k(\lambda_k(s)x(s-\tau_k(s))) \\ & - f_k(\lambda_k(s)x^*(s-\tau_k(s))) \Big\} - \gamma(s)\left\{ h(x(s) - h(x^*(s)) \right\} \Bigg] \Delta s. \end{aligned}$$

Define a function X^* as

$$X^*(t) = \begin{cases} \|X(t)\|_{\omega,c}, & \text{when } t \in (-\infty, t_0]_{\mathbb{T}}, \\ \|X(t_0)\|_{\omega,c} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}(t, t_0) + \frac{1}{1-\overline{\mu}\underline{d}} \int_{t_0}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}(t, s) & \\ \quad \left(\sum_{k=1}^{m} -f_k(\lambda_k(s)x^*(t-\tau_k(s)))\|_{\omega,c} |\beta_k(s)| \; \|f_k(\lambda_k(s)x(t-\tau_k(s))) \right. & \\ \quad \left. + |\gamma(s)| \; \|\{h(x(s) - h(x^*(s))\}\|_{\omega,c} \right) \Delta s, & \text{when } t > t_0. \end{cases}$$

Taking Δ-derivative with respect to t both sides, for $t > t_0$ the above expression yields

$$\begin{aligned} X^{*\Delta}(t) &\leq \frac{-\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}{1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)} X^*(t) + \frac{1}{(1-\overline{\mu}\underline{d})\left(1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)\right)} \\ &\quad \left(\sum_{k=1}^{m} |\beta_k(t)| \; \|f_k(\lambda_k(t)x(t-\tau_k(t))) - f_k(\lambda_k(t)x^*(t-\tau_k(t)))\|_{\omega,c} \right. \\ &\quad \left. + |\gamma(t)| \; \|h(x(t) - h(x^*(t))\|_{\omega,c} \right) \\ &\leq \frac{-\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}{1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)} X^*(t) + \frac{1}{(1-\overline{\mu}\underline{d})\left(1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)\right)} \\ &\quad \left(\sum_{k=1}^{m} |(\beta_k\lambda_k)(t)| l_{f_k} |c|^{-\tau_k(t)/\omega} \|X(t-\tau_k(t))\|_{\omega,c} + |\gamma(t)| l_h \; \|X(t)\|_{\omega,c} \right) \\ &\leq \frac{-\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}{1 + \overline{\mu}\left(\frac{\ln|c|}{\omega} + \underline{d}\right)} X^*(t) + \frac{1}{(1-\overline{\mu}\underline{d})\left(1 + \mu(t)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)\right)} \left(\sum_{k=1}^{m} |\beta_k\lambda_k(t)| \right. \end{aligned}$$

$$l_{f_k}|c|^{-\tau_k(t)/\omega} \sup_{s\in[t-\tau_k(t),t]_{\mathbb{T}_\omega}} X^*(s) + |\gamma(t)| l_h X^*(t)\Big).$$

In view of assumption **A5**, all the conditions of Lemma 1.4.83 are satisfied. Thus, by the Halanay inequality (Lemma 1.4.83), there exists a positive constant M such that

$$\|x(t) - x^*(t)\|_{\omega,c} \le \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} \|x(s) - x^*(s)\|_{\omega,c} \right) e_{\ominus M}(t, t_0), \quad t \in (t_0, \infty)_{\mathbb{T}_\omega}.$$

Hence, x is exponentially stable. □

Theorem 2.4.4. *Suppose that all the assumptions of Theorem 2.3.12 are satisfied. Then the (ω, c)-periodic solution of (2.1.2) is exponentially stable, provided $\underline{d} \oplus \frac{\ln|c|}{\omega} > \sum_{k=1}^{m} |c|^{-\tau_k(t)} L_{F_k}(t) + L_H(t)$ for all $t > t_0$.*

Proof. Let x be the (ω, c)-periodic solution of (2.1.2) with the initial function φ, and let x^* denote a perturbed solution of (2.1.2) corresponding to an arbitrary initial function $\varphi^* \in \mathcal{C}_{\omega,c}$.

Consider $X(t) := x(t) - x^*(t)$. Then, being a solution, X satisfies (2.1.2), and we have

$$X^\Delta(t) = -d(t)X^\sigma(t) + \sum_{k=1}^{m}\{F_k(t, x(t - \tau_k(t))) - F_k(t, x^*(t - \tau_k(t)))\} + \{H(t, x(t)) - H(t, x^*(t))\}.$$

In consequence, for $t \ge t_0$, we have

$$X(t) = X(t_0)e_{\ominus d}(t, t_0) + \int_{t_0}^{t} e_{\ominus d}(t, s)\Bigg[\sum_{k=1}^{m}\{F_k(s, x(s - \tau_k(s))) - F_k(s, x^*(s - \tau_k(s)))\} + \{H(s, x(s)) - H(s, x^*(s))\}\Bigg]\Delta s. \tag{2.4.4}$$

Let

$$X^*(t) := e_{\ominus\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}(t, t_0)\left(\|X(t_0)\|_{\omega,c} + \int_{t_0}^{t} e_{\ominus\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}(t_0, s)\mathcal{F}(s)\Delta s \right) \quad \text{for all } t > t_0,$$

where $\mathcal{F}$ is defined in (2.4.3), and $X^* = \|X\|_{\omega,c}$ when $t \le t_0$. Taking Δ-derivative of the above with respect to t, for $t > t_0$ we get

$$\begin{aligned} X^{*\Delta}(t) &= \frac{-\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}X^*(t) + \frac{\mathcal{F}(t)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)} \\ &\le \frac{-\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}X^*(t) + \frac{1}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}\Bigg[\sum_{k=1}^{m}\overline{L}_{F_k}|c|^{-\tau_k{}^*/\omega} \\ &\qquad \sup_{s\in[t-\tau_k(t),t]_{\mathbb{T}_\omega}} X^*(s) + \overline{L}_H X^*(t)\Bigg]. \end{aligned}$$

Thus, by the Halanay inequality (Lemma 1.4.83), there exists a positive constant M such that

$$X^*(t) \le \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} X^*(s) \right) e_{\ominus M}(t, t_0), \quad t \in (t_0, \infty),$$

which yields

$$\|x(t) - x^*(t)\|_{\omega,c} \le \left(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} \|x(s) - x^*(s)\|_{\omega,c} \right) e_{\ominus M}(t, t_0), \quad t \in (t_0, \infty).$$

Hence, x is exponentially stable. □

Theorem 2.4.5. *Suppose that all the assumptions of Theorem 2.3.14 are satisfied. Then the (ω, c)-periodic solution of (2.3.7) is exponentially stable.*

Proof. Let us define $X(t) := x(t) - x^*(t)$, where x is the (ω, c)-periodic solution of the delayed dynamic equation (2.3.7) with the initial function φ, and x^* is the perturbed solution of (2.3.7) corresponding to an arbitrary initial function $\varphi^* \in \mathcal{C}_{\omega,c}$. Then, being a solution, X satisfies (2.3.7), and we have

$$X^\Delta(t) = -d(t)X^\sigma(t)+\sum_{k=1}^{m}\beta_k(t)\left\{f_k(\lambda_k(t)x(t-\tau_k(t))) - f_k(\lambda_k(t)x^*(t-\tau_k(t)))\right\}-\gamma(t)\{h(x(t)-h(x^*(t))\}.$$

Consequently, for $t \ge t_0$, we have

$$\begin{aligned} X(t) = X(t_0)e_{\ominus d}(t, t_0) + \int_{t_0}^{t} e_{\ominus d}(t, s)\Bigg[\sum_{k=1}^{m}\beta_k(s)\Big\{f_k(\lambda_k(s)x(s-\tau_k(s))) \\ - f_k(\lambda_k(s)x^*(s-\tau_k(s)))\Big\} - \gamma(s)\{h(x(s) - h(x^*(s))\}\Bigg]\Delta s. \end{aligned}$$

Defining $X^* := \|X\|_{\omega,c}$ for $t \le t_0$, and

$$\begin{aligned} X^*(t) &:= \|X(t_0)\|_{\omega,c}\; e_{\ominus\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}(t, t_0) \\ &\quad + \int_{t_0}^{t}\Bigg[\sum_{k=1}^{m}|\beta_k(s)|\; \|f_k(\lambda_k(s)x(s-\tau_k(s))) - f_k(\lambda_k(s)x^*(s-\tau_k(s)))\|_{\omega,c} \\ &\quad + |\gamma(s)|\; \|h(x(s) - h(x^*(s))\|_{\omega,c}\Bigg]e_{\ominus\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}(t, s)\Delta s, \quad \text{for } t > t_0. \end{aligned}$$

We can obtain

$$\begin{aligned} X^{*\Delta}(t) &= \tfrac{-\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}X^*(t) + \tfrac{1}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}\Bigg[\sum_{k=1}^{m}|\beta_k(t)|\|f_k(\lambda_k(t)x(t-\tau_k(t))) \\ &\quad -f_k(\lambda_k(t)x^*(t-\tau_k(t)))\|_{\omega,c} + |\gamma(t)|\; \|h(x(t) - h(x^*(t))\|_{\omega,c}\Bigg] \\ &\le \tfrac{-\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}X^*(t) + \tfrac{1}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}\Bigg[\sum_{k=1}^{m} l_{f_k}|(\beta_k\lambda_k)(t)||c|^{-\tau_k(t)/\omega} \\ &\quad \|X(t-\tau_k(t))\|_{\omega,c} + l_h|\gamma(t)|\; \|X(t)\|_{\omega,c}\Bigg] \\ &\le \tfrac{-\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}X^*(t) + \tfrac{1}{1+\mu(t)\left(\underline{d}\oplus\frac{\ln|c|}{\omega}\right)}\Bigg[\sum_{k=1}^{m} l_{f_k}|c|^{-\tau_k(t)/\omega}|(\beta_k\lambda_k)(t)| \end{aligned}$$

$$\left. \sup_{s \in [t-\tau_k(t),t]_{\mathbb{T}_\omega}} X^*(s) + l_h|\gamma(t)|X^*(t) \right] \quad \text{for all} \quad t > t_0.$$

In view of assumption **A8**, we can apply Lemma 1.4.83. Thus, by the Halanay inequality (Lemma 1.4.83), there exists a positive constant M such that

$$\|x(t) - x^*(t)\|_{\omega,c} \leq \left(\sup_{s \in [t_0-\tau^*,t_0]_{\mathbb{T}_\omega}} \|x(s) - x^*(s)\|_{\omega,c} \right) e_{\ominus M}(t, t_0), \quad t \in (t_0, \infty).$$

Hence, x is exponentially stable. □

2.5 Some applications and numerical examples

In this section, we explore some applications of the results discussed in previous sections of this chapter and present numerical examples based on these findings. Similar to the previous section, we have used the notations $\overline{\alpha}$ to denote $\sup_{t \in [t_0,t_0+\omega]_{\mathbb{T}_\omega}} \alpha(t)$ and $\underline{\alpha}$ to denote $\inf_{t \in [t_0,t_0+\omega]_{\mathbb{T}_\omega}} \alpha(t)$, where t_0 is the smallest nonnegative element of $\mathbb{T}_\omega$.

Note that the nonlinear terms in the Nicholson, Lasota–Ważewska, and Mackey– Glass kind models are xe^{-x}, e^{-x}, and $\frac{x}{1+x^n}$, $n > 1$, respectively. Clearly, these functions are Lipschitz continuous on $\mathbb{R}^+ = [0, \infty)$. Thus, Theorems 2.3.8, 2.3.14, 2.4.3, and 2.4.5 can be applied to these models if other necessary as well as sufficient conditions are also taken into account. This leads us to state the following results.

Theorem 2.5.1. *Consider the following dynamic equation on $\mathbb{T}_\omega$ that comprises variable delay:*

$$x^\Delta(t) = -d(t)x(t) + \sum_{k=1}^{3} \beta_k(t) f_k(\lambda_k(t)x(t - \tau_k(t))) - \gamma(t)h(x(t)), \quad t > t_0 \geq 0 \tag{2.5.1}$$

*with a linear harvesting/immigration function $h(\cdot)$ and three nonlinear recruitment functions, respectively of Nicholson, Lasota–Ważewska, and Mackey–Glass kinds. Specifically, consider the global Lipschitz functions $f_1 = xe^{-x}$, $f_2 = e^{-x}$, and $f_3 = \frac{x}{1+x^n}$ for $n(> 1)$ to be a fixed natural number. For $0 < c \leq e^{\frac{\omega}{\overline{\mu}}(1-\underline{d}\overline{\mu})}$, if all the coefficients of (2.5.1) satisfies condition **A1**, $\underline{d} > 0$, $\frac{\ln|c|}{\omega} + \underline{d} > 0$, and $-(\overline{d} + \mathcal{L}_h) \in \mathcal{R}^+$ (where $\mathcal{L}_h$ is defined with condition **A***), then (2.5.1) has a unique, nonnegative (ω, c)-periodic solution provided that the condition **A5** holds true. This solution will also be asymptotically exponentially stable.*

Corollary 2.5.2. *Consider the Nicholson blowflies model on $\mathbb{T}_\omega^{0+}$, given by*

$$x^\Delta(t) = -d(t)x(t) + \beta(t)x(t - \tau(t))e^{-\lambda(t)x(t-\tau(t))} \tag{2.5.2}$$

with the rd-continuous functions d, β, τ and λ such that $d \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$, $\beta \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$, $\tau \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega^{0+}, \Pi_{\mathbb{T}} \cup [0, \infty))$, and $\lambda \in P_{(\omega,1/c)}(\mathbb{T}_\omega^{0+}, \mathbb{R}^+)$, $0 < c \leq 1$. Then, a sufficient condition for the *existence of a unique, bounded, nonnegative, asymptotically stable (ω, c)-periodic solution is that $-\overline{d} \in \mathcal{R}^+$ and $\frac{1}{\frac{\ln|c|}{\omega}+\underline{d}} + \overline{\mu} < |c|^{\frac{\overline{\tau}}{\omega}} \frac{1-d\overline{\mu}}{\overline{\beta}}$.*

Proof. (2.5.2) can be written as

$$x^{\Delta}(t) = -dx(t) + \delta(t)\lambda(t)x(t-\tau)e^{-\lambda(t)x(t-\tau(t))},$$

where $\delta = \frac{\beta}{\lambda} \in \mathrm{P}^{+}_{\omega,c}(\mathbb{T}, \mathbb{R})$. As xe^{-x} is Lipschitz continuous in $\mathbb{R}^+$ with Lipschitz constant 1, the statement follows from Theorem 2.5.1. □

Corollary 2.5.3. *Consider the Lasota–Ważewska model*

$$x^{\Delta}(t) = -d(t)x(t) + \beta(t)e^{-\lambda(t)x(t-\tau)} \tag{2.5.3}$$

with rd-continuous functions d, τ, β, λ such that $d \in \mathrm{P}_{\omega,1}(\mathbb{T}^{0+}_{\omega}, \mathbb{R}^+)$, $\tau \in \mathrm{P}_{\omega,1}(\mathbb{T}^{0+}_{\omega}, \Pi_{\mathbb{T}} \cup [0,\infty))$, $\beta \in \mathrm{P}_{\omega,c}(\mathbb{T}^{0+}_{\omega}, \mathbb{R}^+)$, and $\lambda \in P_{(\omega,1/c)}(\mathbb{T}^{0+}_{\omega}, \mathbb{R}^+)$, $0 < c \leq 1$. Then there exists a unique, bounded, nonnegative, and exponentially stable (ω, c)-periodic solution of (2.5.3) whenever $-\overline{d} \in \mathcal{R}^+$ and $\frac{1}{\frac{\ln|c|}{\omega}+\underline{d}} + \overline{\mu} < |c|^{\frac{\overline{\tau}}{\omega}} \frac{1-d\overline{\mu}}{\overline{\beta\lambda}}$.

Remark 2.5.4. *If $\beta_3(t) \equiv 0$, then (2.5.1) represents a mixed blowflies model, where the Nicholson model is modified by adding eggs and harvesting adult flies every day. The rate of addition of the eggs corresponds to the second term in the summation, which is a Lasota–Ważewska type term.*

Remark 2.5.5. *If we consider $\beta_2(t) \equiv 0$, then the model represented by (2.5.1) can be interpreted in at least two ways, either as a population of flies or as a red blood cell model, with two nonlinear terms of production.*

Remark 2.5.6. *If we consider $\beta_1(t) \equiv 0$, then (2.5.1) will represent a mixed circulating blood cells model with two nonlinear production control functions: one Mackey–Glass and the other Lasota–Ważewska.*

Theorem 2.5.7. *Consider the following equation on the time scale $\mathbb{T}_{\omega}$ with a delay multiple of c-period:*

$$x^{\Delta}(t) = -d(t)x^{\sigma}(t) + \beta(t)x(t-m\omega)e_{-1}(\lambda(t)x(t-m\omega), t_0), \tag{2.5.4}$$

where the regressive function d and β are positive ω-periodic, λ is a nonnegative $(\omega, 1/c)$-periodic function, $c \in \mathbb{R}^+$, d, β, and λ are continuous and delta differentiable on $\mathbb{T}^{0+}_{\omega}$, $d(t) \geq \underline{d} > 0$, $\overline{d} \in \mathcal{R}^+$, $m \in \mathbb{N}$, and $\lambda(t)x(t-m\omega) \in \mathbb{T}$ for all $t \in \mathbb{T}^{0+}_{\omega}$. Additionally, assume that $0 < \frac{|c|^{-m}M\overline{\beta}}{\frac{\ln|c|}{\omega}\oplus\underline{d}} < 1$. Then there exists a unique exponentially stable nonnegative (ω, c)-periodic solution of (2.5.4).

Proof. Here, $F(t, x) = \beta(t)xe_{-1}(\lambda(t)x, t_0) \in \mathrm{P}_{\omega,c}(\mathbb{T}_{\omega}, \mathbb{B})$. Also, applying the mean value theorem on time scales, for some $\delta \in [x, y] \cap \mathbb{T}^+$, we get

$$\|F(t,x) - F(t,y)\|_{\mathbb{B}} \;\; = \;\; \frac{\beta(t)}{|\lambda(t)|}|e_{-1}(\delta, t_0) - \delta^{\sigma}e_{-1}(\delta, t_0)|\|\lambda x - \lambda y\|_{\mathbb{B}}$$

$$\begin{aligned} &\leq \beta(t)(e^{-(\delta-t_0)}+\delta e^{-(\delta-t_0)})\|x-y\|_{\mathbb{B}} \\ &\leq M\beta(t)\|x-y\|_{\mathbb{B}}, \end{aligned}$$

for $M=(1+t_0)e^{-2t_0}$. The rest is a consequence of Theorem 2.3.12, Theorem 2.3.13 and Theorem 2.4.4. □

Remark 2.5.8. *If we consider $\mathbb{T}_\omega=\mathbb{R}$, then (2.5.4) will represent the classic (ω,c)-periodic Nicholson's model on the time scale with a delay multiple of c-period ω. Therefore, we have a sufficient condition for the existence of a unique nonnegative exponentially stable (ω,c)-periodic solution for the same, provided that $|c|^{-m}\overline{\beta}<\frac{\ln|c|}{\omega}+\underline{d}$.*

Example 2.5.9. *Consider the delay dynamic equation*

$$x^{\Delta}(t)=-d(t)x(t)+\sum_{k=1}^{2}\beta_k(t)f_k(\lambda_k(t)x(t-\tau_k(t)))-\gamma(t)h(x(t)), \tag{2.5.5}$$

where

$$\begin{gathered} d(t)=0.45-\frac{m\ln|c|}{2\pi}+0.4|\sin(kt)|,\ f_1=xe^{-x},\ f_2=e^{-x}, \\ \beta_1(t)=c^{mt/2\pi}(0.17+0.012e^{\cos(mt)}),\ \beta_2(t)=c^{mt/2\pi}(0.19+0.06\sin(mt)), \\ \lambda_1(t)=c^{-mt/2\pi}(0.13-0.1\sin(mt)),\ \lambda_2(t)=c^{-mt/2\pi}(0.24+0.13\cos(mt)), \\ \gamma(t)=0.0157\ e^{\sin\cos(mt)},\ \textit{and } h(x)=x. \end{gathered}$$

We can consider any $m\in(0,\pi]$ such that $\frac{2\pi}{m}+t\in\mathbb{T}$ for all $t\in\mathbb{T}$ and $c\in[\frac{1}{2},1]$. (2.5.5) represents a model of a Lucila Cuprina culture operating on a time scale $\mathbb{T}$ with a harvesting function and two nonlinear recruitment functions. This modified Nicholson blowflies model incorporates daily addition of eggs (corresponding to the term of Lasota–Ważewska kind) and includes daily harvesting of adult flies as well. As $\omega=\frac{2\pi}{m}$ (or its multiple), computing all bounds, we get

$$\begin{gathered} \underline{d}=0.45-\frac{m\ln|c|}{2\pi},\ \overline{d}=0.85-\frac{m\ln|c|}{2\pi},\ |\overline{\beta_1\lambda_1}|=0.026341,\ |\overline{\beta_2\lambda_2}|=0.0925, \\ |\overline{\gamma}|=0.042677,\ l_{f_1}=l_{f_2}=l_h=\mathcal{L}_h=1. \end{gathered}$$

Now, for any time scale $\mathbb{T}$ with $\overline{\mu}<\frac{1}{\underline{d}}$, and any delay terms $\tau:\mathbb{T}\to\Pi_{\mathbb{T}}^{+}$ with upper bound 5, we have

$$\begin{aligned} 0<\frac{1+\overline{\mu}\left(\frac{m\ln|c|}{2\pi}+\underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{m\ln|c|}{2\pi}+\underline{d}\right)\right)\left(\frac{m\ln|c|}{2\pi}+\underline{d}\right)}&\left(\sum_{k=1}^{2}|c|^{-m\overline{\tau}_k/2\pi}|\overline{\beta_k\lambda_k}|+|\overline{\gamma}|\right) \\ &<\frac{0.9137(1+0.45\overline{\mu})}{(1+0.45\underline{\mu})\left(1-\overline{\mu}\left(0.45-\frac{m\ln|c|}{2\pi}\right)\right)}. \end{aligned}$$

Thus,

- *For $\mathbb{T} = \mathbb{R}$:*

$$0 < \frac{1+\overline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}\left(\sum_{k=1}^{2} l_{f_k}|c|^{-\overline{\tau}_k/\omega}|\overline{\beta_k\lambda_k}|+l_h|\overline{\gamma}|\right)$$
$$< 0.9137 < 1.$$

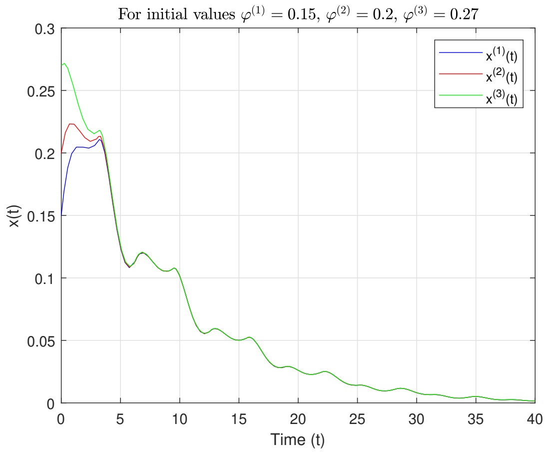


Figure 2.1: The solution trajectories of system (2.5.5) on the time scale $\mathbb{R}$ with coefficients given in Example 2.5.9 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes the $(2\pi, 0.5)$-periodic solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $c = 0.5$, $m = 1$, and delay terms as $\tau_1 = 1.73 + 2.39|\sin(\cos(mt))|$ and $\tau_2 = 3.87 + |\cos(\sin(mt))|$.

- *For $\mathbb{T} = h\mathbb{Z}$, $\quad 0 < h \leq 0.128 < \dfrac{0.0863}{\underline{d}}$:*

$$0 < \frac{1+\overline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}\left(\sum_{k=1}^{2} l_{f_k}|c|^{-\overline{\tau}_k/\omega}|\overline{\beta_k\lambda_k}|+l_h|\overline{\gamma}|\right)$$
$$< \frac{0.9137}{1-h\underline{d}} < 1.$$

- *For $\mathbb{T} = \bigcup_{n\in\mathbb{Z}}[n(a+b), n(a+b)+b]$, $\quad a \neq -b, \quad 0 < a \leq 0.097 < \dfrac{0.0863}{0.216225+\underline{d}}$:*

$$0 < \frac{1+\overline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{d}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{d}\right)}\left(\sum_{k=1}^{2} l_{f_k}|c|^{-\overline{\tau}_k/\omega}|\overline{\beta_k\lambda_k}|+l_h|\overline{\gamma}|\right)$$

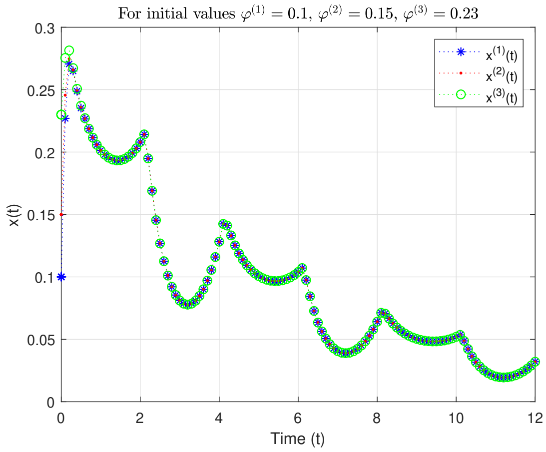


Figure 2.2: The solution trajectories of system (2.5.5) on the time scale $0.1\mathbb{Z}$ with coefficients given in Example 2.5.9 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes the $(4, 0.5)$-periodic solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $c = 0.5$, $m = \pi/2$, and delay terms as $\tau_1 = 0.5$ and $\tau_2 = 0.2$.

$$< \frac{(1 + 0.45a)}{(1 - a\underline{d})} \times 0.9137 < 1.$$

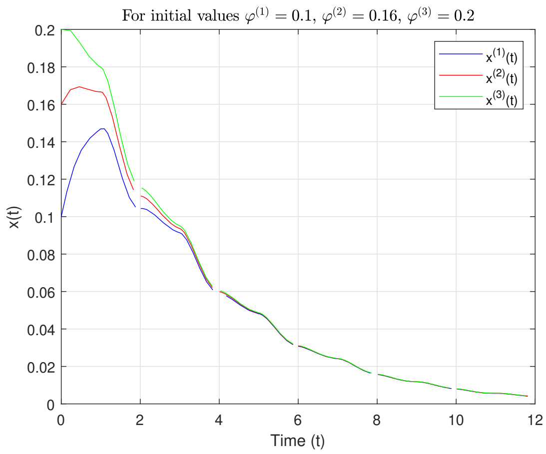


Figure 2.3: The solution trajectories of system (2.5.5) on the time scale $\bigcup\limits_{n\in\mathbb{Z}} [n(a + b), n(a + b) + b]$ with coefficients given in Example 2.5.9 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes the $(2, 0.5)$-periodic solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $a = 0.1$, 1.9, $c = 0.5$, $m = \pi$, and delay terms as $\tau_1 = 2$ and $\tau_2 = 4$.

Consequently, all the conditions of Theorem 2.3.4 hold. Moreover, for all considered cases, we have $\overline{\mu} < 0.8983 < \frac{1}{\overline{d}+\overline{\gamma}}$*, and this implies* $-(\overline{d} + \mathcal{L}_h\overline{\gamma}) \in \mathcal{R}^+$*. Therefore, all considered model have a unique non-*

negative (ω, c)*-periodic solution on the considered translation-invariant time scales, which is also asymptotically exponentially stable. See Figures (2.1), (2.2), and (2.3), which show* $(2\pi, 0.5)$, $(4, 0.5)$, *and* $(2, 0.5)$ *periodic solution trajectories on the time scales* $\mathbb{R}$, $0.1\mathbb{Z}$, *and* $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1.9]$, *respectively. It is evident from the figures that the solutions are also exponentially stable.*

Example 2.5.10. *Consider the following example which corresponds to a model for circulating blood cells on a periodic time scale* $\mathbb{T}$ *with an input function representing immigration of cells and two nonlinear production control functions given by Mackey–Glass and Lasota–Ważewska:*

$$x^{\Delta}(t) = -d(t)x(t) + \sum_{k=1}^{2} \beta_k(t) f_k(\lambda_k(t)x(t-\tau_k(t))) + \gamma(t)h(x(t)), \tag{2.5.6}$$

where

$$\begin{gathered} d(t) = 0.5\ m - \frac{m\ln|c|}{\pi} + \cos^2(mt),\ f_1 = \frac{x}{1+x^4},\ f_2 = e^{-x}, \\ \beta_1(t) = c^{mt/\pi}(0.25 + 0.0025\cos(2mt)),\ \beta_2(t) = c^{mt/\pi}(0.37 + 0.07\sin^2(mt)), \\ \lambda_1(t) = c^{-mt/\pi}(0.11 + 0.2e^{-\sin^2(mt)}),\ \lambda_2(t) = c^{-mt/\pi}(0.5 + 0.01\cos(2mt)), \\ \gamma(t) = 0.05e^{\sin^2(mt)},\ \text{and } h(x) = x. \end{gathered}$$

Here, $m \geq 1$ *and* $c \in \mathbb{C}^*$ *such that* $\frac{\pi}{m} + t \in \mathbb{T}$, $t \in \mathbb{T}$, *and* $1 \leq c \leq 4.8$. *Therefore, we have*

$$\begin{gathered} \omega = \frac{\pi}{m},\ \underline{d} = 0.5m - \frac{m\ln|c|}{\pi},\ \overline{d} = 1 + 0.5m - \frac{m\ln|c|}{\pi},\ |\overline{\beta_1\lambda_1}| = 0.078275, \\ |\overline{\beta_2\lambda_2}| = 0.15811349,\ |\overline{\gamma}| = 0.135914091,\quad l_{f_1} = l_{f_2} = l_h = 1. \end{gathered}$$

This implies that for any time scale $\mathbb{T}$ *with* $\overline{\mu} < \frac{1}{\overline{d}} < \frac{1}{\underline{d}}$, *and any choice of delay functions* $\tau_k : \mathbb{T} \to \Pi_{\mathbb{T}_\omega^{0+}}$,

$$\begin{aligned} 0 &< \frac{1 + \overline{\mu}\left(\frac{m\ln|c|}{\pi} + \underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1 + \underline{\mu}\left(\frac{m\ln|c|}{\pi} + \underline{d}\right)\right)\left(\frac{m\ln|c|}{\pi} + \underline{d}\right)} \left(\sum_{k=1}^{2} |c|^{-m\underline{\tau_k}/\pi}|\overline{\beta_k\lambda_k}| + |\overline{\gamma}|\right) \\ &< \frac{0.450577581\,(1 + 0.5m\overline{\mu})}{\left(1 - \overline{\mu}(0.5m - \frac{m\ln|c|}{\pi})\right)(1 + 0.5m\underline{\mu})\,m}. \end{aligned}$$

Thus,

- *For* $\mathbb{T} = \mathbb{R}$*:*

$$0 < \frac{1 + \overline{\mu}\left(\frac{\ln|c|}{\omega} + \underline{d}\right)}{(1-\overline{\mu}\underline{d})\left(1 + \underline{\mu}\left(\frac{\ln|c|}{\omega} + \underline{d}\right)\right)\left(\frac{\ln|c|}{\omega} + \underline{d}\right)} \left(\sum_{k=1}^{2} l_{f_k}|c|^{-\underline{\tau_k}/\omega}|\overline{\beta_k\lambda_k}| + l_h|\overline{\gamma}|\right)$$

$$< \frac{0.450577581}{m} < 1.$$

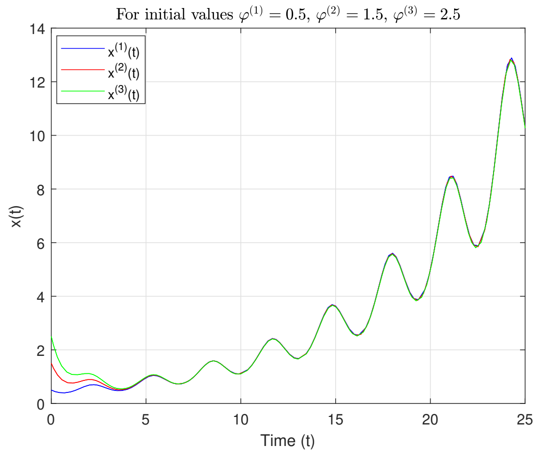


Figure 2.4: The solution trajectories of system (2.5.6) on the time scale $\mathbb{R}$ with coefficients given in Example 2.5.10 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes the $(\pi, 1.5)$-periodic solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $c = 1.5$, $m = 1$, and delay terms as $\tau_1(t) = 2 + \cos^2(mt)$ and $\tau_2(t) = 3 + |\sin(2mt)|$.

- *For* $\mathbb{T} = h\mathbb{Z}$, $\quad h < \frac{1}{\underline{d}} < \dfrac{m - 0.450577581}{m\underline{d}}$*:*

$$0 < \frac{1 + \overline{\mu}\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)}{(1 - \overline{\mu}\underline{d})\left(1 + \underline{\mu}\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)\right)\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)}\left(\sum_{k=1}^{2} l_{f_k}|c|^{-\underline{\tau_k}/\omega}|\overline{\beta_k\lambda_k}| + l_h|\overline{\gamma}|\right)$$
$$< \frac{0.450577581}{(1 - h\underline{d})m} < 1.$$

- *For* $\mathbb{T} = \bigcup_{n\in\mathbb{Z}}[n(a+b), n(a+b)+b]$*,* $a \neq -b$*,* $a < h < \frac{1}{\underline{d}} < \dfrac{m - 0.450577581}{m(0.225288791 + \underline{d})}$*:*

$$0 < \frac{1 + \overline{\mu}\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)}{(1 - \overline{\mu}\underline{d})\left(1 + \underline{\mu}\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)\right)\left(\dfrac{\ln|c|}{\omega} + \underline{d}\right)}\left(\sum_{k=1}^{2} l_{f_k}|c|^{-\underline{\tau_k}/\omega}|\overline{\beta_k\lambda_k}| + l_h|\overline{\gamma}|\right)$$
$$< \frac{(1 + 0.5ma)}{(1 - a\underline{d})m} \times 0.450577581 < 1.$$

Moreover, for the immigration term $h(\cdot)$, we can always choose a positive $\mathcal{L}_h < \frac{\overline{\mu} - \overline{d}}{\overline{\gamma}}$ that satisfies condition **A***. *Consequently, the model considered has a unique positive (ω, c)-periodic solution on the given time*

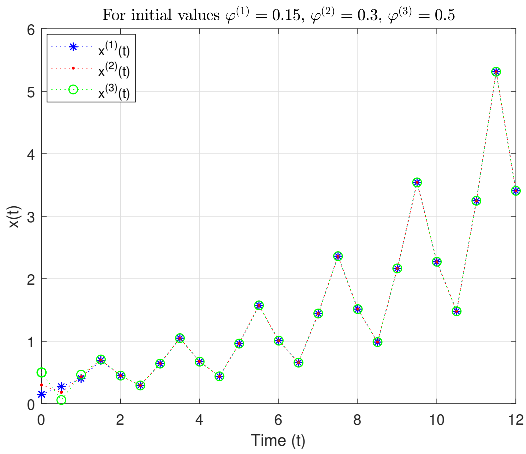


Figure 2.5: The solution trajectories of system (2.5.6) on the time scale $h\mathbb{Z}$ with coefficients given in Example 2.5.10 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes the $(2, 1.5)$-periodic solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $h = 0.5$, $c = 1.5$, $m = \pi/2$, and delay terms as $\tau_1(t) = 1$ and $\tau_2(t) = 0.5$.

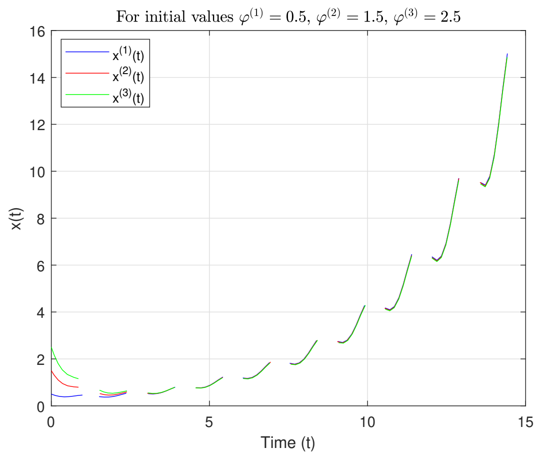


Figure 2.6: The solution trajectories of system (2.5.6) on the time scale $\bigcup_{n\in\mathbb{Z}}[n(a+b), n(a+b)+b]$ with coefficients given in Example 2.5.10 corresponding to different initial values, where for $k = 1, 2, 3$, $x^{(k)}(t)$ denotes $(1.5, 1.5)$-periodic the solution of the system corresponding to the initial term $\varphi^{(k)}(t)$. We have taken $a = 0.5$, $b = 1$, $c = 1.5$, $m = \pi/1.5$, and delay terms as $\tau_1(t) = 5$ and $\tau_2(t) = 2.5$.

scales, and this solution is also asymptotically exponentially stable. See Figures (2.4), (2.5), and (2.6), which show $(\pi, 1.5)$, $(2, 1.5)$, *and* $(1.5, 1.5)$ *periodic solution trajectories on the time scales* $\mathbb{R}$, $0.5\mathbb{Z}$, *and* $\bigcup_{n\in\mathbb{Z}}[1.5n, 1.5n+1]$, *respectively. It is evident from the figures that the solutions are also exponentially stable.*

2.6 Conclusion

Bridging the gap between continuous and discrete systems, in this chapter, the concept of (ω, c)-periodic functions on periodic time scales has been discussed. Several families of delayed population models have been investigated for the existence, uniqueness, and exponential stablity of (ω, c)-periodic solutions. In particular, Nicholson-type and Lasota–Ważewska type delayed systems, and population models combining Nicholson and Lasota–Ważewska type nonlinear terms, have been analyzed for the existence of stable (ω, c)-periodic solutions. Numerical examples are presented to validate the theoretical results and to illustrate them graphically, confirming that the solutions exhibit the expected (ω, c)-periodic behavior. These findings provide a framework for further investigations into more general delayed population models on time scales.

Chapter 3

Existence and Stability of (ω, c)-Periodic Mild Solutions for Nonautonomous Abstract Dynamic Equations on Time Scales [1]

This chapter investigates nonautonomous abstract dynamic equations on ω-periodic time scales, focusing on the existence of exponentially stable (ω, c)-periodic mild solutions. To establish the results we employ a fixed point theorem, Gronwall's inequality, semigroup theory, dichotomy theory, properties of (ω, c)-periodicity, and calculus on time scales. This approach provides new criteria to study the behaviour of (ω, c)-periodic solutions in various models.

[1] The content of this chapter has been communicated with title "*Exponential stability of* (ω, c)*-periodic solutions for nonautonomous abstract dynamic equations on time scales with applications*".

3.1 Introduction

The analysis carried out in Chapter 2 demonstrates that the notion of (ω, c)-periodicity provides an effective framework for studying qualitative properties of dynamic equations on time scales, particularly in the context of delayed population models formulated in finite-dimensional spaces. While these results highlight the relevance of (ω, c)-periodic solutions for modeling hybrid continuous–discrete dynamics, many phenomena arising in applied sciences, such as evolution equations, functional dynamic equations, and systems governed by linear operators on time scales, are naturally posed in infinite-dimensional or abstract settings. In such cases, classical solutions are often no longer adequate and must be replaced by weaker notions, such as mild solutions, with the analysis relying heavily on operator-theoretic techniques [119, 120, 121, 123]. In this direction, the theory of (ω, c)-periodic solutions in continuous time has been developed for abstract differential equations. In particular, assuming $\mathcal{A}(t)$ satisfies the Acquistapace-Terreni conditions and the associated homogeneous problem exhibits an integrable dichotomy, Abadias et al.[154] recently investigated a class of semilinear abstract differential equation

$$x'(t) = \mathcal{A}x(t) + f(t, x(t), \varphi[\alpha(t, x(t))]), \quad t \in \mathbb{R},$$

and established results for existence of unique (ω, c)-periodic mild solutions. However, the literature currently lacks such an investigation for the dynamic equation on time scales (DETS). Motivated by this gap, the present chapter focuses on employing the concept of (ω, c)-periodic functions on time scales to investigate the existence of unique (ω, c)-periodic mild solutions of abstract dynamic equation on time scales. In particular, we consider the nonautonomous abstract dynamic equation on an ω-periodic time scale $\mathbb{T}_\omega$, given by

$$x^\Delta(t) = \mathcal{A}x(t) + f(t), \quad t \in \mathbb{T}_\omega,$$

where $\mathcal{A}$ is taken as the infinitesimal generator of a C_0-semigroup of operators defined on $\mathbb{T}_\omega$, and establish conditions that guarantee the existence of (ω, c)-periodic mild solutions. However, this assumption requires $\mathcal{A}$ to be time independent and the underlying time scale $\mathbb{T}_\omega$ to possess a semigroup structure, which significantly restricts the scope of the study. For instance, mixed time scales of the form $\bigcup_{k=0}^{\infty}[(a+b)k, (a+b)k+a]$, which arise naturally in many practical applications, do not satisfy this requirement. To overcome this limitation, we extend our analysis by allowing $\mathcal{A}$ to be nonautonomous and considering it to be the generator of an evolution family of operators (EFO). We first investigate the case where this family is exponentially stable and then broaden our analysis by allowing the EFO to admit a general dichotomy on time scales. Since the notion of an integrable dichotomy generalizes exponential stability, this framework enables us to obtain regularity results for nonautonomous DETS under more general assumptions. Consequently, we further extend our analysis to the cases where the EFO associated with the homogeneous linear problem admits an integrable dichotomy. Furthermore, we consider the semilinear abstract dynamic equation on $\mathbb{T}_\omega$, given by

$$x^\Delta(t) = \mathcal{A}x(t) + f(t, x(t)),\ t \in \mathbb{T}_\omega,$$

and establish criteria sufficient for the existence and uniqueness of an (ω, c)-periodic mild solution of the above. We also examine the stability of the (ω, c)-periodic solutions of the considered classes of abstract DETS.

Since abstract dynamic equations provide a more general framework, the results obtained in this chapter naturally encompass finite-dimensional systems as particular cases. In this sense, the present work extends and generalizes the results of Chapter 2, whereas the converse does not hold. To further illustrate this inclusion, we consider some dynamic equations on finite-dimensional spaces and validate our theoretical results in these cases. In particular, we investigate interesting models for the existence of a unique and stable (ω, c)-periodic mild/exact solution on $\mathbb{T}_\omega$. These models include the Lasota–Ważewska model [12], Hopfield neural networks (HNN), the Keynesian-Cross model [155], a cellular neural network (CNN), and a second-order dynamic equation on time scales. All the models mentioned are well-established in their respective areas of study. For instance, the Lasota-Ważewska model, an ecological model, is popularly used to study the survival of animal red blood cells. HNNs are recurrent neural networks with many applications, such as optimization, memory association, pattern recognition, etc. The Keynesian-Cross model (or expenditure-output model) is an economic model that shows how the level of aggregate expenditure varies with the level of economic output and is popularly used to identify the equilibrium point in a nation's real gross domestic product. CNN, a parallel computing paradigm, finds applications in image processing, 3D surface analysis, solving PDEs, reducing non-visual problems to geometric maps, modeling biological vision, and other sensory-motor organs. We also provide examples for specific time scales and offer graphical representations to support our theoretical results.

The oraganization of this chapter is as follows. In Section 3.2, we investigate the existence and stablity of (ω, c)-periodic solutions for the abstract dynamic equations under consideration. Section 3.3 is devoted to applications of the obtained results, where illustrative examples on different time scales are presented together with graphical analyses to elucidate the theoretical findings. Finally, concluding remarks are given at the end of the chapter.

3.2 Main Results

In this section, we establish the existence and exponential stability of a unique (ω, c)-periodic solution for certain abstract dynamic equations on $\mathbb{T}_\omega$ under appropriate sufficient conditions.

Theorem 3.2.1. *Let $\mathbb{T}$ be an ω-periodic STS. Consider the linear dynamic equation on $\mathbb{T}$ given by*

$$x^{\Delta}(t) = \mathcal{A}x(t) + f(t), \quad t \geq t_0 \quad \text{and} \quad x(t_0) \in \mathfrak{D}(\mathcal{A}), \tag{3.2.1}$$

where $\mathfrak{D}(\mathcal{A})$ denotes the domain of $\mathcal{A}$. Assume that $\mathcal{A}$ generates a C_0-semigroup $\mathscr{F}$, and for $c \in \mathbb{F}^*_{\mathbb{B}}$ such that $e^{-\omega/\bar{\mu}} < |c| < e^{\omega/\bar{\mu}}$ (where $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$), $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$. Then (3.2.1) admits a unique (ω, c)-periodic mild solution. Also, if we consider $\mathscr{F}$ to be exponentially stable with an exponential convergence rate α (> 0) such that $\ominus\alpha < \frac{\ln|c|}{\omega}$, then this solution will be exponentially stable.

Proof. In view of Definition 1.4.43, the function defined by

$$x(t) = \mathscr{F}(t - t_0)x(t_0) + \int_{t_0}^{t} \mathscr{F}(t - \sigma(s))f(s)\Delta s, \quad t \geq t_0 \tag{3.2.2}$$

is a mild solution of (3.2.1). We need to show that it is (ω, c)-periodic. Let us consider

$$u(t) := \int_{-\infty}^{t} \mathscr{F}(t - \sigma(s))f(s)\Delta s.$$

Then, using w-periodicity of $\mathbb{T}$ and (ω, c)-periodicity of f, we obtain

$$\begin{aligned} u(t+\omega) &= \int_{-\infty}^{t+\omega} \mathscr{F}(t + \omega - \sigma(s))f(s)\Delta s = \int_{-\infty}^{t} \mathscr{F}(t + \omega - \sigma(s + \omega))f(s + \omega)\Delta s \\ &= cu(t), \end{aligned}$$

implies that $u \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$. Now, for a given $t_0 \in \mathbb{T}$, we have $u(t_0) = \int_{-\infty}^{t_0} \mathscr{F}(t_0 - \sigma(s))f(s)\Delta s$. Therefore, using the property of semigroup, for any $t \geq t_0$, we can obtain $\mathscr{F}(t - t_0)u(t_0) = \int_{-\infty}^{t_0} \mathscr{F}(t - \sigma(s))f(s)\Delta s$. This implies that

$$\begin{aligned} \int_{t_0}^{t} \mathscr{F}(t - \sigma(s))f(s)\Delta s &= \int_{-\infty}^{t} \mathscr{F}(t - \sigma(s))f(s)\Delta s - \int_{-\infty}^{t_0} \mathscr{F}(t - \sigma(s))f(s)\Delta s \\ &= u(t) - \mathscr{F}(t - t_0)u(t_0). \end{aligned}$$

Thus, we have

$$u(t) = \mathscr{F}(t - t_0)u(t_0) + \int_{t_0}^{t} \mathscr{F}(t - \sigma(s))f(s)\Delta s.$$

Considering $x(t_0) = u(t_0)$, from above we have $x(t) = u(t)$, that is $x \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$.

Next, we discuss the uniqueness of the (ω, c)-periodic solution of (3.2.1). Let us consider that x and y are two (ω, c)-periodic mild solution solutions of (3.2.1). Let $z = x - y$. Then, $z \in \mathrm{P}_{(\omega,\mathrm{c})}(\mathbb{T}, \mathbb{B})$ and it satisfies

$$z^{\Delta}(t) = \mathcal{A}z(t), \quad t \geq t_0.$$

Given that $\mathcal{A}$ generates the C_0-semigroup $\mathscr{F}$, the aforementioned equation has a unique solution given by $z(t) = \mathscr{F}(t - t_0)z(t_0)$, where $z(t_0) = 0$. This implies that $z \equiv 0$, i.e., $x = y$.

Now, let us examine the stability of the uniquely existing (ω, c)-periodic mild solution, $x(t)$, of (3.2.1). Let

$\tilde{x}(t)$ be a perturbed mild solution of (3.2.1). Also, since the C_0-semigroup $\mathscr{F}$ is exponentially stable with a convergence rate α (> 0), for some constants $K \geq 1$, we have

$$\|\mathscr{F}(t-t_0)\|_{\mathbb{B}} \leq K e_{\ominus\alpha}(t,t_0) \quad \text{for all} \quad t, t_0 \in \mathbb{T}, \quad t > t_0.$$

Further, $e^{-\omega/\bar{\mu}} < |c| \leq 1$ implies that $-\frac{\ln|c|}{\omega} > 0$ and $1 + \mu(t)\frac{\ln|c|}{\omega} > 0$, i.e., $\frac{\ln|c|}{\omega} \in \mathcal{R}^+(\mathbb{T}, \mathbb{B})$. Therefore, employing the inequality of generalized exponentials on time scales given in Lemma 1.4.29, we can have

$$1 \leq |c|^{-(t-t_0)/\omega} = e^{-\frac{\ln|c|}{\omega}(t-t_0)} \leq e_{\ominus\frac{\ln|c|}{\omega}}(t,t_0) \quad \text{for all} \quad t, t_0 \in \mathbb{T}, \quad t > t_0. \tag{3.2.3}$$

Similarly, for $1 \leq |c| < e^{\omega/\bar{\mu}}$, we get $\frac{\ln|c|}{\omega} > 0$ and $-\frac{\ln|c|}{\omega} \in \mathcal{R}^+(\mathbb{T}, \mathbb{B})$. Following the inequality in Lemma 1.4.29, this yields

$$|c|^{-(t-t_0)/\omega} = e^{-\frac{\ln|c|}{\omega}(t-t_0)} \leq e_{\ominus\frac{\ln|c|}{\omega}}(t,t_0) \leq 1 \quad \text{for all} \quad t, t_0 \in \mathbb{T}, \quad t > t_0. \tag{3.2.4}$$

Thus,

$$\begin{aligned}
\|x(t) - \tilde{x}(t)\|_{\omega,c} &\leq \|\mathscr{F}(t-t_0)(x(t_0) - \tilde{x}(t_0))\|_{\omega,c} \\
&\leq |c|^{-(t-t_0)/\omega}\|\mathscr{F}(t-t_0)\|_{\mathbb{B}}\|x(t_0) - \tilde{x}(t_0)\|_{\omega,c} \\
&\leq K e_{\ominus\frac{\ln|c|}{\omega}}(t,t_0) e_{\ominus\alpha}(t,t_0)\|x(t_0) - \tilde{x}(t_0)\|_{\omega,c} \\
&\leq K e_{\ominus\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)}(t,t_0)\|x(t_0) - \tilde{x}(t_0)\|_{\omega,c}.
\end{aligned}$$

Since, we have $\ominus\alpha < \frac{\ln|c|}{\omega}$, i.e., $\frac{\ln|c|}{\omega} \oplus \alpha > 0$ for all $t \in \mathbb{T}$, we can conclude from the above inequality that the (ω, c)-periodic mild solution, $x(t)$, of (3.2.1) is exponentially stable for all $t \geq t_0$. □

Remark 3.2.2. $\mathbb{R}$ *is a STS. Since,* $\sigma(t+\omega) = t+\omega$ *for all* $t \in \mathbb{R}$ *and for any* $\omega \in \mathbb{R}$. *Thus, Theorem 3.2.1 is valid for* $\mathbb{T} = \mathbb{R}$ *as well.*

Corollary 3.2.3. *Let* $\mathcal{A}$ *generate a* C_0*-semigroup* $\mathscr{F}$ *that is exponentially stable with a convergence rate* α. *Also, consider that* $f : \mathbb{R} \to \mathbb{B}$ *is an* (ω, c)*-periodic function, where* $|c| > e^{-\alpha\omega}$, $\omega \in \mathbb{R}^+$. *Then, we have a unique exponentially stable,* (ω, c)*-periodic mild solution to the linear differential equation* $x'(t) = \mathcal{A}x(t) + f(t), \quad t \geq 0$.

Corollary 3.2.4. *Let* $\mathcal{A}$ *generate a* C_0*-semigroup* $\mathscr{F}$ *on* $\mathbb{Z}$ *that is exponentially stable with an exponential convergence rate* α, *and* $f \in P_{\omega,c}(\mathbb{Z}, \mathbb{B})$. *Then, we have a unique exponentially stable,* ω*-periodic mild solution to the linear difference equation* $x(n+1) - x(n) = \mathcal{A}x(n) + f(n),\ n \geq 0,\ n \in \mathbb{Z}$, *provided that* $e^{-\alpha\omega/(1+\alpha)} < |c| < e^{\omega}$, $\omega \in \mathbb{Z}^+$.

Since the STS restricts us to only a few specific time scales, we next present a result that achieves the same outcome without requiring the time scale to be a semigroup.

Theorem 3.2.5. *Consider that $\mathcal{A}$ generates an EFO $\mathscr{F}$, which has an integrable dichotomy $(\mathscr{P}, \lambda)$ that satisfies*

$$B := \sup_{t \in \mathbb{T}_\omega} \int_{-\infty}^{\infty} |c|^{-(t-s)/\omega} \lambda(t,s) \Delta s < \infty.$$

For $c \in \mathbb{F}_{\mathbb{B}}^*$, assume that $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ and $\mathscr{G}$ is ω-bi-periodic, i.e., $\mathscr{G}(t+\omega, s+\omega) = \mathscr{G}(t,s)$ for all $s, t \in \mathbb{T}_\omega$. Then (3.2.1) admits a unique (ω, c)-periodic mild solution given by

$$x(t) = \int_{-\infty}^{\infty} \mathscr{G}(t,s) f(s) \Delta s.$$

Proof. We first define

$$x(t) := \int_{-\infty}^{\infty} \mathscr{G}(t,s) f(s) \Delta s = \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta s - \int_{t}^{\infty} \mathscr{F}(t, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s.$$

Note that

$$\begin{aligned} x(t+\omega) &= \int_{-\infty}^{t+\omega} \mathscr{F}(t+\omega, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta s - \int_{t+\omega}^{\infty} \mathscr{F}(t+\omega, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s \\ &= \int_{-\infty}^{t} \mathscr{F}(t+\omega, \sigma(s)+\omega) \mathscr{P}(\sigma(s)+\omega) f(s) \Delta s \\ &\quad - \int_{t}^{\infty} \mathscr{F}(t+\omega, \sigma(s)+\omega) \mathscr{Q}(\sigma(s+\omega)+\omega) f(s+\omega) \Delta s \\ &= c \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta s - c \int_{t}^{\infty} \mathscr{F}(t, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s \\ &= cx(t). \end{aligned}$$

Also, using the property of the EFO, we can write

$$\begin{aligned} \mathscr{F}(t, t_0) x(t_0) &= \int_{-\infty}^{t_0} \mathscr{F}(t, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta s - \int_{t_0}^{\infty} \mathscr{F}(t, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s \\ &= \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta s - \int_{t_0}^{t} \mathscr{F}(t, \sigma(s)) \mathscr{P}(\sigma(s)) f(s) \Delta \\ &\quad - \int_{t}^{\infty} \mathscr{F}(t, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s - \int_{t_0}^{t} \mathscr{F}(t, \sigma(s)) \mathscr{Q}(\sigma(s)) f(s) \Delta s \\ &= x(t) - \int_{t_0}^{t} \mathscr{F}(t, \sigma(s)) f(s) \Delta s, \end{aligned}$$

i.e., we have

$$x(t) = \mathscr{F}(t, t_0) u(t_0) + \int_{t_0}^{t} \mathscr{F}(t, \sigma(s)) f(s) \Delta s.$$

Hence, we conclude that x is an (ω, c)-periodic mild solution of system (3.2.1).

Let $\mathcal{B}_0 \subset \mathbb{B}$ be the collection of initial conditions and $\zeta \in \mathbb{B}$ represent the (ω, c)-periodic solutions of the homogeneous system corresponding to (3.2.1). For $\zeta \in \mathcal{B}_0$, we first consider that $(\mathscr{I} - \mathscr{P})\zeta \neq 0$. Now, define

$$\phi^{-1}(t) := \|\mathscr{F}(t, \cdot)(\mathscr{I} - \mathscr{P})(\cdot)\zeta\|_{\omega,c}.$$

By the property of the evolution operator, we can write

$$\mathscr{F}(t,\cdot)(\mathscr{I}-\mathscr{P})\zeta = \mathscr{F}(t,s)\,\mathscr{F}(s,\cdot)(\mathscr{I}-\mathscr{P})(\cdot)\zeta,$$

Since $\mathscr{P}$ is a projection, we have $(\mathscr{I}-\mathscr{P})^2 = \mathscr{I}-\mathscr{P}$, and therefore we have

$$\|\mathscr{F}(t,\cdot)(\mathscr{I}-\mathscr{P})(\cdot)\zeta\|_{\omega,c} \le \|c^{-(t-s)/\omega}\mathscr{F}(t,s)(\mathscr{I}-\mathscr{P})(s)\|_{\mathbb{B}}\|\mathscr{F}(s,\cdot)(\mathscr{I}-\mathscr{P})(\cdot)\zeta\|_{\omega,c}.$$

This implies

$$\phi^{-1}(t)\phi(s) = \frac{\phi(s)}{\phi(t)} = \frac{\|\mathscr{F}(t,\cdot)(\mathscr{I}-\mathscr{P}(\cdot))\zeta\|_{\omega,c}}{\|\mathscr{F}(s,\cdot)(\mathscr{I}-\mathscr{P}(\cdot))\zeta\|_{\omega,c}} \le \|c^{-(t-s)/\omega}\mathscr{F}(t,s)(\mathscr{I}-\mathscr{P})(s)\|_{\mathbb{B}}.$$

Consequently, by the given assumption, we have

$$\int_t^\infty \phi^{-1}(t)\phi(s)\Delta s \le \int_t^\infty \|c^{-(t-s)/\omega}\mathscr{F}(t,s)(\mathscr{I}-\mathscr{P})(s)\|_{\mathbb{B}}\Delta s \le B < \infty,$$

which finally implies $\liminf\limits_{s\in[t,\infty)_{\mathbb{T}_\omega}} \phi(s) = 0$ (by [156, Lemma 1]), which contradicts the boundedness of $t \mapsto \mathscr{F}(t,\cdot)(\mathscr{I}-\mathscr{P})(\cdot)\zeta$. Thus, $(\mathscr{I}-\mathscr{P})\zeta = 0$. Next, consider $\mathscr{P}\zeta \ne 0$. Again, defining $\phi^{-1}(t) := \|\mathscr{F}(t,\cdot)\mathscr{P}(\cdot)\zeta\|_{\omega,c}$, and following a similar procedure we can obtain

$$\int_{-\infty}^t \phi^{-1}(t)\phi(s)\Delta s \le \int_{-\infty}^t \|c^{-(t-s)/\omega}\mathscr{F}(t,s)\mathscr{P}(s)\|_{\mathbb{B}}\Delta s \le B < \infty,$$

which implies that $\liminf\limits_{s\in(-\infty,t]_{\mathbb{T}_\omega}} \phi(s) = 0$, which suggests $\mathscr{P}\zeta = 0$. Thus, the unique possibility is $B_0 = \{0\}$, i.e., $\zeta = 0$.

Now, let x and y be any two (ω,c)-periodic mild solutions of (3.2.1), then $x-y$ will be an (ω,c)-periodic mild solution to the corresponding homogeneous system of (3.2.1). It follows that $x-y=0$, or $x=y$. □

Next, we establish sufficient conditions for the existence of a stable (ω,c)-periodic mild solution to abstract semilinear dynamic equations on $\mathbb{T}_\omega$.

Theorem 3.2.6. *Let $\mathbb{T}$ be an ω-periodic STS. Suppose $\mathcal{A}$ generates a C_0-semigroup $\mathscr{F}$ that is exponentially stable with exponential convergence rate α (>0) and convergence constant K (≥ 1). Furthermore, assume that for $c \in \mathbb{F}_{\mathbb{B}}^*$ satisfying $e^{-\alpha\omega/(1+\bar{\mu}\alpha)} < |c| < e^{\omega/\bar{\mu}}$, $f : \mathbb{T}\times\mathbb{B} \to \mathbb{B}$ is an (ω,c)-periodic function that satisfies the Lipschitz condition in $x \in \mathbb{B}$ uniformly for $t \in \mathbb{T}$, i.e., $\|f(t,x)-f(t,y)\|_{\mathbb{B}} \le L\|x-y\|_{\mathbb{B}}$ for all $x,y \in \mathbb{B}$, where $0 < L < \frac{\frac{\ln|c|}{\omega}\oplus\alpha}{K(1+\alpha\bar{\mu})}$ for all $t \in \mathbb{T}$. Then, the semilinear dynamic equation*

$$x^\Delta(t) = \mathcal{A}x(t) + f(t,x(t)), \quad t \in \mathbb{T} \tag{3.2.5}$$

admits a unique (ω,c)-periodic mild solution that is also exponentially stable.

Proof. Let $x(t)$ be a mild solution of (3.2.5). Then, $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{B})$, and it satisfies

$$x(t) = \mathscr{F}(t-t_0)x(t_0) + \int_{t_0}^{t} \mathscr{F}(t-\sigma(s))f(s, x(s))\Delta s, \quad \text{for all} \quad t, t_0 \in \mathbb{T},\ t \geq t_0.$$

Next, we consider the nonlinear operator $\psi : \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B}) \to \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$, defined as

$$\psi(\varphi)(t) := \int_{-\infty}^{t} \mathscr{F}(t-\sigma(s))f(s, \varphi(s))\Delta s \quad \text{for all} \quad t \in \mathbb{T}.$$

Since $f \in P_{\omega,c}(\mathbb{T}\times\mathbb{B}, \mathbb{B})$, in view of Theorem 2.2.17, for each $\varphi \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$, $f(\cdot, \varphi(\cdot))$ belongs to $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$. Thus, following the proof of Theorem 3.2.1, the operator ψ can be shown to be (ω, c)-periodic. Hence, we conclude that the operator ψ is well defined.

Next, we have assumed that $\mathcal{A}$ generates an exponentially stable C_0-semigroup $\mathscr{F}$. Thus, we have

$$\|\mathscr{F}(t-t_0)\|_{\mathbb{B}} \leq Ke_{\ominus\alpha}(t, t_0) \quad \text{for all} \quad t, t_0 \in \mathbb{T}, \quad t > t_0. \tag{3.2.6}$$

Also, note that $e^{-\alpha\omega/(1+\bar{\mu}\alpha)} > e^{-\omega/\bar{\mu}}$, which enables us to apply the inequality (3.2.3) and (3.2.4) for all given ranges of $|c|$. Furthermore, $e^{-\alpha\omega/(1+\bar{\mu}\alpha)} < |c| < 1$ implies $0 < \frac{\ln|c|}{\omega} + \bar{\mu}\frac{\ln|c|}{\omega}\alpha + \alpha < \frac{\ln|c|}{\omega} \oplus \alpha$ for all $t \in \mathbb{T}$.

We have seen in Chapter 2 that the space $(\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B}), \|\cdot\|_{\omega,\mathrm{c}})$ is a Banach space. Moreover, for any $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$,

$$\sup_{t\in\mathbb{T}} \|c^{-t/\omega}f(t)\|_{\mathbb{B}} = \sup_{t\in[t_0,t_0+\omega]_{\mathbb{T}}} \|c^{-t/\omega}f(t)\|_{\mathbb{B}}.$$

Now, by applying some identities and inequalities concerning generalized exponential functions on time scale given in Theorems 1.4.25 and 1.4.26, for any nonidentical elements $\varphi_1, \varphi_2 \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{T}} \|\psi\varphi_1(t) - \psi\varphi_2(t)\|_{\omega,c} &= \sup_{t\in\mathbb{T}} \|c^{-t/\omega} \int_{-\infty}^{t} \mathscr{F}(t-\sigma(s))(f(s, \varphi_1(s)) - f(s, \varphi_2(s)))\Delta s\|_{\mathbb{B}} \\
&\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} |c|^{-t/\omega} \|\mathscr{F}(t-\sigma(s))\|_{\mathbb{B}}\ \|f(s, \varphi_1(s)) - f(s, \varphi_2(s))\|_{\mathbb{B}}\Delta s \\
&\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} |c|^{-t/\omega} Ke_{\ominus\alpha}(t, \sigma(s))L\|\varphi_1(s) - \varphi_2(s)\|_{\mathbb{B}}\Delta s \\
&\leq KL \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} |c|^{-(t-s)/\omega} (1+\alpha\mu(s))\, e_{\ominus\alpha}(t, s)\|c^{-s/\omega}(\varphi_1(s) - \varphi_2(s))\|_{\mathbb{B}}\Delta s \\
&\leq KL(1+\alpha\bar{\mu}) \sup_{t\in\mathbb{T}} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c} \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega}|\oplus\alpha\right)}(t, s)\Delta s \\
&\leq KL(1+\alpha\bar{\mu}) \sup_{t\in\mathbb{T}} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c} \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} \frac{e_{\ominus\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)}(t, \sigma(s))}{1+\mu(s)\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)}\Delta s \\
&\leq \frac{KL(1+\alpha\bar{\mu})}{\inf\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)} \sup_{t\in\mathbb{T}} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c} \\
&< \sup_{t\in\mathbb{T}} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c},
\end{aligned}$$

implies that ψ is a contraction map. Hence, the Banach fixed point theorem ensures that there exists a unique $x \in \mathrm{P}_{\omega,c}(\mathbb{T},\mathbb{B})$ such that $\psi x = x$, that is, $x(t) = \int_{-\infty}^{t} \mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s$. Furthermore, for $t_0 \in \mathbb{T}$, $t \geq t_0$, we have

$$\begin{aligned}\int_{t_0}^{t} \mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s &= \int_{-\infty}^{t} \mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s - \int_{-\infty}^{t_0} \mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s \\ &= x(t) - \mathscr{F}(t-t_0)x(t_0).\end{aligned}$$

Thus, we have

$$x(t) = \mathscr{F}(t-t_0)x(t_0) + \int_{t_0}^{t} \mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s,$$

which is the unique (ω,c)-periodic mild solution of (3.2.5). Finally, we show that the solution $x(t)$ is exponentially stable. Let $\tilde{x}(t)$ be a mild solution of perturbed equation (3.2.5) assigned with the initial condition $\tilde{x}(t_0)$. Then,

$$\begin{aligned}\|x(t)-\tilde{x}(t)\|_{\omega,c} &\leq \|\mathscr{F}(t-t_0)(x(t_0)-\tilde{x}(t_0))\|_{\omega,c} + \int_{t_0}^{t} \|T(t-\sigma(s))(f(s,x(s))-f(s,\tilde{x}(s)))\|_{\omega,c}\Delta s \\ &\leq |c|^{-(t-t_0)/\omega}\|\mathscr{F}(t-t_0)\|_{\mathbb{B}}\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c} \\ &\quad + \int_{t_0}^{t} |c|^{-(t-s)/\omega}\|\mathscr{F}(t-\sigma(s))\|_{\mathbb{B}} L\|x(s)-\tilde{x}(s)\|_{\omega,c}\Delta s \\ &\leq K|c|^{-(t-t_0)/\omega} e_{\ominus\alpha}(t,t_0)\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c} \\ &\quad + LK\int_{t_0}^{t} |c|^{-(t-s)/\omega} e_{\ominus\alpha}(t,\sigma(s))\|x(s)-\tilde{x}(s)\|_{\omega,c}\Delta s \\ &\leq K e_{\ominus(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,t_0)\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c} \\ &\quad + KL(1+\alpha\bar{\mu})\int_{t_0}^{t} e_{\ominus(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,s)\|x(s)-\tilde{x}(s)\|_{\omega,c}\Delta s\end{aligned}$$

This implies

$$\begin{aligned}e_{(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,0)\|x(t)-\tilde{x}(t)\|_{\omega,c} &\leq K e_{(\frac{\ln|c|}{\omega}\oplus\alpha)}(t_0,0)\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c} \\ &\quad + KL(1+\alpha\bar{\mu})\int_{t_0}^{t} e_{(\frac{\ln|c|}{\omega}\oplus\alpha)}(s,0)\|x(s)-\tilde{x}(s)\|_{\omega,c}\Delta s\end{aligned}$$

Thus, using Gronwall's inequality (see Lemma 1.4.84), we obtain

$$e_{(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,0)\|x(t)-\tilde{x}(t)\|_{\omega,c} \leq K e_{(\frac{\ln|c|}{\omega}\oplus\alpha)}(t_0,0) e_{KL(1+\alpha\bar{\mu})}(t,t_0)\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c},$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. This yields

$$\|x(t)-\tilde{x}(t)\|_{\omega,c} \leq K e_{\ominus(\frac{\ln|c|}{\omega}\oplus\alpha\ominus KL(1+\alpha\bar{\mu}))}(t,t_0)\|x(t_0)-\tilde{x}(t_0)\|_{\omega,c} \quad \text{for all} \quad t \geq t_0. \tag{3.2.7}$$

Note that $\underline{\mu}KL(1+\alpha\bar{\mu}) \geq 0$ implies $\frac{KL(1+\alpha\bar{\mu})}{1+\underline{\mu}KL(1+\alpha\bar{\mu})} \leq KL(1+\alpha\bar{\mu}) < \frac{\ln|c|}{\omega} \oplus \alpha$ for all $t \in \mathbb{T}$. Thus, we have $\frac{\ln|c|}{\omega}\oplus\alpha\ominus KL(1+\alpha\bar{\mu}) = \frac{\ln|c|}{\omega}\oplus\alpha - \frac{KL(1+\alpha\bar{\mu})}{1+\mu KL(1+\alpha\bar{\mu})} > 0$ for all t. Hence, (3.2.7) states that x is exponentially stable for all $t \geq t_0$. The proof is complete. □

Corollary 3.2.7. *Let $\mathcal{A}$ generate a C_0-semigroup $\mathscr{F}$ that is exponentially stable with an exponential convergence rate α and convergence constant K. Also, assume that, $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{R} \times \mathbb{B}, \mathbb{B})$ is a Lipschitz continuous function with Lipschitz constant L. Then, there always exists a unique exponentially stable, (ω, c)-periodic mild solution to the semilinear differential equation*

$$x'(t) = \mathcal{A}x(t) + f(t, x(t)), \quad t \geq 0$$

provided that $0 < KL < \frac{\ln|c|}{\omega} + \alpha$, $\omega \in \mathbb{R}^+$.

Corollary 3.2.8. *Let $\mathcal{A}$ generate a C_0-semigroup $\mathscr{F}$ that is exponentially stable with an exponential convergence rate α and convergence constant K. Also, assume that, $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{Z} \times \mathbb{B}, \mathbb{B})$ is a Lipschitz continuous function with Lipschitz constant L. Then, for $e^{-\alpha\omega/(1+\alpha)} < |c| < e^{\omega}$, $\omega \in \mathbb{Z}^+$, there always exists a unique exponentially stable, (ω, c)-periodic mild solution to the semilinear difference equation*

$$x(n+1) - x(n) = \mathcal{A}x(n) + f(n, x(n)), \ n \geq 0, \ n \in \mathbb{Z}$$

provided that $0 < KL < \frac{\ln|c|}{\omega} + \frac{\alpha}{1+\alpha}$.

Theorem 3.2.9. *Consider the semi-linear dynamic equation (3.2.5) on $\mathbb{T}_\omega$. Assume that $\mathcal{A}$ has an associated ω-bi-periodic EFO $\{\mathscr{F}(t,s) : t \geq s\}$ with $t, s \in \mathbb{T}_\omega$, which is uniformly exponentially stable with exponential convergence rate α (> 0) and convergence constant K (≥ 1). In addition to this, assume that for $c \in \mathbb{F}_{\mathbb{B}}^*$ with $e^{-\alpha\omega/(1+\bar{\mu}\alpha)} < |c| < e^{\omega/\bar{\mu}}$, $f : \mathbb{T}_\omega \times \mathbb{B} \to \mathbb{B}$ is an (ω, c)-periodic function that satisfies $\|f(t,x) - f(t,y)\|_{\mathbb{B}} \leq L\|x - y\|_{\mathbb{B}}$ for all $x, y \in \mathbb{B}$, where $0 < L < \frac{\frac{\ln|c|}{\omega} \oplus \alpha}{K(1+\alpha\bar{\mu})}$ for all $t \in \mathbb{T}_\omega$. Then (3.2.5) has a unique, exponentially stable, (ω, c)-periodic mild solution.*

Proof. Let us define an operator as

$$\psi(\varphi)(t) := \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s)) f(s, \varphi(s)) \Delta s \quad \text{for all} \quad t \in \mathbb{T}_\omega.$$

As for $\varphi(t) \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, we can have

$$\begin{aligned}
\psi(\varphi)(t+\omega) &= \int_{-\infty}^{t+\omega} \mathscr{F}(t+\omega, \sigma(s)) f(s, \varphi(s)) \Delta s \\
&= \int_{-\infty}^{t} \mathscr{F}(t+\omega, \sigma(s)+\omega) f(s+\omega, c\varphi(s)) \Delta s \\
&= c \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s)) f(s, \varphi(s)) \Delta s,
\end{aligned}$$

which suggests that $\psi : \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}) \to \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. Now, for any $\varphi_1, \varphi_2 \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ with $\varphi_1 \neq \varphi_2$, we have

$$\sup_{t \in \mathbb{T}_\omega} \|\psi\varphi_1(t) - \psi\varphi_2(t)\|_{\omega,c} = \sup_{t \in \mathbb{T}_\omega} \|c^{-t/\omega} \int_{-\infty}^{t} \mathscr{F}(t, \sigma(s))(f(s, \varphi_1(s)) - f(s, \varphi_2(s))) \Delta s\|_{\mathbb{B}}$$

$$\leq \sup_{t\in\mathbb{T}_\omega}\int_{-\infty}^{t}|c|^{-t/\omega}Ke_{\ominus\alpha}(t,\sigma(s))L\|\varphi_1(s)-\varphi_2(s)\|_{\mathbb{B}}\Delta s$$
$$\leq \frac{KL(1+\alpha\bar{\mu})}{\inf\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)}\sup_{t\in\mathbb{T}_\omega}\|\varphi_1(t)-\varphi_2(t)\|_{\omega,c}$$
$$< \sup_{t\in\mathbb{T}_\omega}\|\varphi_1(t)-\varphi_2(t)\|_{\omega,c}.$$

This implies that there exists a unique $x\in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ such that $x(t)=\int_{-\infty}^{t}\mathscr{F}(t-\sigma(s))f(s,x(s))\Delta s$, which can be shown to be exponentially stable under the given conditions following a similar proof as of Theorem 3.2.6. □

Corollary 3.2.10. *Consider the linear dynamic equation (3.2.1) on $\mathbb{T}_\omega$. Assume that $\mathcal{A}$ generates an ω-bi-periodic EFO $\{\mathscr{F}(t,s)\}_{t>s,\ t,s\in\mathbb{T}_\omega}$, and for $c\in\mathbb{F}_{\mathbb{B}}^*$ with $e^{-\omega/\bar{\mu}}<|c|<e^{\omega/\bar{\mu}}$, let $f\in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. Then (3.2.1) admits a unique (ω,c)-periodic mild solution. Also, if we consider $\mathscr{F}$ to be exponentially stable with an exponential convergence rate α (>0) such that $\ominus\alpha<\frac{\ln|c|}{\omega}$, then the solution will be exponentially stable.*

Theorem 3.2.11. *Suppose that we consider $f(t,x)=h(x)+\phi(t)$ in (3.2.5), where $\phi\in P_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ and $h:\mathbb{B}\to\mathbb{B}$ is a Lipschitz function with Lipschitz constant L. Also, assume that $h(cx)=ch(x)$. Then, the unique (ω,c)-periodic solution $x_\phi(t)$ given by Theorem 3.2.6 (or Theorem 3.2.9) depends continuously on ϕ.*

Proof. Let $\phi_1,\phi_2\in P_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ and x_{ϕ_1},x_{ϕ_2} be the (ω,c)-periodic solutions of (3.2.5) given by Theorem 3.2.6. Then,

$$x_{\phi_1}(t)-x_{\phi_2}(t)=\int_{-\infty}^{t}\mathscr{F}(t-\sigma(s))(h(x_{\phi_1}(s))-h(x_{\phi_2}(s)))\Delta s+\int_{-\infty}^{t}\mathscr{F}(t-\sigma(s))(\phi_1(s)-\phi_2(s))\Delta s.$$

Thus,

$$\begin{aligned}\sup_{t\in\mathbb{T}_\omega}\|x_{\phi_1}(t)-x_{\phi_2}(t)\|_{\omega,c} &\leq \sup_{t\in\mathbb{T}_\omega}\int_{-\infty}^{t}|c|^{-(t-s)/\omega}Ke_{\ominus\alpha}(t,\sigma(s))L\|x_{\phi_1}(s)-x_{\phi_2}(s)\|_{\omega,c}\Delta s\\ &\quad+\sup_{t\in\mathbb{T}_\omega}\int_{-\infty}^{t}|c|^{-(t-s)/\omega}Ke_{\ominus\alpha}(t,\sigma(s))\|\phi_1(s)-\phi_2(s)\|_{\omega,c}\Delta s\\ &\leq KL(1+\alpha\bar{\mu})\sup_{t\in\mathbb{T}_\omega}\int_{-\infty}^{t}e_{\ominus(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,s)\|x_{\phi_1}(s)-x_{\phi_2}(s)\|_{\omega,c}\Delta s\\ &\quad+K(1+\alpha\bar{\mu})\sup_{t\in\mathbb{T}_\omega}\int_{-\infty}^{t}e_{\ominus(\frac{\ln|c|}{\omega}\oplus\alpha)}(t,s)\|\phi_1(s)-\phi_2(s)\|_{\omega,c}\Delta s\\ &\leq \frac{KL(1+\alpha\bar{\mu})}{\frac{\ln|c|}{\omega}\oplus\alpha}\sup_{t\in\mathbb{T}_\omega}\|x_{\phi_1}(t)-x_{\phi_2}(t)\|_{\omega,c}+\frac{K(1+\alpha\bar{\mu})}{\frac{\ln|c|}{\omega}\oplus\alpha}\sup_{t\in\mathbb{T}_\omega}\|\phi_1(t)-\phi_2(t)\|_{\omega,c},\end{aligned}$$

which implies

$$\sup_{t\in\mathbb{T}_\omega}\|x_{\phi_1}(t)-x_{\phi_2}(t)\|_{\omega,c}\leq\frac{K(1+\alpha\bar{\mu})}{\inf\left(\frac{\ln|c|}{\omega}\oplus\alpha\right)-KL(1+\alpha\bar{\mu})}\sup_{t\in\mathbb{T}_\omega}\|\phi_1(t)-\phi_2(t)\|_{\omega,c}.$$

The proof is complete. $\square$

Theorem 3.2.12. *Consider that $\mathcal{A}$ generates an EFO $\mathscr{F}$, which has an integrable dichotomy $(\mathscr{P}, \lambda)$ that satisfies $B = \sup_{t\in\mathbb{T}_\omega} \int_{-\infty}^{\infty} |c|^{-(t-s)/\omega}\lambda(t,s)\Delta s < \infty$. For $c \in \mathbb{F}_{\mathbb{B}}^*$, assume that $f \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ satisfies the Lipschitz condition in $x \in \mathbb{B}$ uniformly for $t \in \mathbb{T}_\omega$ with Lipschitz constant L, and $\mathscr{G}$ is ω-bi-periodic on $\mathbb{T}_\omega$. Then (3.2.1) has a unique (ω, c)-periodic mild solution if we are provided with $BL < 1$.*

Proof. Define

$$\psi(\varphi)(t) := \int_{-\infty}^{t} \mathscr{F}(t,\sigma(s))\mathscr{P}(\sigma(s))f(s,\varphi(s))\Delta s - \int_{t}^{\infty} \mathscr{F}(t,\sigma(s))\mathscr{Q}(\sigma(s))f(s,\varphi(s))\Delta s \quad \text{for all} \quad t \in \mathbb{T}_\omega.$$

Now, for $\varphi(t) \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, we can have

$$\begin{aligned}
\psi(\varphi)(t+\omega) &= \int_{-\infty}^{t+\omega} \mathscr{F}(t+\omega,\sigma(s))\mathscr{P}(\sigma(s))f(s,\varphi(s))\Delta s - \int_{t+\omega}^{\infty} \mathscr{F}(\tau+\omega,\sigma(s))\mathscr{Q}(\sigma(s))f(s,\varphi(s))\Delta s \\
&= \int_{-\infty}^{t} \mathscr{F}(t+\omega,\sigma(s)+\omega)\mathscr{P}(\sigma(s)+\omega)f(s+\omega,c\varphi(s))\Delta s \\
&\quad - \int_{t}^{\infty} \mathscr{F}(t+\omega,\sigma(s)+\omega)\mathscr{Q}(\sigma(s+\omega)+\omega)f(s+\omega,c\varphi(s))\Delta s \\
&= c\left[\int_{-\infty}^{t} \mathscr{F}(t,\sigma(s))\mathscr{P}(\sigma(s))f(s,\varphi(s))\Delta s - \int_{t}^{\infty} \mathscr{F}(\tau,\sigma(s))\mathscr{Q}(\sigma(s))f(s,\varphi(s))\Delta s\right] \\
&= c\psi(\varphi)(t).
\end{aligned}$$

This implies $\psi : \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) \to \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Now, for any $\varphi_1, \varphi_2 \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ with $\varphi_1 \neq \varphi_2$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{T}_\omega} \|\psi\varphi_1(t) - \psi\varphi_2(t)\|_{\omega,c} &\leq \sup_{t\in\mathbb{T}_\omega} \|c^{-t/\omega} \int_{-\infty}^{t} \mathscr{F}(t,\sigma(s))\mathscr{P}(\sigma(s))(f(s,\varphi_1(s)) - f(s,\varphi_2(s)))\Delta s\|_{\mathbb{B}} \\
&\quad + \sup_{t\in\mathbb{T}_\omega} \|c^{-t/\omega} \int_{t}^{\infty} \mathscr{F}(\tau,\sigma(s))\mathscr{Q}(\sigma(s))(f(s,\varphi_1(s)) - f(s,\varphi_2(s)))\Delta s\|_{\mathbb{B}} \\
&\leq \sup_{t\in\mathbb{T}_\omega} \int_{-\infty}^{\infty} |c|^{-(t-s)/\omega}\lambda(t,\sigma(s))L\|\varphi_1(s) - \varphi_2(s)\|_{\omega,c}\Delta s \\
&\leq BL \sup_{t\in\mathbb{T}_\omega} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c} \\
&< \sup_{t\in\mathbb{T}_\omega} \|\varphi_1(t) - \varphi_2(t)\|_{\omega,c}.
\end{aligned}$$

This implies that there exists a unique $x \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ with $x(t) = \psi(\varphi)(t)$. $\square$

Remark 3.2.13. *If, in Theorem 3.2.12, the EFO generated by $\mathcal{A}$ admits an exponential dichotomy on $\mathbb{T}_\omega$, then we can choose $\alpha, K > 0$ such that $\lambda(t,s) = Ke_{\ominus\alpha}(t,s)$. Hence, (3.2.5) admits a unique, exponentially stable, (ω, c)-periodic mild solution, provided that the conditions on α, c, and f given in Theorem 3.2.9 are also satisfied in this setting.*

3.3 Applications and Examples

This section presents several applications and illustrative examples that demonstrate the theoretical results established in this chapter.

3.3.1 Lasota–Ważewska model on time scales

In healthy animals, red blood cells (RBC) levels are typically maintained within a narrow physiological range. An (ω, c)-periodic solution to Lasota–Ważewska model with $|c| \neq 1$ implies unbounded growth or decay in the RBC level over time, which is not biologically realistic under normal steady-state conditions. Therefore, in a healthy state, only the values of $c \approx 1$ are appropriate, as they suggest bounded oscillatory behavior. However, certain hematological disorders or viral infections may exhibit (ω, c)-periodic oscillations in RBC counts with $|c| < 1$, indicating a gradual depletion of cells. In contrast, external perturbations or compensatory mechanisms in the body (such as after blood loss or during treatment) can lead to transient phases with $|c| > 1$, representing a temporary (ω, c)-periodic increase in RBC levels.

Consider the Lasota–Ważewska model on $\mathbb{T}_\omega$, given as

$$\begin{aligned} x^{\Delta}(t) &= -dx(t) + \beta(t)e^{-\lambda(t)x(t)}, \quad t \in {\mathbb{T}_\omega}^{+} \\ x(0) &= x_0, \end{aligned} \tag{3.3.1}$$

where d is a positive constant; $-d \in \mathcal{R}^+$, $\beta : [0, \infty)_{\mathbb{T}_\omega} \to \mathbb{R}^+$ is an (ω, c)-periodic function, and $\lambda : [0, \infty)_{\mathbb{T}_\omega} \to \mathbb{R}^+$ is an $(\omega, \frac{1}{c})$-periodic function, where $e^{-\alpha\omega/(1+\bar{\mu}d)} < |c| < e^{\omega/\bar{\mu}}$.
Let us define $f(t, x) = \beta(t)e^{-\lambda(t)x}$, then

$$|f(t, x_1) - f(t, x_2)| \leq |\beta(t)||e^{-\lambda(t)x_1} - e^{-\lambda(t)x_2}| \leq |\beta(t)\lambda(t)||x_1 - x_2| \leq \overline{\beta\lambda}\ |x_1 - x_2|,$$

where $\overline{\beta\lambda} = \sup_{t \in \mathbb{T}_\omega} |\beta(t)\lambda(t)|$. Also, for the considered dynamic equation $\mathscr{F}(t, s) = e_{-d}(t, s)$, and $|\mathscr{F}(t, s)| = e_{-d}(t, s) \leq e_{\ominus d}(t, s)$, whenever $t \geq s$. Thus, if $\overline{\beta\lambda} < \left(\frac{\ln|c|}{\omega} \oplus d\right) \Big/ (1 + d\bar{\mu})$ for all $t \in \mathbb{T}_\omega$, then all conditions of Theorem 3.2.9 are satisfied, and (3.3.1) has a unique (ω, c)-periodic solution which is exponentially stable.

To provide a more precise understanding, we discuss some examples with fixed numerical values for the coefficient of (3.3.1) on specific time scales.

Example 3.3.1. *With $\mathbb{T} = \mathbb{R}$, consider the following differential equation:*

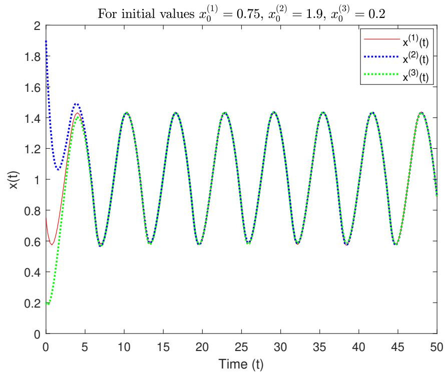
$$x'(t) = -0.7\ x(t) + 1.25\ c^{t/2\pi}|\sin(t/2)|\ e^{-0.2\ c^{-t/2\pi}|\cos(t/2)|x(t)}, \quad x(0) = 0.75 \tag{3.3.2}$$

with $c > 0$. In this case, we have $\mu \equiv 0$, $\alpha = 0.7 > 0$, $\beta(t) = 1.25\ c^{t/2\pi}|\sin(t/2)|$, which is $(2\pi, c)$-periodic function and $\lambda(t) = 0.2\ c^{-t/1.4\pi}|\cos(t/2)|$ is a $(2\pi, \frac{1}{c})$-periodic function. It is easy to observe that

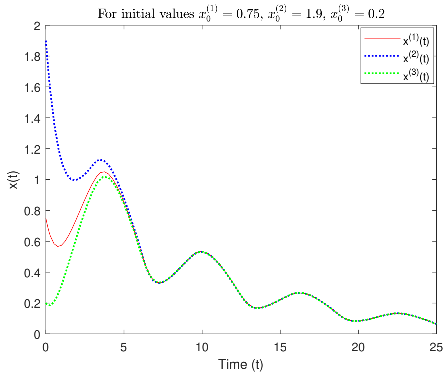
$\overline{\beta\gamma} = 0.25$. *We will see one example from all the possible cases, i.e., when* $|c| = 1$, $e^{-1.4\pi} < |c| < 1$, *and* $e^{2\pi} > |c| > 1$. *Thus,*

- *for* $|c| = 1$, *we have* $\frac{\ln 1}{2\pi} + d = 0.7 > \overline{\beta\lambda}$.
- *for* $|c| = 0.5$, *we have* $\frac{\ln 0.5}{2\pi} + d > 0.5896 > \overline{\beta\gamma}$.
- *for* $|c| = 2$, *we have* $\frac{\ln 2}{2\pi} + d > 0.8103 > \overline{\beta\lambda}$.

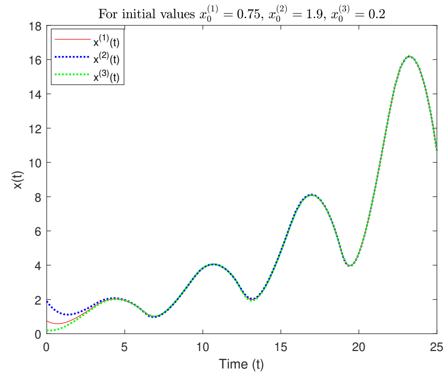
In each case, all the conditions of Theorem 3.2.6 are well satisfied. Hence, we conclude that for given values of c, (3.3.2) *has a unique* $(2\pi, c)$*-periodic solution, which is also exponentially stable. (See Figure 3.1.)*



(a) Solution of (3.3.2) with $c = 1$, plotted over the range $[0, 50]$.



(b) Solution of (3.3.2) with $c = 0.5$, plotted over the range $[0, 25]$.



(c) Solution of (3.3.2) with $c = 2$, plotted over the range $[0, 25]$.

Figure 3.1: Figures (3.1a), (3.1b), and (3.1c) depict the solution trajectories of (3.3.2) corresponding to the initial value 0.75 and perturbed initial values 1.9 and 0.2, under the values of c as 1, 0.5, and 2, respectively. For $k = 1, 2, 3$, in each figures $x^{(k)}(t)$ represents the behavior of the solution corresponding to the initial condition $x_0^{(k)}$.

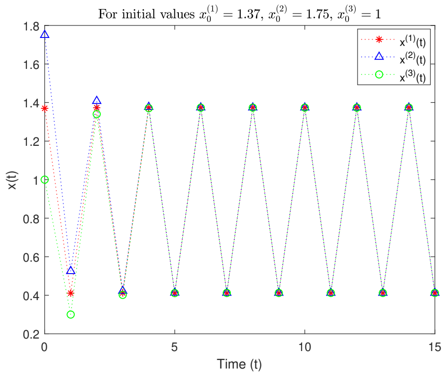
Example 3.3.2. *On the time scale* $\mathbb{Z}$, *let us consider the following difference equation:*

$$x(t+1) - x(t) = -0.7\ x(t) + 1.25\ c^{t/2}|\sin(\pi n/2)|\ e^{-0.2\ c^{-t/2}|\cos(\pi n/2)|x(t)}, \quad x(0) = 1.37 \tag{3.3.3}$$

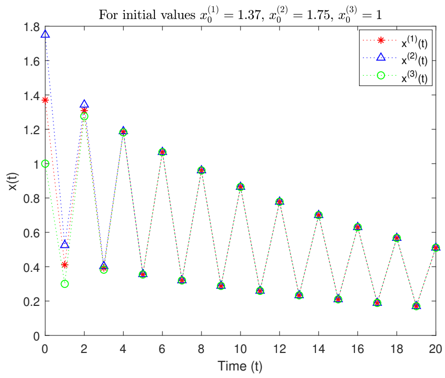
with $c > 0$, and $t(\geq 0) \in \mathbb{Z}$. In this case, we have $\mu \equiv 1$, $\alpha = 0.7 > 0$, $1-\alpha > 0$, $\beta(t) = 1.25\ c^{t/2}|\sin(\pi t/2)|$, which is $(2, c)$-periodic function and $\gamma(t) = 0.2\ c^{-t/2}|\cos(\pi t/2)|$ is a $(2, \frac{1}{c})$-periodic function. It is easy to observe that $\overline{\beta\gamma} = 0.25$. Thus,

- *for $|c| = 1$, we have $\left(\frac{\ln 1}{2} + \alpha + \frac{\ln 1}{2}\alpha\right) \Big/ (1+\alpha) = 0.4117 > \overline{\beta\gamma}$.*
- *for $|c| = 0.9 > e^{-0.8235}$, we have $\left(\frac{\ln 0.9}{2} + \alpha + \alpha\frac{\ln 0.9}{2}\right) \Big/ (1+\alpha) > 0.3064 > \overline{\beta\gamma}$.*
- *for $|c| = 2 < e^2$, we have $\left(\frac{\ln 2}{2} + \alpha + \frac{\ln 2}{2}\alpha\right) \Big/ (1+\alpha) > 0.7833 > \overline{\beta\gamma}$.*

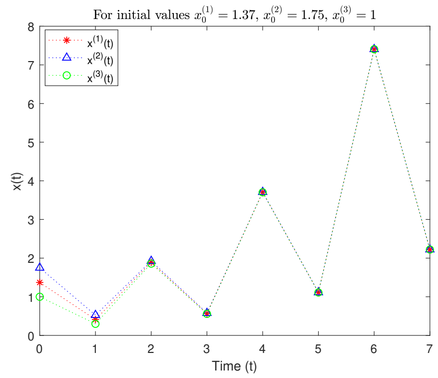
Hence, we conclude that for given values of c, (3.3.3) has a unique $(2, c)$-periodic mild solution, which is also exponentially stable. (See Figure 3.2.)



(a) Solution of (3.3.3) with $c = 1$, plotted over the range $[0, 15]$.



(b) Solution of (3.3.3) with $c = 0.9$, plotted over the range $[0, 20]$.



(c) Solution of (3.3.3) with $c = 2$, plotted over the range $[0, 7]$.

Figure 3.2: Figures (3.2a), (3.2b), and (3.2c) depict the solution trajectories of (3.3.3) associated with the initial value 1.37 and perturbed initial values 1.75 and 1, under the values of c as 1, 0.9, and 2, respectively. For $k = 1, 2, 3$, in each figure, $x^{(k)}(t)$ represents the behaviour of the solution corresponding to the initial condition $x_0^{(k)}$.

Example 3.3.3. *With* $\mathbb{T} = \bigcup_{n\in\mathbb{Z}}[n(a+b), n(a+b)+b]$, *with* $a = 0.2$ *and* $b = 0.4$, *let us consider the following dynamic equation:*

$$x^{\Delta}(t) = -0.7\ x(t) + 1.25\ c^{5t/3}|\sin(5\pi t/3)|\ e^{-0.2\ c^{-5t/3}|\cos(5\pi t/3)|x(t)}, \quad x(0) = 1.01 \quad c > 0. \tag{3.3.4}$$

In this case, we have $\mu(t) = \begin{cases} 0, & \text{when } t \in [0.6n, 0.6n+0.4), \\ 0.2, & \text{when } t \in \{0.6n+0.4,\ n \in \mathbb{Z}\}. \end{cases}$ *Also,* $\alpha = 0.7 > 0$, $1 - \mu(t)\alpha > 0$ *for all* $t \in \bigcup_{n\in\mathbb{Z}}[0.6n, 0.6n+0.4]$, $\beta(t) = 1.25\ c^{5t/3}|\sin(5\pi t/3)| + \alpha$, *which is* $(\frac{3}{5}, c)$*-periodic function and* $\gamma(t) = 0.2\ c^{-5t/3}|\cos(5\pi t/3)|$ *is a* $(\frac{3}{5}, \frac{1}{c})$*-periodic function. It is easy to observe that* $\overline{\beta\gamma} = 0.25$. *Thus,*

- *for* $|c| = 1$, *we have* $\left(\frac{5\ln 1}{3} \oplus \alpha\right)/(1+\alpha\bar{\mu}) > 0.614 > \overline{\beta\gamma}$.
- *for* $|c| = 0.81 > e^{-0.3684}$, *we have* $\left(\frac{5\ln 0.81}{3} \oplus \alpha\right)/(1+\alpha\bar{\mu}) > 0.263 > \overline{\beta\gamma}$.
- *for* $|c| = 1.2 < e$, *we have* $\left(\frac{5\ln 1.2}{3} \oplus \alpha\right)/(1+\alpha\bar{\mu}) > 0.9179 > \overline{\beta\gamma}$.

Hence, for given c, (3.3.4) *has a unique* $(\frac{3}{5}, c)$*-periodic mild solution, which is also exponentially stable. (See Figure 3.3)*

3.3.2 Hopfield neural networks (HNNs) on time scales

The concept of (ω, c)-periodicity enables the HNN model to accommodate attenuating or amplifying oscillations, which may correspond to phenomena such as fatigue, adaptation, or learning in neural systems. Our analysis provides a framework for studying how memory states evolve and maintain stability, particularly when external inputs or synaptic weights vary (ω, c)-periodically over time. Moreover, (ω, c)-periodic solutions of HNNs can represent computational cycles that adapt to varying external stimuli or internal decay mechanisms.

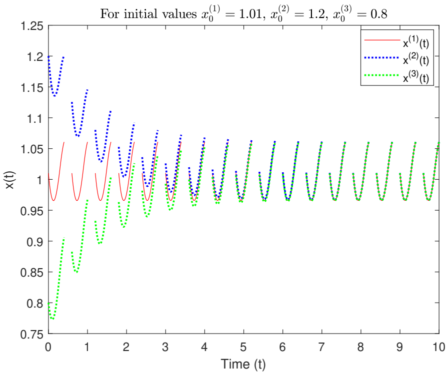
Consider the following high-order HNN model on $\mathbb{T}_\omega$:

$$\begin{aligned} x_i^{\Delta}(t) \quad &= \quad -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t)g_j(\lambda_j(t)x_j(t))g_k(\lambda_k(t)x_k(t)) + I_i(t),\ t \geq 0, \\ & x_i(0) = \phi_i, \quad i = 1, 2, ..., n, \end{aligned} \tag{3.3.5}$$

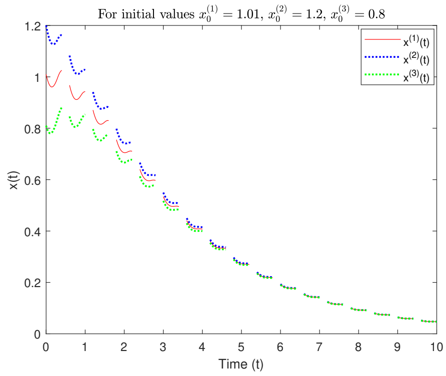
where n is the number of units in the neural network; $x_i(t)$ is the state vector of the i^{th} unit at the time t; $d_i(t)$ is the self-feedback connection weight; $a_{ij}(t)$ and $b_{ijk}(t)$ are the network's first and 2^{nd} order connection weights; f_i and g_i are the activation functions of signal transmission; and $I_i(t)$ are the external inputs to the network at time t.

We assume a list of conditions to be satisfied, given as follows:

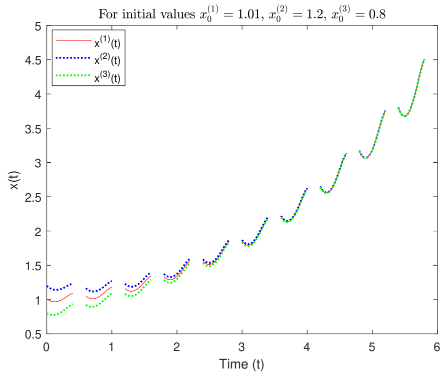
(A1) There exists a positive constant $\underline{d}$, such that $-|\underline{d}| = -\|\underline{d}\|_{\mathbb{B}} \in \mathcal{R}^{+}(\mathbb{T}_\omega, \mathbb{R})$, and the functions d_i satisfy $0 < \underline{d} \leq d_i(t)$ for all $t \in \mathbb{T}_\omega$.



(a) Solution of (3.3.4) with $c = 1$, plotted over the range $[0, 10]$.



(b) Solution of (3.3.4) with $c = 0.81$, plotted over the range $[0, 10]$.



(c) Solution of (3.3.4) with $c = 1.2$, plotted over the range $[0, 6]$.

Figure 3.3: Figures (3.3a), (3.3b), and (3.3c) depict the solution trajectories of (3.3.4) associated with the initial value 1.01 and perturbed initial values 1.2 and 0.8, under the values of c as 1, 0.81, and 1.2, respectively. For $k = 1, 2, 3$, in each figure, $x^{(k)}(t)$ represents the behavior of the solution corresponding to the initial condition $x_0^{(k)}$.

(A2) There exist $c \in \mathbb{F}_{\mathbb{B}}^*$ with $e^{-|\underline{d}|\omega/(1+\bar{\mu}|\underline{d}|)} < |c| < e^{\omega/\bar{\mu}}$, such that for every $i, j, k = 1, 2, ..., n$, $d_i, a_{ij} \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, $b_{ijk} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, $\lambda_i \in \mathrm{P}_{\omega,\frac{1}{\mathrm{c}}}(\mathbb{T}_\omega, \mathbb{B})$ and $I_i \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.

(A3) $f_i : \mathbb{B} \to \mathbb{B}$ is a linear homogeneous functions, i.e., $f_i(cx(t)) = cf_i(x(t))$ for all $c \in \mathbb{F}_{\mathbb{B}}$ and $x \in \mathbb{B}$.

(A4) Function $g_i : \mathbb{B} \to \mathbb{B}$ is Lipschitz continuous with Lipschitz constant L_{g_i}. Also, there exists a constant G_i such that $\|g_i(x(t))\|_{\mathbb{B}} \leq G_i$.

(A5) For all $t \in \mathbb{T}_\omega$, $0 < nL < \left(\left|\frac{\ln|c|}{\omega}\right| \oplus |\underline{d}|\right) \Big/ (1+|\underline{d}|\bar{\mu})$, where $L = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\bar{a}_{ij}L_{f_j} + \sum_{j=1}^{n}\sum_{k=1}^{n}\overline{b_{ijk}\lambda_k}G_jL_{g_j}\right)$, $\bar{a}_{ij} = \sup_{t\in\mathbb{T}_\omega}|a_{ij}(t)|$, $\overline{b_{ijk}\lambda_k} = \sup_{t\in\mathbb{T}_\omega}|b_{ijk}(t)\lambda_k(t)|$, and L_{f_i} is the Lipschitz constant of function f_i.

We can write (3.3.5) as

$$x^\Delta(t) = -Ax(t) + F(t, x(t)), \quad x(0) = \phi$$

with $x = [x_1, ..., x_n]^T$, $A = \mathrm{diag}[\mathrm{d_1(t), ..., d_n(t)}]$ and

$$F(t, x(t)) = \left[\sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t) g_j(\lambda_j(t)x_j(t)) g_k(x_k(t)) + I_i(t)\right]_{i=1}^{n}.$$

Now, assumptions **(A2)** and **(A3)** lead us to

$$\begin{aligned} F(t+\omega, cx) &= \left[\sum_{j=1}^{n} a_{ij}(t) f_j(cx_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} cb_{ijk}(t) g_j\left(\frac{1}{c}\lambda_j(t) cx_j(t)\right) g_k\left(\frac{1}{c}\lambda_k(t) cx_k(t)\right) + cI_i(t)\right]_{i=1}^{n} \\ &= \left[\sum_{j=1}^{n} a_{ij}(t) cf_j(x_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} cb_{ijk}(t) g_j(\lambda_j(t)x_j(t)) g_k(\lambda_k(t)x_k(t)) + cI_i(t)\right]_{i=1}^{n} \\ &= cF(t, x(t)), \end{aligned}$$

implies that $F(t, x(t)) \in P_{\omega,c}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$.

Let us define a norm on the Banach space $\mathbb{B}^n$ as $\|\cdot\|_{\mathbb{B}^n} := \sum_{i=1}^{n} \|\cdot\|_{\mathbb{B}}$. Then, using assumptions **(A3)** and **(A4)**, we can conclude that

$$\begin{aligned} \|F(t,x) - F(t,y)\|_{\mathbb{B}^n} &\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n} |a_{ij}(t)| \|f_j(x_j) - f_j(y_j)\|_{\mathbb{B}} + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}(t)| G_j \|g_k(\lambda_k x_k) - g_k(\lambda_k y_k)\|_{\mathbb{B}}\right) \\ &\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \bar{a}_{ij} L_{f_j} \|x_j - y_j\|_{\mathbb{B}} + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}| G_j L_{g_k} \|\lambda_k x_k - \lambda_k y_k\|_{\mathbb{B}}\right) \\ &\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \bar{a}_{ij} L_{f_j} + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}\lambda_k| G_j L_{g_j}\right) \|x-y\|_{\mathbb{B}^n}, \end{aligned}$$

i.e., $F(t, x(t))$ is Lipschitz continuous with a Lipschitz constant $L = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \bar{a}_{ij} L_{f_j} + \sum_{j=1}^{n}\sum_{k=1}^{n} \overline{b_{ijk}\lambda_k} G_j L_{g_j}\right)$. Furthermore, in view of assumption **(A1)**, for $t \ge s$ we have $\mathscr{F}(t,s) = \mathrm{diag}[\mathrm{e}_{\ominus \mathrm{d}_1}(\mathrm{t,s}), ..., \mathrm{e}_{\ominus \mathrm{d_n}}(\mathrm{t,s})]$. Thus, $\|\mathscr{F}(t,s)\| \le n e_{\ominus \underline{d}}(t,s) \le n e_{\ominus |\underline{d}|}(t,s)$. Thus, using Theorem 3.2.9, we can say that the system (3.3.5) has an exponentially stable (ω, c)-periodic solution when **(A5)** is true.

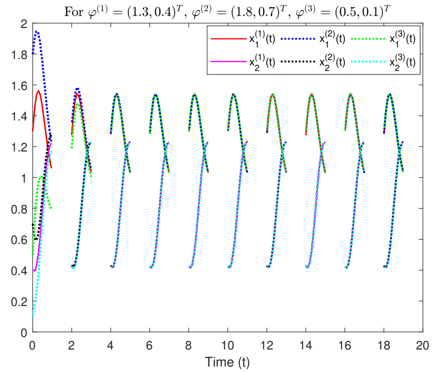
Example 3.3.4. *Consider a system of HNNs with two neurons ($n = 2$) on the time scale $\mathbb{T} = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1]$, with the following activation functions and coefficients:*

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} 0.1x \\ 0.074x \end{bmatrix}, \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} e^{-|x|} \\ \sin(x) \end{bmatrix}, \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} = \begin{bmatrix} c^{t/2} e^{\cos(\pi t)} \\ c^{t/2} e^{\sin(\pi t)} \end{bmatrix}, \begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \end{bmatrix} = \begin{bmatrix} 0.07c^{-t/2}\cos(\pi t) \\ 0.06c^{-t/2}\sin(\pi t) \end{bmatrix},$$

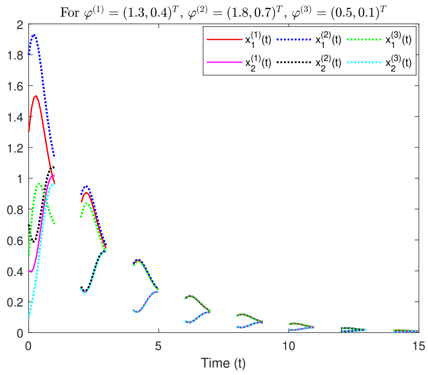
$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0.8 + 0.5|sin(\pi t)| \\ 0.5 + e^{|cos(\pi t)|} \end{bmatrix}, [a_{ij}] = \begin{bmatrix} 0.02e^{\sin \pi t} & -0.017\cos \pi t \\ 0.015 \sin \pi t & 0.01 e^{\cos \pi t} \end{bmatrix},$$

$$[b_{1jk}] = \begin{bmatrix} 0.01c^{t/2}\cos\pi t & 0.03c^{t/2}e^{\sin\pi t} \\ 0.05c^{t/2}\cos(\sin\pi t) & -0.07c^{t/2}\cos\pi t \end{bmatrix}, [b_{2jk}] = \begin{bmatrix} 0.01c^{t/2}\sin\pi t & 0.03c^{t/2}e^{\cos\pi t} \\ 0.05c^{t/2}\sin(\cos\pi t) & -0.07c^{t/2}\sin\pi t \end{bmatrix},$$

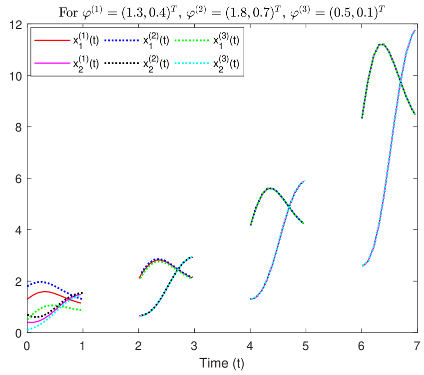
and supplemented with an initial condition $x(0) = \varphi = [1.3, 0.4]^T$.



(a) Solution of Example 3.3.4 with $c = 1$ in the range $[0, 20]$.



(b) Solution of Example 3.3.4 with $c = 0.5$ in the range $[0, 15]$.



(c) Solution of Example 3.3.4 with $c = 2$ in the range $[0, 7]$.

Figure 3.4: Figures (3.4a), (3.4b), and (3.4c) depict the solutions of Example 3.3.4 for the initial value $(1.3, 0.4)^T$ and perturbed initial values $(1.8, 0.7)^T$ and $(0.5, 0.1)^T$, with $c = 1$, $c = 0.5$, and $c = 2$, respectively. Each figure shows $x^{(k)}(t)$, representing the behaviour of the solution corresponding to the initial condition $\varphi^{(k)}$ for $k = 1, 2, 3$.

We have $\omega = 2$, *and* $\mu = 0$ *for all* $t \in [2n, 2n+1)$, $\mu = 1$ *for all* $t \in \{2n+1\}$, $n \in \mathbb{Z}$. *Also, for* $i, j, k = 1, 2$, *we can easily compute the bounds as*

$$\underline{d} = 0.8, \quad \bar{a}_{ij} \le 0.02, \quad \overline{b_{ijk}\lambda_k} \le 0.005, \quad G_1 = 1, \quad G_2 = 1,$$

$$\bar{L}_{f_1} = 0.1, \quad \bar{L}_{f_2} = 0.074, \quad \bar{L}_{g_1} = 1, \quad \bar{L}_{g_2} = 1.$$

Thus,

- *for* $|c| = 1$, *we have* $\left(\frac{\ln 1}{2} \oplus \underline{d}\right) \Big/ (1 + \underline{d}\bar{\mu}) > 0.44 > 2L \ (= 0.096)$.

- *for* $|c| = 0.5 > e^{-1}$, *we have* $\left(\frac{\ln 0.5}{2} \oplus \underline{d}\right) \Big/ (1 + \underline{d}\bar{\mu}) > 0.097 > 2L$.
- *for* $|c| = 2 < e^{2}$, *we have* $\left(\frac{\ln 2}{2} \oplus \underline{d}\right) \Big/ (1 + \underline{d}\bar{\mu}) > 0.79 > 2L$.

Since all the conditions of assumptions (A1)-(A5) are satisfied, for given c, system (3.3.1) *has a unique exponentially stable (2,c)-periodic solution on* $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]$. *(See Figure 3.4.)*

Example 3.3.5. *The system of HNN on the time scale* $\mathbb{R}$, *given as*

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t)g_j(\lambda_j(t)x_j(t))g_k(\lambda_k(t)x_k(t)) + I_i(t),$$

with the same coefficient as given in Example 3.3.4 and the initial condition $\varphi = [1.3, 0.4]^T$, *has a unique exponentially stable (2,c)-periodic solution provided that* $|c| > e^{2(2L-\underline{d})} = e^{-1.408} \approx 0.2446$.

3.3.3 Keynesian-Cross model on time scales

The (ω, c)-periodic Keynesian-Cross model provides a framework to analyze business cycles with structural shifts, where output oscillates with an amplitude that changes over time due to factors such as policy lags, shifts in consumer behavior, or investment dynamics. Fiscal components like government expenditure, taxation, and consumption may exhibit (ω, c)-periodic trends driven by inflation, debt accumulation, or political cycles. Studying (ω, c)-periodic solutions can enhance our understanding of whether output stabilizes or diverges in non-stationary economic environments. This approach can help to determine whether the economy tends to maintain or converge to a bounded trajectory under time-varying, cyclically amplifying fiscal influences, offering valuable insight into the conditions for long-run macroeconomic stability.

Consider the following Keynesian-Cross model with lagged income in $\mathbb{T}_\omega$:

$$\begin{aligned} D(t) &= C(t) + I(t) + G(t), \\ C(t) &= \beta S(t) + \gamma y(t), \\ y^{\Delta}(t) &= \delta[D^\sigma - y], \quad t \geq t_0, \end{aligned} \tag{3.3.6}$$

where y represents the aggregate income, D represents the aggregate demand, C is the aggregate consumption, I is the aggregate investment, G represents the government spending, and S is the aggregate savings. The positive constant δ (< 1) is the speed of adjustment term, and β and γ (< 1) are non-negative constants. In (3.3.6), $D^\sigma(t)$ denotes the forward shift of $D(t)$, that is, $D^\sigma(t) := D(\sigma(t))$. We have considered that current consumption depends on savings and current income, and the change in income is a fraction of excess demand at $\sigma(t)$ over income at t. Also, consider the following hypotheses:

(H1) $1 - \delta\gamma\mu(t) > 0$ for $t > t_0$.

(H2) $e^{-\omega/\bar{\mu}} < |c| < e^{\omega/\bar{\mu}}$.

(H3) $I, G, S \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{R}^+)$.

We can rewrite system (3.3.6) as follows:

$$y^\Delta = \delta[\beta S(\sigma(t)) + \gamma y^\sigma(t) + I(\sigma(t)) + G(\sigma(t)) - y]. \tag{3.3.7}$$

Now, using $y^\sigma = y + \mu y^\Delta$ and considering (H1), we have

$$\begin{aligned} y^\Delta &= -\frac{\delta(1-\gamma)}{(1-\gamma\delta\mu(t))}y + \frac{\delta(\beta S(\sigma(t)) + I(\sigma(t)) + G(\sigma(t)))}{1-\gamma\delta\mu(t)} \\ &= -\alpha(t)y + f(t). \end{aligned} \tag{3.3.8}$$

Given (H1), we have $1-\mu(t)\alpha(t) > 0$, i.e., $-\alpha(t) \in \mathcal{R}^+(\mathbb{T}_\omega, \mathbb{R}^+)$. Thus, $|\mathscr{F}(t,s)| = e_{\ominus\alpha}(t,s) \le e_{\ominus\underline{\alpha}}(t,s)$ for $t \ge s$. In view of (H3), $f(t)$ is an (ω, c)-periodic function. Thus, using Corollary 3.2.10, we can conclude that the model (3.3.6), along with hypotheses (H1)-(H3), has an exponentially stable (ω, c)-periodic solution, provided that $\frac{\ln|c|}{\omega} \oplus \underline{\alpha} > 0$.

Example 3.3.6. *The Keynesian-Cross model on the time scale* $\mathbb{T} = \bigcup_{n\in 2\mathbb{Z}} [0.5n, 0.5n+0.5]$*, given as*

$$\begin{aligned} D(t) &= C(t) + 2c^t e^{\cos 2\pi t} + 0.3c^t \sin(\cos 2\pi t); \\ C(t) &= 2c^t e^{\sin 2\pi t} + 0.08y(t); \\ y^\Delta(t) &= 0.7[D^\sigma - y], \quad t \ge 0 \quad \textit{and} \quad y(0) = 3.43, \end{aligned}$$

has an exponentially stable $(1, c)$*-periodic mild solution when* $e^{-0.487} < |c| < e^2$*. (See Figure 3.5).*

Example 3.3.7. *The Keynesian-Cross model on the time scale* $\mathbb{R}$*, given as*

$$\begin{aligned} D(t) &= C(t) + 2c^t e^{\cos 2\pi t} + 0.3c^t \sin(\cos 2\pi t); \\ C(t) &= 2c^t e^{\sin 2\pi t} + 0.08y(t); \\ y'(t) &= 0.7[D - y], \quad t \ge 0 \quad \textit{and} \quad y(0) = 3.43, \end{aligned}$$

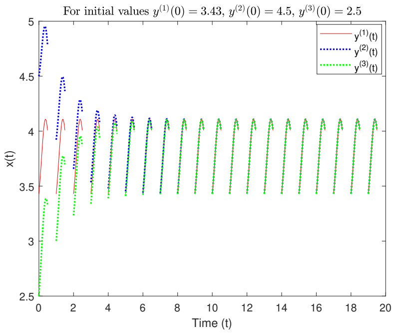
has an exponentially stable $(1, c)$*-periodic mild solution when* $|c| > e^{-0.64} \approx 0.53$.

3.3.4 A cellular neural network (CNN) on time scales

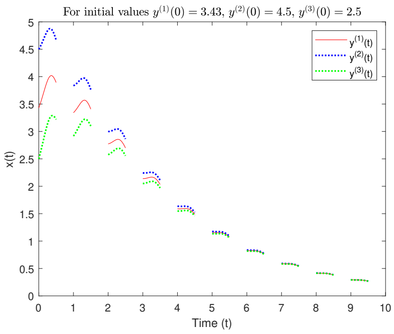
Consider a dynamic equation on $\mathbb{T}_\omega$, given as

$$x^\Delta = -\alpha x(t) + \beta(t)\tanh(\lambda(t)x(t)) + \phi(t), \quad x(0) = x_0, \tag{3.3.9}$$

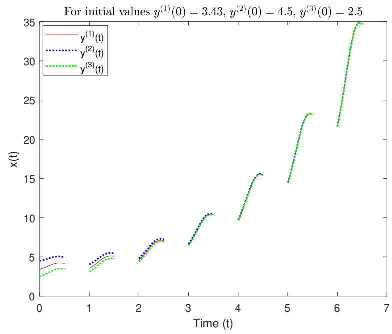
where $\alpha > 0$, and



(a) Solution of Example 3.3.6 with $c = 1$ in the range $[0, 20]$.



(b) Solution of Example 3.3.6 with $c = 0.7$ in the range $[0, 10]$.



(c) Solution of Example 3.3.6 with $c = 1.5$ in the range $[0, 7]$.

Figure 3.5: Figures (3.5a), (3.5b), and (3.5c) depict the solutions of Example 3.3.6 for the initial value 3.43 and perturbed initial values 4.5 and 2.5, with $c = 1$, $c = 0.7$, and $c = 1.5$, respectively. Each figure shows $y^{(k)}(t)$, representing the behaviour of the solution corresponding to the initial condition $y^{(k)}(0)$ for $k = 1, 2, 3$.

(A*1) $-\alpha \in \mathcal{R}^+(\mathbb{T}_\omega, \mathbb{R})$,

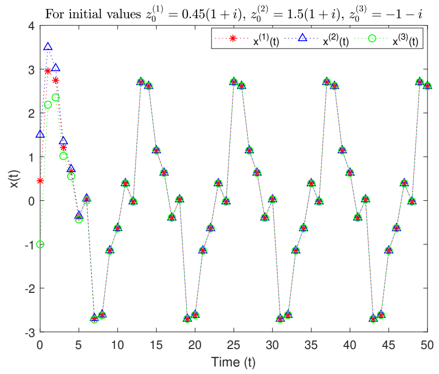
(A*2) $e^{-\omega\alpha/(1+\alpha\bar{\mu})} < |c| < e^{-\omega/\bar{\mu}}$,

(A*3) $\beta, \phi \in P_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, $\lambda \in P_{\omega,\frac{1}{c}}(\mathbb{T}_\omega, \mathbb{B})$,

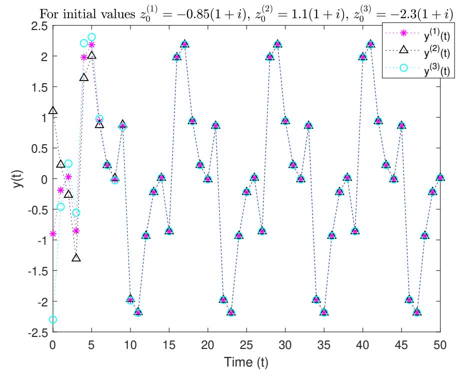
(A*4) for all $t \in \mathbb{T}$, $\overline{\beta\lambda} < \left(\frac{\ln|c|}{\omega} \oplus \alpha\right) \Big/ (1 + \alpha\bar{\mu})$, where $\overline{\beta\lambda} = \sup_{t\in\mathbb{T}} |\beta(t)\lambda(t)|$.

Notice that $f(t, x) = \beta \tanh(\lambda x) + \phi(t)$ satisfies $\|f(t, x) - f(t, y)\|_{\mathbb{B}} \le \overline{\beta\lambda}\|x - y\|_{\mathbb{B}}$. Therefore, if **(A*1)**–**(A*4)** hold, then (3.3.9) admits a unique mild solution that is (ω, c)-periodic and exponentially stable.

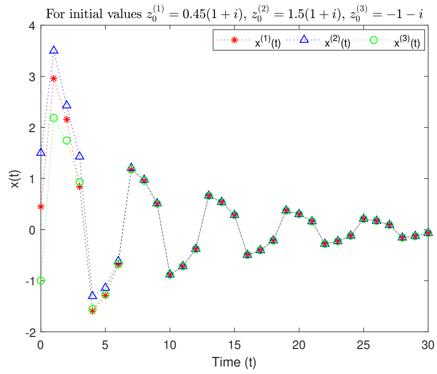
Remark 3.3.8. (3.3.9) *is commonly used to represent the single self-excited neuron on time scales without delayed excitation. Here x represents the neuron's voltage, α is the capacitance to resistance ratio, and β is the feedback strength function. The function ϕ represents the neuron's external input.*



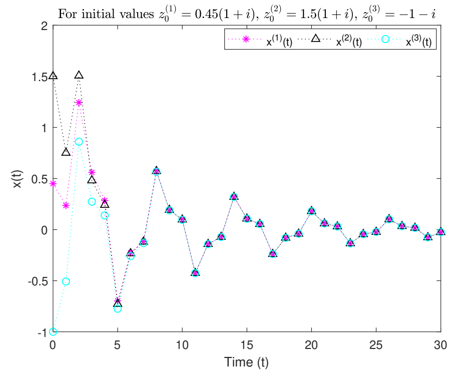
(a) Real part of the solution of Example 3.3.9 with $c = i$, plotted over the range $[0, 50]$.



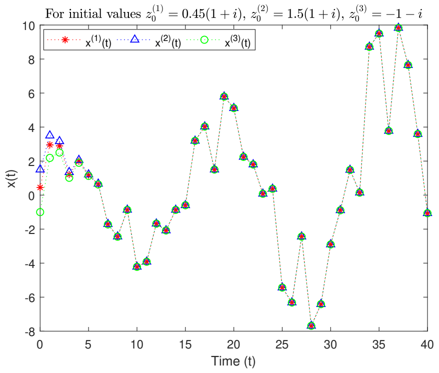
(b) Imaginary part of the solution of Example 3.3.9 with $c = i$, plotted over the range $[0, 50]$.



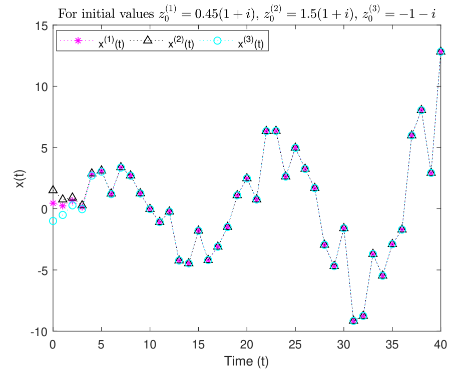
(c) Real part of the solution of Example 3.3.9 with $c = -0.75$, plotted over the range $[0, 30]$.



(d) Imaginary part of the solution of Example 3.3.9 with $c = -0.75$, plotted over the range $[0, 30]$.



(e) Real part of the solution of Example 3.3.9 with $c = 0.5 + i$, plotted over the range $[0, 40]$.



(f) Imaginary part of the solution of Example 3.3.9 with $c = 0.5 + i$, plotted over the range $[0, 40]$.

Figure 3.6: Figures (3.6a), (3.6c), and (3.6e) show the real parts of the solution trajectories for Example 3.3.9, with initial values $0.45(1+i)$, $1.5(1+i)$, and $-1-i$, respectively. Figures (3.6b), (3.6d), and (3.6f) display the corresponding imaginary parts of these trajectories. The solutions are analyzed under different values of the parameter c: i, -0.75, and $0.5+i$, respectively. For $k = 1, 2, 3$, each figure depicts the behaviour of the solution $x^{(k)}(t)$ or $y^{(k)}(t)$) corresponding to the initial condition $z_0^{(k)}$.

Example 3.3.9. *The complex valued discrete CNN given as*

$$\begin{aligned} z(t+1)-z(t) &= -0.5z(t)+\frac{1}{13e}c^{t/3}e^{i\sin(2\pi t/3)}\tanh(c^{-t/3}\cos(2\pi t/3)z(t)) \\ &\quad +c^{t/3}(e^{\cos(2\pi t/3)}+i\sin(2\pi t/3)), \quad z(t)=x(t)+iy(t),\ t\in\mathbb{Z}^+, \\ z(0)\ (=z_0) &= 0.45(1+i), \end{aligned}$$

has a unique exponentially stable $(3,c)$-periodic mild solution for $e^{-10/13}<|c|<e^3$. (See Figure 3.6).

Example 3.3.10. *The complex valued CNN on the time scale $\mathbb{R}$ given as*

$$\begin{aligned} z'(t) &= -0.5z(t)+\frac{1}{13e}c^{t/3}e^{i\sin(2\pi t/3)}\tanh(c^{-t/3}\cos(2\pi t/3)z(t))+c^{t/3}(e^{\cos(2\pi t/3)}+i\sin(2\pi t/3)),\ t\in\mathbb{R}^+, \\ z(0) &= 0.45(1+i), \end{aligned}$$

has a unique exponentially stable $(3,c)$-periodic mild solution for all c such that $|c|>e^{-11/13}$.

3.3.5 A second-order dynamic equation

As a final application, we consider a dynamic equation on $\mathbb{T}=\bigcup_{n\in\mathbb{Z}}[n(a+b),n(a+b)+b]$ given as

$$x^{\Delta\Delta}+a_1x^{\Delta}+a_2x=\alpha c^{kt/(2\pi))}\sin(kt)(x+1) \tag{3.3.10}$$
$$x(0)=x_0,\quad x^{\Delta}(0)=x_0^{\Delta},$$

where for $\lambda=\frac{(a_1-\sqrt{a_1^2-4a_2})}{2}$, $e^{-\lambda 2\pi/k(1+\lambda a)}<c\le 1$, and $k\in\mathbb{R}$ is such that $\frac{2\pi}{k}\in\Pi_{\mathbb{T}}^+$.

Using $y=x^{\Delta}$, (3.3.10) can be transformed in a first-order system of dynamic equations, given as

$$u^{\Delta}=Au+f(t,u),\quad u(0)=u_0, \tag{3.3.11}$$

where $u=\begin{bmatrix} x \\ y\end{bmatrix}$, $A=\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1\end{bmatrix}$ and $f=\begin{bmatrix} 0 \\ \alpha c^{kt/2\pi}\sin(kt)(x+1)\end{bmatrix}$.

If the eigenvalues of the matrix A, i.e., $\frac{-a_1\pm\sqrt{a_1^2-4a_2}}{2}$ are real and distinct, we have

$$\mathscr{F}(t,s)=e_A(t,s)=\begin{bmatrix} e_{(-a_1+\sqrt{a_1^2-4a_2})/2}(t,s) & e_{(-a_1-\sqrt{a_1^2-4a_2})/2}(t,s) \\ \frac{-a_1+\sqrt{a_1^2-4a_2}}{2}e_{(-a_1+\sqrt{a_1^2-4a_2})/2}(t,s) & \frac{-a_1-\sqrt{a_1^2-4a_2}}{2}e_{(-a_1-\sqrt{a_1^2-4a_2})/2}(t,s)\end{bmatrix},$$

provided that $\frac{-a_1\pm\sqrt{a_1^2-4a_2}}{2}\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$.

If we consider $a_2>0$, then we have

$$\|\mathscr{F}(t,s)\|\le Ke_{(-a_1+\sqrt{a_1^2-4a_2})/2}(t,s)\le Ke_{\ominus(a_1-\sqrt{a_1^2-4a_2})/2}(t,s),\ \text{for some } K\ge 1.$$

Also, $\|f(t,u_1)-f(t,u_2)\| \leq |\alpha c^{kt/2\pi}\sin(kt)(x_1-x_2)| \leq |\alpha|\|u_1-u_2\|$, when $|c|\leq 1$.

Thus, if $K|\alpha| < \left(\frac{k\ln|c|}{2\pi} \oplus \frac{(a_1-\sqrt{a_1^2-4a_2})}{2}\right) \Big/ \left(1+a\frac{(a_1-\sqrt{a_1^2-4a_2})}{2}\right)$ for all $t\in\mathbb{T}$, then (3.3.10) has a unique $(2\pi/k,c)$-periodic mild solution that will also be exponentially stable.

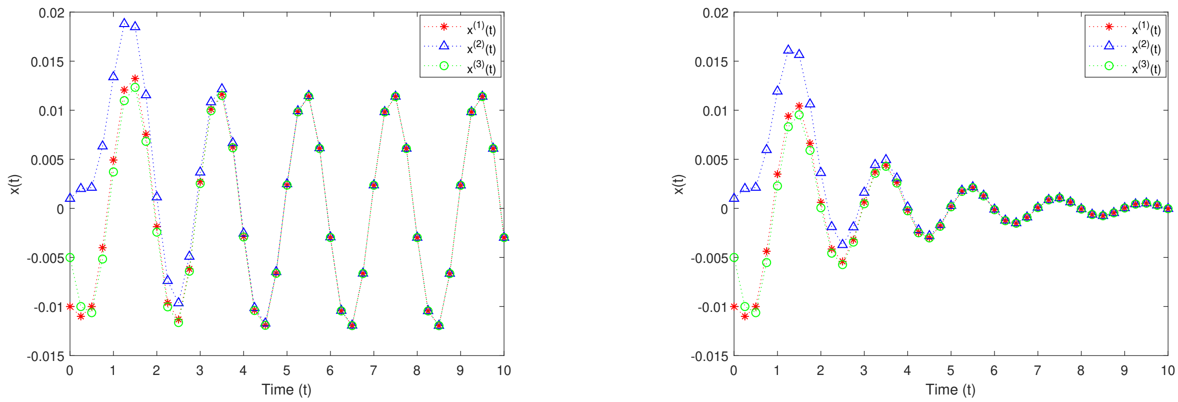
Example 3.3.11. *Consider $a=\frac{1}{4}$ and $b=0$, i.e, $\mathbb{T}=\frac{1}{4}\mathbb{Z}$, and the following dynamic equation on $\mathbb{T}$:*

$$x^{\Delta\Delta}+3x^{\Delta}+2x=\frac{1}{4}c^{t/2}(\sin(\pi t))(x+1), \tag{3.3.12}$$
$$x(0)=-0.01, \quad x^{\Delta}(0)=-0.011,$$

which can be written as

$$u^{\Delta}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}u+\begin{bmatrix}0\\ 0.1c^{t/2}\sin(\pi t)(x(t)+1)\end{bmatrix}, \quad \textit{with} \quad u=\begin{bmatrix}x\\ x^{\Delta}\end{bmatrix}.$$

We can easily compute $\mathscr{F}(t,s)=\begin{bmatrix}e_{-1}(t,s) & e_{-2}(t,s)\\ -e_{-1}(t,s) & -2e_{-2}(t,s)\end{bmatrix}$, which implies $\|\mathscr{F}(t,s)\|\leq 3e_{\ominus 1}(t,s)$ for $t\geq s$. Thus, for $e^{-1.25}<|c|\leq 1$, (3.3.12) has a unique $(2,c)$-periodic mild solution that will also be exponentially stable and can be seen in Figure 3.7



(a) Solution of Example 3.3.11 with $c=1$, plotted over the range $[0,10]$.

(b) Solution of Example 3.3.11 with $c=0.5$, plotted over the range $[0,10]$.

Figure 3.7: Figures (3.7a), and (3.7b) show the solution trajectories of Example 3.3.11 corresponding to the initial values $[x_0,x_0^{\Delta}]^{(1)}=[-0.01,-0.011]$ and perturbed initial values $[x_0,x_0^{\Delta}]^{(2)}=[0.001,0.002]$ and $[x_0,x_0^{\Delta}]^{(3)}=[-0.005,-0.01]$, under the values of c as 1, and 0.5, respectively. For $k=1,2,3$, in each figure, $x^{(k)}(t)$ represents the behavior of the solution corresponding to the initial condition $[x_0,x_0^{\Delta}]^{(k)}$.

Example 3.3.12. *Consider $a=0$ and $b=1$, i.e, $\mathbb{T}=\mathbb{R}$, and the following differential equation:*

$$x''+3x'+2x=0.1c^{t/2}\sin(\pi t)(x+1) \tag{3.3.13}$$

$$x(0) = 0.28, \quad x'(0) = 0.003.$$

Then, for all $c \in (e^{-1.4}, 1]$*, we have a unique* $(2, c)$*-periodic mild solution for* (3.3.13) *that will also be exponentially stable.*

3.4 Conclusion

In this chapter, we have established sufficient conditions ensuring the existence and uniqueness of (ω, c)-periodic mild solutions for linear and semilinear abstract dynamic equations on ω-periodic time scales. In addition, the exponential stability of these solutions has been thoroughly analyzed. We have also investigated the continuous dependence of exponentially stable (ω, c)-periodic mild solutions on the source term within a class of nonautonomous abstract semilinear dynamic equations. To illustrate the practical relevance of our theoretical results, we applied them to several dynamic equations on ω-periodic time scales, including the Lasota–Ważewska model, Hopfield neural networks, the Keynesian-Cross model, a cellular neural network, and a second-order dynamic equation. For each model, the existence, uniqueness, and stability of (ω, c)-periodic mild solutions are examined. These applications demonstrate the versatility, robustness, and broad applicability of the developed framework.

CHAPTER 4

(ω, c)-Asymptotically Periodic Oscillation in Delayed Cellular Neural Networks on Time Scales [1]

This chapter introduces the notion of (ω, c)-asymptotic periodicity on ω-periodic time scales as a natural extension of (ω, c)-periodicity. Fundamental properties of this new class of functions are systematically investigated. As an application, the theoretical framework is employed to analyze delayed cellular neural networks (DCNNs) defined on time scales. By utilizing appropriate analytical techniques, sufficient conditions are derived to guarantee the existence and uniqueness of (ω, c)-asymptotically periodic solutions for the considered networks. Furthermore, the global exponential stability of these solutions is established, demonstrating the effectiveness of the proposed approach in studying the long-term dynamical behavior of DCNNs on arbitrary time domains.

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4.1 Introduction

Neural networks (NNs) have gathered significant attention due to their successful applications in various fields such as image processing, signal processing, fixed point computation, connected component detection, automatic control, disease diagnosis, associative memory pattern classification, solving nonlinear algebraic equations and optimal control problems of nonlinear large-scale systems [157]. These diverse applications largely depend on the dynamical properties of the networks, making analysis of their dynamic behavior a critical aspect of neural network design. Some recent studies on the dynamics of neural networks can be found in [158, 159, 160]. Cellular neural networks (CNNs), introduced by Chua and Yang in 1988 [161, 162], are composed of cells containing linear and nonlinear circuit elements, including capacitors, resistors, controlled sources, and independent sources, and are organized in a cellular automaton–like structure with only local interconnections. CNNs have demonstrated effectiveness in image processing [163], classification tasks [164, 165], and semantic segmentation [166], and are widely employed in computer vision applications [163]. From a practical perspective, the analog and continuous-time dynamics of CNNs offer advantages such as reduced power consumption [164]. Mathematically, CNNs are modeled by systems of differential equations that capture the interactions among interconnected cells, where complex global dynamics may emerge despite the simple structure of individual cells. Incorporating delays into CNN model is essential for accurately capturing the finite speed of signal transmission between cells; however, the presence of delays may induce instability, bifurcations, or even the destruction of oscillatory behavior. Since the introduction of delayed cellular neural networks (DCNNs) in 1990 [167], significant research interest has been directed toward investigating the dynamical behavior of DCNNs with constant as well as time-varying delays. Discrete delays are commonly used to model signal transmission in neural networks, reflecting the dependence of neurons on past state values. Nevertheless, since signal propagation often occurs over a finite time interval, distributed delays must also be incorporated to obtain a more realistic description. Consequently, many neural network models involve mixed delays, combining discrete time-varying delays with continuously distributed delays [168]. It has been observed that time delays appearing in the stabilizing negative feedback term, commonly referred to as the leakage term, can significantly degrade system stability and exert a crucial influence on the dynamical behavior of CNNs. Motivated by this observation, delays have been explicitly introduced into the leakage term, prompting extensive research aimed at establishing delay-dependent sufficient conditions for the stability of equilibria in CNN systems.

In practical implementations of neural networks, it is often necessary to consider discrete-time systems as analogues of continuous-time models, particularly since discrete-time formulations are more convenient for computer-based processing. Nevertheless, the dynamics of discrete-time neural networks can differ significantly from those of their continuous-time counterparts. In particular, criteria established for continuous-time CNNs to guarantee the existence and stability of periodic or generalized periodic solutions may fail to apply in discrete-time settings, highlighting the need for separate analysis. Moreover, various discrete time domains arise naturally in applications, making it impractical to investigate each discrete case individually and derive comparative results. This challenge motivates the study of neural network dynamics on time scales, which provide a unified framework for continuous and discrete systems and are therefore of substantial practical and theoretical significance.

The dynamics of CNNs has been extensively studied due to their relevance to a wide range of applications, with research focusing on fundamental qualitative properties such as the existence and stability of equilibrium points, periodic solutions, and generalized periodic solutions (see [92, 169, 170, 171, 172]). Motivated by the observation that many natural and artificial neural systems exhibit periodic oscillatory behavior, considerable attention has been devoted to the study of periodic and generalized periodic solutions and their stability. Notably, the stability analysis of periodic solutions is more general than that of equilibrium points, as the latter can be viewed as a special case of periodic solutions. Consequently, the existence and stability of (generalized) periodic solutions in DCNNs has been widely studied in continuous-time [91, 173, 174, 178, 179, 180], discrete-time [175, 176], and time-scale settings [177, 181, 182, 183]. In addition to periodic behavior, the transmission of signals in neural networks is often characterized by antiperiodic dynamics, making antiperiodic solutions important in the analysis, design, and control of neural networks. Such solutions have applications in automatic control, artificial intelligence, disease diagnosis, and other engineering fields. Motivated by these applications, antiperiodic and generalized antiperiodic solutions of CNNs have also been extensively investigated, and sufficient conditions for the existence and stability of antiperiodic solutions in various CNNs have been established using a variety of analytical techniques (see [184, 185, 186, 187, 188]), covering models with leakage delays, oscillating coefficients, time-varying transmission delays, as well as quaternion and Clifford-valued CNNs. The importance of both periodic and antiperiodic behaviors in CNNs motivates the study of their dynamics within a generalized analytical framework. In particular, Bloch-type periodic behavior may become relevant in CNNs when phase-shifted oscillations, wave-like patterns, or amplitude-modulated responses arise due to delays, spatial coupling, or feedback mechanisms, situations in which classical periodicity may be too restrictive. In this direction, the concept of (ω, c)-periodicity provides a generalized framework for modeling and analyzing CNN dynamics by incorporating classical periodicity and antiperiodicity as special cases, while allowing for a broader class of dynamical structures. The case $|c| \neq 1$ may be of particular interest in this context, as it could represent amplitude scaling effects induced by leakage terms, delays, or nonlinear interactions.

Although (ω, c)-periodicity provides a robust framework for analyzing wide range of phenomena, real-world systems may deviate from exact (ω, c)-periodicity. Nevertheless, the concept remains useful for the study of behaviors that approximate this structure within a controlled margin of error, which motivated the introduction of (ω, c)-asymptotic periodicity (see [101]). This class encompasses functions that may not initially exhibit (ω, c)-periodicity but eventually acquire this property. To the best of our knowledge, the discrete counterpart of (ω, c)-asymptotic periodicity and its extension to the time scale remain largely unexplored in the literature. In addition, there is a notable gap in existing work concerning (ω, c)-asymptotically periodic solutions to CNNs, including in the continuous-time setting. Accordingly, the aim of this chapter is to introduce and investigate the concept of (ω, c)-asymptotic periodicity on time scales and to establish a foundational understanding of the basic properties of (ω, c)-asymptotically periodic functions within this framework. Rather than addressing the existence and exponential stability of (ω, c)-asymptotically periodic solutions of CNNs described separately by differential or difference equations, we focus on CNN systems formulated through dynamic equations on time scales, thereby ensuring the applicability of the results across a broad class of time domains. Moreover, the criteria for the existence and stability of solutions depend explicitly on the magnitude of the delays, shedding light on the roles of leakage delays and mixed time-varying delays in shaping the dynamical behavior of the network.

The organization of this chapter is as follows. In Section 4.2, the concept of (ω, c) asymptotic periodicity on time scales is introduced, the properties of (ω, c)-asymptotically periodic functions are investigated, and the corresponding function space is analyzed. Section 4.3 presents a system of (ω, c)-asymptotically periodic CNNs with leakage delays and mixed time-varying delays on time scales. Moreover, under appropriate conditions, the existence and uniqueness of (ω, c)-asymptotically periodic solutions are established, and the global exponential stability of the solution is proved in this section. In Section 4.4, numerical examples and simulations on various time scales are provided to illustrate the theoretical results. Finally, the chapter is concluded.

4.2 (ω, c)-asymptotically periodic functions on ω-periodic time scales

Throughout this chapter, let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach space over the field $\mathbb{F}_{\mathbb{B}} = \mathbb{R}$ (or $\mathbb{C}$). Furthermore, let $(\mathrm{BC}_{\mathrm{rd}})_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B})$ represent the space of $\mathbb{B}$-valued rd-continuous functions f defined on $\mathbb{T}$ such that $c^{-t/\omega} f(t)$ is bounded for $t \in \mathbb{T}$, and let $\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ denote the collection of functions $f : \mathbb{T}_\omega \to \mathbb{B}$ such that f is (ω, c)-periodic (with the same c-period ω), as defined in Chapter 2. Here, $\mathbb{T}_\omega$ respresnts a time scale $\mathbb{T}$ that is ω periodic. We have also seen that the space $(\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}), \sup_{\mathbb{T}_\omega} \|\cdot\|_{\omega,\mathrm{c}})$ form a Banach space, where $\|\cdot\|_{\omega,c}$ is define as

$$\|f\|_{\omega,c} := \|c^{-t/\omega} f(t)\|_{\mathbb{B}} \tag{4.2.1}$$

Before introducing the concept of (ω, c)-asymptotic periodicity on time scales, we first present asymptotic functions on time scales.

Definition 4.2.1. *A function $f \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ is said to be (ω, c)-asymptotic function if for each $\varepsilon > 0$, there exists a $\mathrm{T} \in \mathbb{T}_\omega^{0+}$ such that $\|f\|_{\omega,c} < \varepsilon$ for all $t \geq \mathrm{T}$. The collection of (ω, c)-asymptotic functions is evidently a vector space and is denoted by $\mathcal{A}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.*

We now introduce the concept of (ω, c)-asymptotic periodicity on $\mathbb{T}_\omega$ as a generalization of the corresponding notion on $\mathbb{R}$ defined in [101].

Definition 4.2.2. *A function $f \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ is said to be (ω, c)-asymptotically periodic if it can be decomposed as the sum of an (ω, c)-periodic function and an (ω, c)-asymptotic function, that is, if*

$$f = f_{op} + f_{oa} \text{ with } f_{op} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}) \text{ and } \mathrm{f_{oa}} \in \mathcal{A}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}).$$

We denote the collection of $\mathbb{B}$-valued (ω, c)-asymptotically periodic functions on $\mathbb{T}_\omega$ that share the same c-period ω for their periodic part by $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.

Example 4.2.3. *Consider a function $f : \mathbb{T} \to \mathbb{B}$, defined as*

$$f(t) = a^t \cos\left(\frac{\pi t}{k}\right) + b^t g(t),$$

where $a, b \in \mathbb{B}$, $k \in \Pi_{\mathbb{T}}^+$, $|b| < |a|$ and $g : \mathbb{T} \to \mathbb{B}$ is a bounded function. Then, as

$$a^{(t+k)} \cos\left(\frac{\pi}{k}(t+k)\right) = -a^k a^t \cos\left(\frac{\pi t}{k}\right) \quad \textit{and}$$

$$\|(-a^k)^{-t/k} b^t g(t)\|_{\mathbb{B}} = \left\| \left(\frac{b}{a}\right)^t g(t) \right\|_{\mathbb{B}} \to 0, \ \textit{as } t \to \infty,$$

f is a $(k, -a^k)$-asymptotically periodic function on $\mathbb{T}$, with an associated $(k, -a^k)$-periodic part $a^t \cos\left(\frac{\pi t}{k}\right)$ and $(k, -a^k)$-asymptotic part $b^t g(t)$.

It is important to note that while every (ω, c)-periodic function is inherently asymptotically (ω, c)-periodic, the converse is not always true. Let $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$. We know that f will be asymptotically periodic (with a period ω) if $f = f_p + f_a$, where f_p is an ω-periodic function and f_a is an asymptotic function, i.e., if $f_p(t+\omega) = f_p(t)$ for all $t \in \mathbb{T}_\omega$, and $\lim_{t\to\infty} f_a(t) = 0$. Clearly, we can also view f as an $(\omega, 1)$-asymptotically periodic function. Thus, $\mathrm{AP}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ (or $\mathrm{AP}_\omega(\mathbb{T}_\omega, \mathbb{B})$) is the collection of asymptotically periodic functions with a period ω. In the same way, $\mathrm{AP}_{\omega,-1}(\mathbb{T}_\omega, \mathbb{B})$ denotes the collection of asymptotically antiperiodic functions with an antiperiod ω, that is, if $f \in \mathrm{AP}_{\omega,-1}(\mathbb{T}, \mathbb{B})$, then $f = f_{ap} + f_a$, where $f_{ap}(t+\omega) = -f_{ap}(t)$ for all $t \in \mathbb{T}_\omega$, and $\lim_{t\to\infty} f_a(t) = 0$. The function $f = f_{bp} + f_a$ is an asymptotically ω-Bloch periodic function if $f_{bp}(t+\omega) = e^{ik\omega} f_{bp}(t)$ for all $t \in \mathbb{T}_\omega$, where $k \in \mathbb{N}$, and $\lim_{t\to\infty} f_a(t) = 0$.

Obviously, such functions can be categorized as $\mathrm{AP}_{\omega,\mathrm{e}^{\mathrm{i}k\omega}}(\mathbb{T}_\omega, \mathbb{B})$. Based on these observations, we can conclude the following:

Remark 4.2.4. *The collection $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ comprises functions that are asymptotically ω-periodic, asymptotically ω-antiperiodic, and asymptotically ω-Bloch periodic. Furthermore, if $|c| \neq 1$, the (ω, c)- asymptotically periodic functions will be unbounded. As a result, the study of $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ allows us to collectively analyze a wide range of functions belonging to different subclasses.*

Now, to provide a characterization of the (ω, c)-asymptotically periodic functions, we present the following proposition.

Proposition 4.2.5. *Let $f \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$. Then $f \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ iff there exists $u \in \mathrm{AP}_\omega(\mathbb{T}_\omega, \mathbb{B})$ such that*

$$f(t) = c^{t/\omega} u(t) \quad \textit{for all} \quad t \in \mathbb{T}.$$

Proof. Let us first assume that there exists an asymptotically ω-periodic function u such that $f(t) = c^{t/\omega} u(t)$ for all $t \in \mathbb{T}_\omega$. Then, by definition, there exists an ω-periodic function u_p and an asymptotic function u_a such that $u = u_p + u_a$. Consequently, we have $f = f_{op} + f_{oa}$, with $f_{op} = c^{t/\omega} u_p$ and $f_{oa} = c^{t/\omega} u_a$, satisfying the condition

$$f_{op}(t + \omega) = c f_{op}(t) \quad \text{for all} \quad t \in \mathbb{T}_\omega.$$

Moreover, for each $\varepsilon > 0$, there exists a positive T such that

$$\|f_{oa}\|_{\omega,c} = \|c^{-t/\omega} f_{oa}\|_{\mathbb{B}} = \|u_a\|_{\mathbb{B}} < \varepsilon \quad \text{whenever} \quad t \geq T.$$

This implies that f is indeed an (ω, c)-asymptotically periodic function on $\mathbb{T}_\omega$. Conversely, consider f to be an (ω, c)-asymptotically periodic function on $\mathbb{T}_\omega$. By its definition, $f = f_{op} + f_{oa}$, such that $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Now, using the characterization of the (ω, c)-periodic functions on $\mathbb{T}_\omega$ (see Proposition 2.2.11), we have $c^{-t/\omega} f_{op} \in \mathrm{P}_\omega(\mathbb{T}_\omega, \mathbb{B})$. Furthermore, the definition of $\mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ clarifies that $\lim_{t \to \infty} c^{-t/\omega} f_{oa} = 0$. Thus, $c^{-t/\omega} f(t) = u(t) \in \mathrm{AP}_\omega(\mathbb{T}_\omega, \mathbb{B})$, i.e., there exists an asymptotically ω-periodic function u on $\mathbb{T}_\omega$ such that $f(t) = c^{t/\omega} u(t)$. The proof is complete. □

Next, suppose that $f \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ and $f = f_{op} + f_{oa} = g + h$, where $f_{op}, g \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa}, h \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Then,

$$f_{op} - g = h - f_{oa} := u.$$

Since u belongs to both $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $\mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, there will be a function v, both ω-periodic and asymptotic, such that $u(t) = c^{-t/\omega} v(t)$ for all $t \in \mathbb{T}_\omega$. This is only possible when $v \equiv 0$, which implies that $u \equiv 0$. Consequently, $f_{op}(t) = g(t)$ and $f_{oa}(t) = h(t)$ for all $t \in \mathbb{T}_\omega$. Thus, we have the following observation:

Remark 4.2.6. *The decomposition of an (ω, c)-asymptotically periodic function as the sum of an (ω, c)-periodic function and an (ω, c)-asymptotic function is unique.*

Now, let us take a look at some basic properties of (ω, c)-asymptotically periodic functions on the time scale $\mathbb{T}$.

Theorem 4.2.7. *If $f, g \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, then*

1. $f + g \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$,
2. $\alpha f \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *for each* $\alpha \in \mathbb{B}$,
3. $f_\tau(t) := f(t + \tau) \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ *for each fixed* $\tau \in \Pi_{\mathbb{T}_\omega}$.

Proof. The proof is a direct consequence of the definition. □

The first two properties of Theorem 4.2.7 are sufficient to claim that $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ forms a linear space. To rigorously study the properties of this space and its functions, we will treat it as a normed linear space endowed with the norm

$$\|f\|_{a\omega c} = \sup_{t\in\mathbb{T}_\omega} \|f\|_{\omega,c}. \tag{4.2.2}$$

Next, we will show that this normed linear space is complete. This enables us to leverage the properties of a Banach space for the analysis of functions defined in $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.

Theorem 4.2.8. $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$*, equipped with the norm* (4.2.2)*, is a Banach space.*

Proof. As $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ is a subspace of the Banach space $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$, equipped with the norm $\sup_{t\in\mathbb{T}_\omega} \|\cdot\|_{\mathbb{B}}$, and thus also with the norm $\|\cdot\|_{a\omega c}$, we only need to show that $\mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ is closed. Let $f_n \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ be a sequence such that $f_n \to f$. We can write $f_n = f_{op_n} + f_{oa_n}$, where $f_{op_n} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa_n} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Let $f_{op_n} \to f_{op}$ and $f_{oa_n} \to f_{oa}$ with respect to the norm $\|\cdot\|_{a\omega c}$ as $n \to \infty$. Then, for each $t \in \mathbb{T}_\omega$, we have

$$f_{op_n}(t) \to f_{op}(t), \quad \text{and} \quad f_{oa_n}(t) \to f_{oa}(t) \quad \text{as} \quad n \to \infty.$$

Furthermore, we observe that

$$f_{op_n}(t+\omega) \to f_{op}(t+\omega), \quad \text{and} \quad c^{-t/\omega} f_{oa_n}(t) \to c^{-t/\omega} f_{oa}(t) \quad \text{as} \quad n \to \infty.$$

However,

$$f_{op_n}(t+\omega) = c f_{op_n}(t) \to c f_{op}(t), \quad \text{and} \quad c^{-t/\omega} f_{oa_n}(t) \to 0 \quad \text{as} \quad t \to \infty.$$

It follows that

$$f_{op}(t+\omega) = c f_{op}(t), \quad \text{and} \lim_{t\to\infty} c^{-t/\omega} f_{oa}(t) = 0,$$

where the limit is in the sense of the norm $\|\cdot\|_{\omega,c}$. Therefore, we conclude that $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, implying that $f_n \to f_{op} + f_{oa} = f$ with $f \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. The proof is complete. $\square$

Theorem 4.2.9. *If $f \in \mathrm{AP}_{\omega,c_1}(\mathbb{T}_\omega, \mathbb{B})$ and $g \in \mathrm{AP}_{\omega,c_2}(\mathbb{T}_\omega, \mathbb{B})$, then $f \cdot g \in \mathrm{AP}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$, provided that $\mathbb{B}$ is a Banach algebra.*

Proof. Let $f = f_{op} + f_{oa}$ and $g = g_{op} + g_{oa}$, where $f_{op} \in \mathrm{P}_{\omega,c_1}(\mathbb{T}_\omega, \mathbb{B})$, $g_{op} \in \mathrm{P}_{\omega,c_2}(\mathbb{T}, \mathbb{B})$, $f_{oa} \in \mathcal{A}_{\omega,c_1}(\mathbb{T}, \mathbb{B})$, and $g_{oa} \in \mathcal{A}_{\omega,c_2}(\mathbb{T}, \mathbb{B})$. Then,

$$h := f \cdot g = f_{op} \cdot g_{op} + f_{oa} \cdot g_{op} + f_{op} \cdot g_{oa} + f_{oa} \cdot g_{oa} = h_{op} + h_{oa},$$

where $h_{op} = f_{op} \cdot g_{op}$ and $h_{oa} = f_{oa} \cdot g_{op} + f_{op} \cdot g_{oa} + f_{oa} \cdot g_{oa}$. We have $h_{op}(t + \omega) = c_1 \cdot c_2 h_{op}(t)$ for all $t \in \mathbb{T}_\omega$. Also, for periodic functions f_p and g_p associated with f_{op} and g_{op}, respectively, we can derive the following inequality:

$$\begin{aligned}\|h_{oa}\|_{\omega,c} &= \|g_p(t)c_1^{-t/\omega} f_{oa}(t) + f_p(t)c_2^{-t/\omega} g_{oa}(t) + c_1^{-t/\omega} f_{oa}(t) c_2^{-t/\omega} g_{oa}(t)\|_{\mathbb{B}} \\ &\le \|g_p\|_{\omega,1}\|f_{oa}\|_{\omega,c_1} + \|f_p\|_{\omega,1}\|g_{oa}\|_{\omega,c_2} + \|f_{oa}\|_{\omega,c_1}\|g_{oa}\|_{\omega,c_2}.\end{aligned}$$

Thus, we conclude that $h_{op} \in \mathrm{P}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$ and $h_{oa} \in \mathcal{A}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$. Hence, $f \cdot g \in \mathrm{AP}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$. $\square$

Remark 4.2.10. *If $f \in \mathrm{AP}_\omega(\mathbb{T}_\omega, \mathbb{B})$ and $g \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, then their product $f \cdot g$ belongs to $\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, provided that $\mathbb{B}$ is a Banach algebra. Furthermore, this result holds even if $f \in \mathrm{P}_\omega(\mathbb{T}_\omega, \mathbb{B})$, as $\mathrm{P}_\omega(\mathbb{T}_\omega, \mathbb{B}) \subseteq \mathrm{AP}_\omega(\mathbb{T}_\omega, \mathbb{B})$.*

Theorem 4.2.11. *If $f \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, then $f^\Delta \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ provided that both the (ω, c)-periodic and (ω, c)-asymptotic components associated with f are in $(\mathrm{BC}^1_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.*

Proof. Let $f = f_{op} + f_{oa}$, where $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. We aim to show that $f^\Delta_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f^\Delta_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Using the properties of (ω, c)-periodic functions on $\mathbb{T}_\omega$, we can establish that f^Δ_{op} remains an (ω, c)-periodic function defined on $\mathbb{T}_\omega$. Now, consider $t \in \mathbb{T}^\kappa_\omega$ with $\mu(t) \neq 0$. Since μ is bounded in an invariant under translation time scale, we can observe that

$$c^{-t/\omega} f^\Delta_{oa}(t) = c^{-t/\omega} \frac{f_{oa}(\sigma(t)) - f_{oa}(t)}{\mu(t)} = c^{-t/\omega} \frac{f_{oa}(t + \mu(t)) - f_{oa}(t)}{\mu(t)} \to 0,$$

as t approaches infinity. Now, for $t \in \mathbb{T}^\kappa_\omega$ with $\mu(t) = 0$, we have

$$\begin{aligned}\lim_{t\to\infty} c^{-t/\omega} f^\Delta_{oa}(t) &= \lim_{t\to\infty} c^{-t/\omega} \lim_{\tau\to 0, \tau\in\Pi_{\mathbb{T}}} \frac{f_{oa}(t+\tau) - f_{oa}(t)}{\tau} \\ &= \lim_{t\to\infty} c^{-t/\omega} \frac{f_{oa}(t+\tau) - f_{oa}(t)}{\tau} \\ &= \lim_{\tau\to 0, \tau\in\Pi_{\mathbb{T}}} \lim_{t\to\infty} c^{-t/\omega} \frac{f_{oa}(t+\tau) - f_{oa}(t)}{\tau}\end{aligned}$$

$$= \lim_{\tau \to 0, \tau \in \Pi_{\mathbb{T}}} \frac{1}{\tau} \left(c^{\tau/\omega} \lim_{(t+\tau) \to \infty} c^{-(t+\tau)/\omega} f_{oa}(t+\tau) - \lim_{t \to \infty} c^{-t/\omega} f_{oa}(t) \right) = 0.$$

This implies that f_{oa}^{Δ} is an (ω, c)-asymptotic function on $\mathbb{T}_\omega$. Hence, we can conclude that f^{Δ} is an (ω, c)-asymptotically periodic function on $\mathbb{T}_\omega$. □

Next, we will outline a class of functions for which the composition with an (ω, c)-asymptotically periodic function remains (ω, c)-asymptotically periodic on a time scale. This result is referred to as the composition theorem for (ω, c)-asymptotically periodic functions, and we state it as follows:

Theorem 4.2.12. *Consider a function $f : \mathbb{B} \to \mathbb{B}$ such that $f = g + h$. Assume that the following conditions hold:*

(a) $g(cx) = cg(x)$ for all $c \in \mathbb{F}_{\mathbb{B}} \setminus \{0\}$ and $x \in \mathbb{B}$.

(b) For z in any bounded subset of $\mathbb{B}$, $\tilde{h}(z) := h(c^{t/\omega} z)$ is uniformly continuous in the sense of the norm $\|\cdot\|_{\omega,c}$ and $\|\tilde{h}(z)\|_{\omega,c} \to 0$, uniformly for $t \in \mathbb{T}_\omega$.

(c) f is Lipschitz continuous, i.e., there exists a positive constant L_f such that

$$\|f(x) - f(y)\|_{\mathbb{B}} \leq L_f \|x - y\|_{\mathbb{B}} \quad \text{for all} \quad x, y \in \mathbb{B}.$$

Then, if $\phi \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, it follows that $f(\phi(\cdot)) \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.

Proof. Let $\phi(t) = \phi_p(t) + \phi_a(t)$, where $\phi_p \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $\phi_a \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Then, we can write

$$f(\phi(t)) = f(\phi(t)) - f(\phi_p(t)) + g(\phi_p(t)) + h(\phi_p(t)).$$

To simplify the notations, we introduce the following abbreviations:

$$\mathcal{F}(t) = f(\phi(t)) - f(\phi_p(t)), \quad \mathcal{G}(t) = g(\phi_p(t)), \quad \text{and} \quad \mathcal{H}(t) = h(\phi_p(t)).$$

Now, let us examine each of these functions. For $\mathcal{G}$, we have

$$\mathcal{G}(t + \omega) = g(\phi_p(t + \omega))(t) = g(c\phi_p(t)) = cg(\phi_p(t)) = c\mathcal{G}(t),$$

which implies that $\mathcal{G} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. For $\mathcal{F}$, we have

$$\begin{aligned}
\|\mathcal{F}\|_{\omega,c} &= \|c^{-t/\omega}(f(\phi(t)) - f(\phi_p(t)))\|_{\mathbb{B}} \\
&= |c|^{-t/\omega} \|f(\phi(t)) - f(\phi_p(t))\|_{\mathbb{B}} \\
&\leq c|^{-t/\omega} L_f \|\phi(t) - \phi_p(t)\|_{\mathbb{B}} \\
&= L_f \|c^{-t/\omega}(\phi(t) - \phi_p(t))\|_{\mathbb{B}} \\
&= L_f \|\phi_a(t)\|_{\omega,c} \to 0 \quad (\text{as } t \to \infty),
\end{aligned}$$

demonstrating that $\mathcal{F} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Now, let us analyze $\mathcal{H}$. Consider the set $\mathcal{K} := \{c^{-t/\omega}\phi_p(t) : t \in \mathbb{T}_\omega\}$. Since $\mathcal{K}$ comprises points of a periodic function mapping to a Banach space $\mathbb{B}$, it is relatively compact. Thus, for any $\delta > 0$, we can find $x_1, \dots, x_k \in \mathbb{B}$ such that $c^{-t/\omega}\phi_p(t) \in \bigcup_{j=1}^{k} \mathcal{B}(x_j, \delta)$ for all $t \in \mathbb{T}_\omega$, where $\mathcal{B}(x_j, \delta)$ denotes an open ball centered at x_j with radius δ. Consequently, for $\varepsilon > 0$, there exists $x_j \in \mathbb{B},\ j = 1, \dots, k$, such that

$$\|c^{-t/\omega}\phi_p(t) - x_j\|_{\mathbb{B}} < \delta = \delta(\varepsilon/2).$$

From the uniform continuity of $\tilde{h}(\cdot) = h(c^{t/\omega}\cdot)$ on $\mathcal{K}$, we have, for all $j = 1, \dots, k$,

$$\|h(\phi_p(t)) - h(c^{t/\omega}x_j)\|_{\omega,c} < \frac{\varepsilon}{2} \quad \text{uniformly for } t \in \mathbb{T}_\omega. \tag{4.2.3}$$

Furthermore, since $\|\tilde{h}\|_{\omega,c} \to 0$ on $\mathcal{K}$ uniformly for $t \in \mathbb{T}_\omega$, thus, there exists $T \in \mathbb{T}_\omega^+$ such that for all $t \geq T$:

$$\|h(c^{t/\omega}x_j)\|_{\omega,c} < \frac{\varepsilon}{2} \quad \text{for all} \quad j = 1, \dots, k. \tag{4.2.4}$$

Thus, for all $t \geq T$:

$$\|h(\phi_p(t))\|_{\omega,c} \leq \|h(\phi_p(t)) - h(c^{t/\omega}x_j)\|_{\omega,c} + \|h(c^{t/\omega}x_j)\|_{\omega,c} < \varepsilon.$$

This implies that $\|\mathcal{H}\|_{\omega,c} \to 0$, and therefore, $\mathcal{H} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Hence, we conclude that $f(\phi(\cdot))$ is an (ω, c)-asymptotically periodic function. □

Remark 4.2.13. *Although the basic properties of (ω, c)-asymptotically periodic functions in the context of time scales are discussed for the first time, the continuous analogous of these properties are presented in [101]. We have used the methodologies for this work to generalize the results for the time scales.*

4.3 Existence and stability of (ω, c)-asymptotically periodic solutions for cellular neural networks

In this section, we investigate the conditions that ensure the existence of (ω, c)-asymptotically periodic solutions for a class of CNNs with mixed time-varying delays and leakage delays defined on an ω-periodic time scale. We also examine the global exponential stability of resulting solution.

4.3.1 Model description

The following mathematical expression describes a system of delayed cellular neural networks (DCNNs) that incorporate leakage delays and multiple mixed (discrete as well as distributed) time-varying delays:

$$\begin{aligned} x_i^{\Delta}(t) &= -a_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} e_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))\Delta s + U_i(t), \quad t \in \mathbb{T}_{\omega}^{+}, \end{aligned} \tag{4.3.1}$$

where $i = 1, 2, \dots, n$, and n is the number of cells in the neural network. Here, $x_i(t)$ is the potential (or voltage) of the i^{th} cell at time t, and $a_i(>0)$ is the self-feedback connection weight (the rate at which the i^{th} cell will reset its potential to the resting state in isolation when disconnected from the network). The discretely delayed connection weights b_{ij}, e_{ij} and the distributively delayed connection weights d_{ij} measure the strength of connectivity between the neuron j and the neuron i. The external inputs f_j, g_j, and h_j are the non-linear activation functions of signal transmission that show how the j^{th} neuron reacts to input. U_i is an external input source introduced to the i^{th} unit outside of the network. $\eta_i(>0)$ denotes the leakage delay, $\tau_{ij}(>0)$ is a discrete time-varying delay required to process and transmit a signal from the j^{th} cell to the i^{th} cell. $\delta_{ij}(>0)$ is a distributed time-varying delay that arises in signal transmission between units i and j. Let us consider that (4.3.1) is supplemented with the initial values given by

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0]_{\mathbb{T}}, \tag{4.3.2}$$

where $i = 1, 2, \dots, n$, $\varphi_i \in \mathbb{B}$, $\theta = \max\{\bar{\eta}, \bar{\tau}, \bar{\delta}\}$, $\bar{\eta} = \max_{1\le i,j\le n}\{\sup_{t\in\mathbb{T}} \eta_i(t)\}$, $\bar{\tau} = \max_{1\le i,j\le n}\{\sup_{t\in\mathbb{T}} \tau_{ij}(t)\}$, and $\bar{\delta} = \max_{1\le i,j\le n}\{\sup_{t\in\mathbb{T}} \delta_{ij}(t)\}$. We assume that the activation functions and the coefficients of the system (4.3.1) satisfy the following conditions:

Assumption 1: There exist $c \in \mathbb{F}_{\mathbb{B}}^{*} = \mathbb{F}_{\mathbb{B}}\backslash\{0\}$ such that $a_i \in \mathrm{P}_{\omega}(\mathbb{T}_{\omega}, \mathbb{R}^{+})$, $\eta_i, \tau_{ij}, \delta_{ij} \in \mathrm{P}_{\omega}(\mathbb{T}_{\omega}, \Pi_{\mathbb{T}_{\omega}}^{0+})$, $b_{ij}, e_{ij}, d_{ij} \in \mathrm{AP}_{\omega}(\mathbb{T}_{\omega}, \mathbb{B})$, and $U_i \in \mathrm{AP}_{\omega,c}(\mathbb{T}_{\omega}, \mathbb{B})$.

Assumption 2: $f_i, g_j, h_j : \mathbb{B} \to \mathbb{B}$ are Lipschitz continuous with respect to norm $\|\cdot\|_{\mathbb{B}}$, with Lipschitz constants $L_{f_j}, L_{g_j}, L_{h_j}$, respectively.

Assumption 3: f_i, g_i, and h_i can be respectively decomposed as $f_i = f_{i1} + f_{i2}$, $g_i = g_{i1} + g_{i2}$, and $h_i = h_{i1} + h_{j2}$, where

(a) for $l \in \{f_{i1}, g_{i1}, h_{i1}\}$, and x in $\mathbb{B}$, $l(cx) = cx$.

(b) for $l \in \{f_{i2}, g_{i2}, h_{i2}\}$, and z in any bounded subset of $\mathbb{B}$, $\tilde{l}(z) := l(c^{t/\omega}z)$ is uniformly continuous in the sense of norm $\|\cdot\|_{\omega,c}$ and $\|\tilde{l}(z)\|_{\omega,c} \to 0$, uniformly for $t \in \mathbb{T}_{\omega}$.

We refer to DCNN (4.3.1) along with the above assumptions as the (ω, c)-asymptotically periodic DCNN. Next, we discuss the existence and stability of its solution.

4.3.2 Existence of solution

We use the Banach fixed point theorem to show the existence of (ω, c)-asymptotically periodic solutions for the system (4.3.1), along with Assumptions **1**–**3**. To be able to apply this theorem, we need to consider certain additional assumptions on the coefficients of (4.3.1). Before delving into the main results, let us first prove some necessary lemmas.

Lemma 4.3.1. *Assume that the function* $\alpha \in \mathrm{P}_\omega(\mathbb{T}_\omega, \mathbb{R}^+)$ *is bounded below by a positive constant* $\underline{\alpha}$ *such that* $-\underline{\alpha} \in \mathcal{R}^+$ *and* $f : \mathbb{T}_\omega \to \mathbb{B}$ *is* (ω, c)*-asymptotically periodic. Then, for each* t *in the domain* $\mathbb{T}_\omega$*, the integral* $\int_{-\infty}^{t} e_{-\alpha}(t, \sigma(s)) f(s)\Delta s$ *belongs to the space* $\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$*, provided that* $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})}$.

Proof. For $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, let us consider the function $f = f_{op} + f_{oa}$. We then have

$$\begin{aligned}\int_{-\infty}^{t} e_{-\alpha}(t, \sigma(s)) f(s)\Delta s &= \int_{-\infty}^{t} e_{-\alpha}(t, \sigma(s)) f_{op}(s)\Delta s + \int_{-\infty}^{t} e_{-\alpha}(t, \sigma(s)) f_{oa}(s)\Delta s \\ &:= I_1(t) + I_2(t).\end{aligned}$$

Using the properties of (ω, c)-periodic functions in $\mathbb{T}_\omega$, we can deduce that $I_1 \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. To complete our proof, we need to show that $\|I_2(t)\|_{\omega,c} \to 0$ as $t \to \infty$. It is important to note that this step is based on the boundedness of the graininess function μ on a periodic time scale. Let $\bar{\mu}$ represent the supremum of $\mu(t)$ over $\mathbb{T}_\omega$, and $\underline{\mu}$ represent the infimum. Indeed, for any $\varepsilon > 0$, there exists a $T > 0$ such that for all $t > T$ we have the following:

$$\begin{aligned}\|I_2(t)\|_{\omega,c} &= \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{-\alpha}(t, \sigma(s)) f_{oa}(s)\Delta s \right\|_{\mathbb{B}} \\ &= \left\| \int_{-\infty}^{t} c^{-(t-s)/\omega} e_{-\alpha}(t, \sigma(s)) c^{-s/\omega} f_{oa}(s)\Delta s \right\|_{\mathbb{B}} \\ &\le \int_{-\infty}^{t} |c|^{-(t-s)/\omega} e_{-\underline{\alpha}}(t, \sigma(s)) \|c^{-s/\omega} f_{oa}(s)\|_{\mathbb{B}} \Delta s \\ &\le \frac{1}{1-\bar{\mu}\underline{\alpha}} \int_{-\infty}^{t} e^{-\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)(t-s)} \|f_{oa}(s)\|_{\omega,c} \Delta s \\ &\le \frac{e^{-\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)(t-T)}}{1-\bar{\mu}\underline{\alpha}} \sup_{t \in (-\infty, T]_{\mathbb{T}_\omega}} \|f_{oa}(t)\|_{\omega c} \int_{-\infty}^{T} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)}(T, s)\Delta s \\ &\quad + \frac{\varepsilon}{1-\bar{\mu}\underline{\alpha}} \int_{T}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)}(t, s)\Delta s\end{aligned}$$

As $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, we have $\sup_{\mathbb{T}_\omega} \|f_{oa}(t)\|_{\omega,c} < \infty$. Furthermore, since the generalized exponential kernel is integrable, that is, $\int_{-\infty}^{T} e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)}(T, s)\Delta s < \infty$, it follows that the first term in the above ineauation tends to 0 as t tends to ∞. Consequently, we have

$$\limsup_{t\to\infty} \|I_2(t)\|_{\omega,c} < \frac{\varepsilon}{1-\bar{\mu}\underline{\alpha}} \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} \frac{e_{\ominus\left(\underline{\alpha}+\frac{\ln|c|}{\omega}\right)}(t,\sigma(s))}{1+\mu(s)\left(\underline{\alpha}+\frac{\ln|c|}{\omega}\right)}\Delta s < \frac{\omega\varepsilon}{(\omega\underline{\alpha}+\ln|c|)(1-\bar{\mu}\underline{\alpha})}.$$

Therefore, we conclude that $\int_{-\infty}^{t} e_{-\alpha}(t,\sigma(s))f(s)\Delta s \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$. $\square$

Lemma 4.3.2. *Consider a function τ belonging to the set $\mathrm{P}_\omega(\mathbb{T}_\omega, \Pi^+_{\mathbb{T}_\omega})$, and let $f \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$. Then, the function $f(\cdot-\tau(\cdot))$ is (ω,c)-asymptotically periodic on $\mathbb{T}_\omega$.*

Proof. Let $f = f_{op} + f_{oa}$, where $f_{op} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. Then,

$$f(t-\tau(t)) = f_{op}(t-\tau(t)) + f_{oa}(t-\tau(t)) := f_1(t) + f_2(t).$$

Note that since $\tau \in \Pi_{\mathbb{T}_\omega}$, it follows that $t-\tau(t) \in \mathbb{T}_\omega$ for all $t \in \mathbb{T}_\omega$. Now, let us examine the function f_1.

$$f_1(t+\omega) = f_{op}(t+\omega-\tau(t+\omega)) = f_{op}(t-\tau(t)+\omega) = cf_{op}(t-\tau(t)) = cf_1(t).$$

This implies that $f_1 \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$. Now, consider the function f_2.

$$\begin{aligned}
\|f_2\|_{\omega,c} &= \|c^{-t/\omega} f_{oa}(t-\tau(t))\|_{\mathbb{B}} \\
&= \|c^{-\tau(t)/\omega} c^{-(t-\tau(t))/\omega} f_{oa}(t-\tau(t))\|_{\mathbb{B}} \\
&\leq \left(\sup_{t\in\mathbb{T}_\omega} |c|^{-\tau(t)/\omega}\right) \|f_{oa}\|_{\omega,c} \\
&\leq \begin{cases} |c|^{-\underline{\tau}/\omega}\|f_{oa}\|_{\omega,c}, & \text{when } |c| \geq 1 \\ |c|^{-\bar{\tau}/\omega}\|f_{oa}\|_{\omega,c}, & \text{when } |c| < 1 \end{cases} \quad \to 0 \quad \text{as } t\to\infty,
\end{aligned}$$

where $\bar{\tau}$ and $\underline{\tau}$ are respectively the supremum and infimum of τ over $\mathbb{T}_\omega$. This shows that $f_2 \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. Hence, we conclude that $f(\cdot-\tau(\cdot)) \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$. $\square$

Lemma 4.3.3. *Consider a function δ belonging to the set $\mathrm{P}^+_\omega(\mathbb{T}_\omega,\Pi_{\mathbb{T}_\omega})$, and let $f : \mathbb{T}_\omega \to \mathbb{B}$ be an (ω,c)-asymptotically periodic function. Then, for each $t \in \mathbb{T}_\omega$, the integral $\int_{t-\delta(t)}^{t} f(s)\Delta s$ will be (ω,c)-asymptotically periodic on $\mathbb{T}_\omega$.*

Proof. Let $f = f_{op} + f_{oa}$, where $f_{op} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$ and $f_{oa} \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. We then have

$$\int_{t-\delta(t)}^{t} f(s)\Delta s = \int_{t-\delta(t)}^{t} f_{op}(s)\Delta s + \int_{t-\delta(t)}^{t} f_{oa}(s)\Delta s := I_1(t) + I_2(t).$$

Now,

$$I_1(t+\omega) = \int_{t-\delta(t)+\omega}^{t+\omega} f_{op}(s)\Delta s = \int_{t-\delta(t)}^{t} f_{op}(s+\omega)\Delta s = c\int_{t-\delta(t)}^{t} f_{op}(s)\Delta s = cI_1(t).$$

This shows that I_1 is an (ω, c)-periodic function defined on $\mathbb{T}_\omega$. Also, there exist a $T \in \mathbb{T}_\omega^+$ such that for $t > T + \bar{\delta}$, we have

$$\begin{aligned}
\|I_2\|_{\omega,c} &= \left\| c^{-t/\omega} \int_{t-\delta(t)}^{t} f_{oa}(s)\Delta s \right\|_{\mathbb{B}} \\
&\leq \int_{t-\delta(t)}^{t} |c|^{-(t-s)/\omega} \|f_{oa}(s)\|_{\omega,c} \Delta s \\
&\leq \varepsilon \int_{t-\delta(t)}^{t} |c|^{-(t-s)/\omega} \Delta s \\
&< \begin{cases} \varepsilon \sup\limits_{t\in\mathbb{T}_\omega} \displaystyle\int_{t-\delta(t)}^{t} \Delta s, & \text{when} \quad |c| = 1 \\ \varepsilon \sup\limits_{t\in\mathbb{T}} \displaystyle\int_{t-\delta(t)}^{t} \frac{|c|^{-t/\omega}\mu(s)}{|c|^{\frac{\mu(s)}{\omega}} - 1} \left(|c|^{s/\omega}\right)^{\Delta} \Delta s, & \text{otherwise} \end{cases} \\
&\leq \begin{cases} \varepsilon \sup\limits_{t\in\mathbb{T}} \delta(t), & \text{when} \quad |c| = 1 \\ \varepsilon |c|^{-\mu^*/\omega} \sup\limits_{t\in\mathbb{T}} \dfrac{\bar{\mu}\left(|c|^{\frac{-\delta(t)}{\omega}} - 1\right)}{|c|^{-\frac{\mu^*}{\omega}} - 1}, & \text{when} \quad \mu \neq 0, |c| \neq 1 \\ \varepsilon \sup\limits_{t\in\mathbb{T}} \dfrac{-\omega\left(|c|^{\frac{-\delta(t)}{\omega}} - 1\right)}{\ln|c|}, & \text{when} \quad \mu = 0,\ |c| \neq 1 \end{cases} \\
&\leq \begin{cases} \bar{\delta}\ \varepsilon, & \text{when} \quad |c| = 1 \\ \frac{\bar{\mu}\left(|c|^{-\frac{\delta^*}{\omega}} - 1\right)}{|c|^{-\frac{\mu^*}{\omega}} - 1} \varepsilon |c|^{-\mu^*/\omega}, & \text{when} \quad \mu \neq 0,\ |c| \neq 1 \\ \frac{\omega\left(1 - |c|^{-\frac{\delta^*}{\omega}}\right)}{\ln|c|} \varepsilon & \text{when} \quad \mu = 0,\ |c| \neq 1, \end{cases}
\end{aligned}$$

where $\mu^* = \underline{\mu}$ when $|c| > 1$, and $\mu^* = \bar{\mu}$ when $|c| < 1$, and similar for the δ^*. Thus, I_2 is (ω, c)-asymptotic function defined on $\mathbb{T}_\omega$. Hence, we conclude that $\int_{t-\delta(t)}^{t} f(s)\Delta s \in \mathrm{AP}_{(\omega,c)}(\mathbb{T}_\omega, \mathbb{B})$. □

Throughout this chapter, $\bar{\mu}$ and $\underline{\mu}$ denote $\sup\limits_{t\in\mathbb{T}_\omega} \mu(t)$ and $\inf\limits_{t\in\mathbb{T}_\omega} \mu(t)$, respectively, as we mentioned earlier. Before we proceed towards the main results, we will adopt the following notations:

$$\bar{a} = \max_{1\leq i\leq n} \sup_{t\in\mathbb{T}_\omega} |a_i(t)|, \quad \underline{a} = \min_{1\leq i\leq n} \inf_{t\in\mathbb{T}_\omega} |a_i(t)|,$$

$$\bar{b} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}_\omega} \|b_{ij}(t)\|_{\mathbb{B}}, \quad \bar{e} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}_\omega} \|e_{ij}(t)\|_{\mathbb{B}}, \quad \bar{d} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}_\omega} \|d_{ij}(t)\|_{\mathbb{B}},$$

$$\bar{\eta} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}_\omega} |\eta_{ij}(t)|, \quad \underline{\eta} = \min_{1\leq i,j\leq n} \inf_{t\in\mathbb{T}_\omega} |\eta_{ij}(t)|, \quad \bar{\delta} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}} |\delta_{ij}(t)|, \quad \underline{\delta} = \min_{1\leq i,j\leq n} \inf_{t\in\mathbb{T}_\omega} |\delta_{ij}(t)|,$$

$$\bar{L}_f = \max_{1\leq j\leq n} L_{f_j}, \quad \bar{L}_g = \max_{1\leq j\leq n} L_{g_j}, \quad \bar{L}_h = \max_{1\leq j\leq n} L_{h_j},$$

$$\bar{\tau} = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{T}_\omega} |\tau_{ij}(t)|, \quad \underline{\tau} = \min_{1\leq i,j\leq n} \inf_{t\in\mathbb{T}_\omega} |\tau_{ij}(t)|.$$

Additionally, to study the existence of the (ω, c)-asymptotically periodic solutions for system (4.3.1), let us consider the following assumptions:

Assumption 4: There exists a positive constant $\underline{a}$ such that $-\underline{a} \in \mathcal{R}^+$, and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{a}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{a}\bar{\mu})}$.

In addition, the regressive functions a_i satisfy $0 < \underline{a} \leq a_i(t)$ for all $t \in \mathbb{T}_\omega$.

Assumption 5: $0 < \frac{\mathcal{C}}{(\frac{\ln|c|}{\omega}+\underline{a})(1-\underline{a}\bar{\mu})}, \left(\frac{\bar{a}}{(\frac{\ln|c|}{\omega}+\underline{a})(1-\underline{a}\bar{\mu})}+1\right)\mathcal{C} < 1$ with

$$\mathcal{C} = \begin{cases} \bar{a}\bar{\eta} + n^2\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g + \bar{d}\,\bar{L}_h\,\bar{\delta}\right), & \text{when} \quad |c| = 1 \\ \frac{\omega\bar{a}(1-|c|^{-\eta^*/\omega})}{\ln|c|} + n^2\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega} + \frac{\omega\bar{d}\bar{L}_h(1-|c|^{-\delta^*/\omega})}{\ln|c|}\right), & \\ \qquad\qquad\qquad \text{when} \quad |c| \neq 1 \text{ and } \mu = 0 \text{ a.e. on } (t-\alpha_i, t)_{\mathbb{T}_\omega} & \\ \frac{\bar{\mu}\bar{a}(1-|c|^{-\eta^*/\omega})}{|c|^{\mu^*/\omega}-1} + n^2\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega} + \frac{\bar{d}\bar{L}_h\bar{\mu}(1-|c|^{-\delta^*/\omega})}{|c|^{\mu^*/\omega}-1}\right), & \\ \qquad\qquad\qquad \text{when} \quad |c| \neq 1 \text{ and } \mu \neq 0 \text{ a.e. on } (t-\alpha_i, t)_{\mathbb{T}_\omega} & \\ \max\left\{\frac{\omega}{\ln|c|}, \frac{\bar{\mu}}{|c|^{\mu^*/\omega}-1}\right\}\bar{a}(1-|c|^{-\eta^*/\omega}) + n^2\Big(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega} & \\ \quad + \max\left\{\frac{\omega}{\ln|c|}, \frac{\bar{\mu}}{|c|^{\mu^*/\omega}-1}\right\}\bar{d}\bar{L}_h(1-|c|^{-\delta^*/\omega})\Big), & \text{otherwise} \end{cases}$$

where $\alpha_i = \max\{\eta_i, \delta_i\}$, for any function α, the notation $\alpha^* = \underline{\alpha}$ when $|c| > 1$, and $\alpha^* = \bar{\alpha}$ when $|c| < 1$, and $t \in \mathbb{T}_\omega$. Also, in the calculation of $\mathcal{C}$, μ^* is the infimum or supremum evaluated on the subintervals of $(t-\eta_i, t)_{\mathbb{T}_\omega}$ or $(t-\delta_i, t)_{\mathbb{T}_\omega}$ where it is not a.e. 0.

Remark 4.3.4. *Assumption 4 plays an important role in using time-scale inequalities to prove our main result. Specifically, we can use*

$$e_{-a_i}(t,s) \leq e_{-\underline{a}}(t,s) \leq e^{\underline{a}(t-s)} \quad \textit{for all} \quad t,s \in \mathbb{T}_\omega, \quad t \geq s.$$

Furthermore, $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{a}\bar{\mu})} < |c| < e^{-\underline{a}\omega}$ *implies that* $-\left(\frac{\ln|c|}{\omega}+\underline{a}\right) > 0$ *and* $1+\mu(t)\left(\frac{\ln|c|}{\omega}+\underline{a}\right) > 0$*, i.e.,* $\frac{\ln|c|}{\omega}+\underline{a} \in \mathcal{R}^+(\mathbb{T}_\omega, \mathbb{B})$*. Therefore, using first inequality of Theorem 1.4.29, we can have*

$$e^{-\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(t-s)} \leq e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,s) \quad \textit{for all} \quad t,s \in \mathbb{T}_\omega, \quad t \geq s.$$

Similarly, for $e^{-\underline{a}\omega} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{a}\bar{\mu})}$*, we have* $\frac{\ln|c|}{\omega}+\underline{a} > 0$ *and* $-\left(\frac{\ln|c|}{\omega}+\underline{a}\right) \in \mathcal{R}^+(\mathbb{T}_\omega, \mathbb{B})$*, which implies from the second inequality of Theorem 1.4.29 that*

$$e^{-\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(t-s)} \leq e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,s) \quad \textit{for all} \quad t,s \in \mathbb{T}_\omega, \quad t \geq s.$$

Also, for $|c| = e^{-\underline{a}\omega}$ *the above inequality is obvious.*

Theorem 4.3.5. *Consider that Assumptions **1**–**5** are fulfilled for* $i,j = 1,\ldots,n$*. Then system (4.3.1) admits a unique* (ω,c)*-asymptotically periodic solution.*

Proof. To prove this theorem, we will employ the Banach fixed point theorem. Let $x = (x_1,\ldots,x_n)^T \in \mathbb{B}^n$, and define the norm $\|\cdot\|_{\mathbb{B}^n} = \sum_{i=1}^{n}\|\cdot\|_{\mathbb{B}}$. Consequently, we have $\|x\|_{a\omega c} = \sum_{i=1}^{n}\|x_i\|_{a\omega c}$. We have seen in

Theorem 4.2.11 that $x \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B})$ implies $x^\Delta \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B})$. Consequently, we can assert that $\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B}^n)$ is a Banach space equipped with the norm $\sup\limits_{t\in\mathbb{T}_\omega^+} \|x\|$, where

$$\|x\| = \max\{\|x(t)\|_{\omega,c}, \|x^\Delta(t))\|_{\omega c}\}. \tag{4.3.3}$$

Now, we initiate the proof by rewriting the system (4.3.1) as follows:

$$\begin{aligned} x_i^\Delta(t) = -a_i(t)x_i(t) + a_i(t)\int_{t-\eta_i(t)}^{t} x_i^\Delta(s)\,\Delta s + \sum_{j=1}^{n} b_{ij}(t)\cdot f_j(x_j(t)) + \\ + \sum_{j=1}^{n} e_{ij}(t)\cdot g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))\,\Delta s + U_i(t), \end{aligned} \tag{4.3.4}$$

which can again be transformed into the following system of integral equations:

$$\begin{aligned} x_i(t) = \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s)) \Bigg(a_i(s)\int_{s-\eta_i(s)}^{s} x_i^\Delta(\nu)\,\Delta\nu + \sum_{j=1}^{n} b_{ij}(s)\cdot f_j(x_j(s)) + \\ \sum_{j=1}^{n} e_{ij}(s)\cdot g_j(x_j(s-\tau_{ij}(s))) + \sum_{j=1}^{n} d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s} h_j(x_j(\nu))\,\Delta\nu + U_i(s) \Bigg)\,\Delta s. \end{aligned} \tag{4.3.5}$$

It can readily be verified that if $x = (x_1, \dots, x_n)^T$ will be a solution of the system (4.3.5), then x will also be a solution of the system (4.3.1), and vice versa. Let us now define an operator $\Psi : \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B}^n) \to \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B}^n)$ as

$$x \mapsto \Psi(x) = (\Psi_1(x), \Psi_2(x), \dots, \Psi_n(x))^T,$$

where for $i = 1, 2, ..., n$, and

$$\Psi_i(x)(t) = \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))\cdot(\Gamma_x)_i(s)\Delta s \in \mathbb{B},$$

with

$$\begin{aligned} (\Gamma_x)_i(s) = a_i(s)\int_{s-\eta_i(s)}^{s} x_i^\Delta(\nu)\,\Delta\nu + \sum_{j=1}^{n} b_{ij}(s)\cdot f_j(x_j(s)) + \sum_{j=1}^{n} e_{ij}(s)\cdot g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s} h_j(x_j(\nu))\,\Delta\nu + U_i(s). \end{aligned}$$

Our proof relies on the fact that the operator defined above is well-defined and is a contraction operator. In fact, because of assumptions **1**–**3**, Theorems 4.2.7, 4.2.9, 4.2.11, 4.2.12 and Lemmas 4.3.1, 4.3.2, 4.3.3 assert that $\Psi(x) \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B}^n)$ for every $x \in \mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+, \mathbb{B}^n)$. Also, in view of assumption **4**, for any

$x, y \in \mathrm{AP}_{\omega,\mathrm{c}}(\mathbb{T}_\omega^+, \mathbb{B}^{\mathrm{n}})$ such that $x \neq y$, we have

$$\begin{aligned}
\|\psi(x)-\psi(y)\|_{a\omega c} &= \sum_{i=1}^{n} \|\psi_i(x)-\psi_i(y)\|_{a\omega c} \\
&= \sum_{i=1}^{n} \sup_{t\in\mathbb{T}_\omega^+} \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))\left((\Gamma_x)_i(s)-(\Gamma_y)_i(s)\right)\Delta s \right\|_{\mathbb{B}} \\
&\leq \sum_{i=1}^{n} \sup_{t\in\mathbb{T}_\omega^+} \left\| c^{-t/\omega} \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s)) \left(a_i(s) \int_{s-\eta_i(s)}^{s} (x_i^\Delta(\nu)-y_i^\Delta(\nu))\,\Delta\nu + \right.\right. \\
&\quad \sum_{j=1}^{n} b_{ij}(s)(f_j(x_j(s))-f_j(y_j(s))) + \sum_{j=1}^{n} e_{ij}(s)(g_j(x_j(s-\tau_{ij}(s))) \\
&\quad \left.\left. - g_j(y_j(s-\tau_{ij}(s)))) + \sum_{j=1}^{n} d_{ij}(s) \int_{s-\delta_{ij}(s)}^{s} (h_j(x_j(\nu))-h_j(y_j(\nu)))\,\Delta\nu \right) \Delta s \right\|_{\mathbb{B}} \\
&\leq \sup_{t\in\mathbb{T}_\omega^+} \sum_{i=1}^{n} \int_{-\infty}^{t} |c|^{-(t-s)/\omega} |c|^{-s/\omega} e_{-\underline{a}}(t,\sigma(s)) \left(\|a_i(s)\|_{\mathbb{B}} \int_{s-\eta_i(s)}^{s} \|x_i^\Delta(\nu)-y_i^\Delta(\nu)\|_{\mathbb{B}}\,\Delta\nu \right. \\
&\quad + \sum_{j=1}^{n} \|b_{ij}(s)\|_{\mathbb{B}} \|f_j(x_j(s))-f_j(y_j(s))\|_{\mathbb{B}} + \sum_{j=1}^{n} \|e_{ij}(s)\|_{\mathbb{B}} \|g_j(x_j(s-\tau_{ij}(s))) \\
&\quad \left. - g_j(y_j(s-\tau_{ij}(s)))\|_{\mathbb{B}} + \sum_{j=1}^{n} \|d_{ij}(s)\|_{\mathbb{B}} \int_{s-\delta_{ij}(s)}^{s} \|h_j(x_j(\nu))-h_j(y_j(\nu))\|_{\mathbb{B}}\,\Delta\nu \right) \Delta s \\
&\leq \sup_{t\in\mathbb{T}_\omega^+} \sum_{i=1}^{n} \int_{-\infty}^{t} \frac{e^{-\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(t-s)}}{1-\underline{a}\mu(s)} \left(\|a_i(s)\|_{\mathbb{B}} \int_{s-\eta_i(s)}^{s} |c|^{-(s-\nu)/\omega} \|c^{-\nu/\omega}(x_i^\Delta(\nu) \right. \\
&\quad - y_i^\Delta(\nu))\|_{\mathbb{B}}\,\Delta\nu + \sum_{j=1}^{n} \|b_{ij}(s)\|_{\mathbb{B}}\, L_{f_j} \|c^{-s/\omega}(x_j(s)-y_j(s))\|_{\mathbb{B}} + \\
&\quad \sum_{j=1}^{n} \|e_{ij}(s)\|_{\mathbb{B}}\, L_{g_j} |c|^{-\tau_{ij}(s)/\omega} \|c^{-(s-\tau_{ij}(s))/\omega}(x_j(s-\tau_{ij}(s))-y_j(s-\tau_{ij}(s)))\|_{\mathbb{B}} \\
&\quad \left. + \sum_{j=1}^{n} \|d_{ij}(s)\|_{\mathbb{B}} \int_{s-\delta_{ij}(s)}^{s} L_{h_j} |c|^{-(s-\nu)/\omega} \|c^{-\nu/\omega}(x_j(\nu)-y_j(\nu))\|_{\mathbb{B}}\,\Delta\nu \right) \Delta s \\
&\leq \sup_{t\in\mathbb{T}_\omega^+} \int_{-\infty}^{t} \frac{e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,s)}{1-\underline{a}\bar{\mu}} \sum_{i=1}^{n} \left(\|a_i(s)\|_{\mathbb{B}} \int_{s-\eta_i(s)}^{s} |c|^{-(s-\nu)/\omega}\,\Delta\nu \cdot \|x_i^\Delta - y_i^\Delta\|_{a\omega c} \right. \\
&\quad + \sum_{j=1}^{n} \left(L_{f_j} \|b_{ij}(s)\|_{\mathbb{B}} + \|e_{ij}(s)\|_{\mathbb{B}}\, L_{g_j} |c|^{-\tau_{ij}(s)/\omega} \right. \\
&\quad \left.\left. + \|d_{ij}(s)\|_{\mathbb{B}}\, L_{h_j} \int_{s-\delta_{ij}(s)}^{s} |c|^{(s-\nu)/\omega}\Delta\nu \right) \|x_j-y_j\|_{a\omega c} \right) \Delta s.
\end{aligned}$$

As, for $|c| \neq 1$, we can obtain

$$\int_{s-\eta_i(s)}^{s} |c|^{\nu/\omega}\,\Delta\nu = \int_{s-\eta_i(s)}^{s} \frac{\mu(\nu)}{|c|^{\frac{\mu(\nu)}{\omega}}-1} \left(|c|^{\nu/\omega}\right)^{\Delta}\,\Delta\nu$$

$$\leq \begin{cases} \frac{\bar{\mu}}{|c|^{\frac{\mu^*}{\omega}}-1}\left(|c|^{s/\omega}-|c|^{(s-\eta_i(s))/\omega}\right) \text{ when } \mu \neq 0 \text{ on } (s-\eta_i, s)_{\mathbb{T}_\omega}, \\ \frac{\omega}{\ln|c|}\left(|c|^{s/\omega}-|c|^{(s-\eta_i(s))/\omega}\right) \text{ when } \mu = 0 \text{ on } (s-\eta_i, s)_{\mathbb{T}_\omega}. \end{cases}$$

Consequently, for $|c| \neq 1$ and $\mu \neq 0$ a.e. on $(s-\max\{\eta_i, \delta_i\}, s)_{\mathbb{T}_\omega}$, $s \in \mathbb{T}_\omega$, we have

$$\begin{aligned}
\|\psi(x)-\psi(y)\|_{a\omega c} \leq & \sup_{t\in\mathbb{T}_\omega^+}\int_{-\infty}^{t}\frac{e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,s)}{1-\underline{a}\bar{\mu}}\sum_{i=1}^{n}\left(\frac{\bar{\mu}\|a_i(s)\|_{\mathbb{B}}}{|c|^{\frac{\mu^*}{\omega}}-1}\left(1-|c|^{\frac{-\eta_i(s)}{\omega}}\right)\|x_i^{\Delta}-y_i^{\Delta}\|_{a\omega c}\right.\\
&+\sum_{j=1}^{n}\Big(\|b_{ij}(s)\|_{\mathbb{B}}\, L_{f_j}+\|e_{ij}(s)\|_{\mathbb{B}}\, L_{g_j}|c|^{-\tau_{ij}(s)/\omega}\\
&\left.+\|d_{ij}(s)\|_{\mathbb{B}}\, L_{h_j}\frac{\bar{\mu}}{|c|^{\frac{\mu^*}{\omega}}-1}\left(1-|c|^{\frac{-\delta_{ij}(s)}{\omega}}\right)\Big)\sum_{j=1}^{n}\|x_j-y_j\|_{a\omega c}\right)\Delta s\\
\leq & \left(\frac{\bar{\mu}\bar{a}}{|c|^{\mu^*/\omega}-1}(1-|c|^{-\eta^*/\omega})\|x^{\Delta}-y^{\Delta}\|_{a\omega c}+n^2\left(\bar{b}\,\bar{L}_f+\bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega}+\right.\right.\\
&\left.\left.\frac{\bar{d}\bar{L}_h\bar{\mu}}{|c|^{\mu^*/\omega}-1}(1-|c|^{-\delta^*/\omega})\right)\|x-y\|_{a\omega c}\right)\frac{\left(1-e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,-\infty)\right)}{\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\\
\leq & \left(\frac{\bar{\mu}\bar{a}}{|c|^{\mu^*/\omega}-1}(1-|c|^{-\eta^*/\omega})+n^2\left(\bar{b}\,\bar{L}_f+\bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega}+\frac{\bar{d}\bar{L}_h\bar{\mu}}{|c|^{\mu^*/\omega}-1}(1-|c|^{-\delta^*/\omega})\right)\right)\\
&\frac{1}{\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\max\{\|x-y\|_{a\omega c}, \|x^{\Delta}-y^{\Delta}\|_{a\omega c}\}\\
\leq & \frac{\frac{\bar{\mu}\bar{a}(1-|c|^{-\eta^*/\omega})}{|c|^{\mu^*/\omega}-1}+n^2\left(\bar{b}\,\bar{L}_f+\bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega}+\frac{\bar{d}\bar{L}_h\bar{\mu}(1-|c|^{-\delta^*/\omega})}{|c|^{\mu^*/\omega}-1}\right)}{\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|.
\end{aligned}$$

Similarly, for $|c| \neq 1$ and $\mu = 0$ a.e. on $(s-\max\{\eta_i, \delta_i\}, s)_{\mathbb{T}_\omega}$, $s \in \mathbb{T}_\omega$, we can obtain,

$$\begin{aligned}
\|\psi(x)-\psi(y)\|_{a\omega c} \leq & \sup_{t\in\mathbb{T}_\omega^+}\int_{-\infty}^{t}\frac{e_{\ominus\left(\frac{\ln|c|}{\omega}+\underline{a}\right)}(t,s)}{1-\underline{a}\bar{\mu}}\sum_{i=1}^{n}\left(\frac{\omega\|a_i(s)\|_{\mathbb{B}}}{\ln|c|}\left(1-|c|^{\frac{-\eta_i(s)}{\omega}}\right)\|x_i^{\Delta}-y_i^{\Delta}\|_{a\omega c}\right.\\
&+\sum_{j=1}^{n}\Big(\|b_{ij}(s)\|_{\mathbb{B}}L_{f_j}+\|e_{ij}(s)\|_{\mathbb{B}}\, L_{g_j}|c|^{-\tau_{ij}(s)/\omega}\\
&\left.+\frac{L_{h_j}\omega\|d_{ij}(s)\|_{\mathbb{B}}}{\ln|c|}\left(1-|c|^{\frac{-\delta_{ij}(s)}{\omega}}\right)\Big)\sum_{j=1}^{n}\|x_j-y_j\|_{a\omega c}\right)\Delta s\\
\leq & \frac{\frac{\omega\bar{a}(1-|c|^{-\eta^*/\omega})}{\ln|c|}+n^2\left(\bar{b}\,\bar{L}_f+\bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega}+\frac{\omega\bar{d}\bar{L}_h(1-|c|^{-\delta^*/\omega})}{\ln|c|}\right)}{\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|.
\end{aligned}$$

For $|c| \neq 1$, but a situation different from the above two cases, we can break the intervals $(s-\eta_i, s)_{\mathbb{T}_\omega}$ or $(s-\delta_i, s)_{\mathbb{T}_\omega}$ into subintervals in such a way that the corresponding integration on these subintervals lies in either of the two cases. Thus, for this case we have

$$\|\psi(x)-\psi(y)\|_{a\omega c} \le \Bigg[\max\left\{\frac{\omega}{\ln|c|}, \frac{\bar{\mu}}{|c|^{\mu^*/\omega}-1}\right\}\bar{a}(1-|c|^{-\eta^*/\omega}) + n^2\left(\bar{b}\,\bar{L}_f\right.$$
$$\left.\left. + \bar{e}\,\bar{L}_g|c|^{-\tau^*/\omega} + \max\left\{\frac{\omega}{\ln|c|}, \frac{\bar{\mu}}{|c|^{\mu^*/\omega}-1}\right\}\bar{d}\bar{L}_h(1-|c|^{-\delta^*/\omega})\right)\right] \frac{\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|}{\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}.$$

When $|c|=1$,

$$\|\psi(x)-\psi(y)\|_{a\omega c} \le \sup_{t\in\mathbb{T}_\omega^+}\int_{-\infty}^{t}\frac{e_{\ominus\underline{a}}(t,s)}{1-\underline{a}\mu(s)}\sum_{i=1}^{n}\Bigg(\|a_i(s)\|_{\mathbb{B}}\int_{s-\eta_i(s)}^{s}\Delta\nu\cdot\|x_i^{\Delta}-y_i^{\Delta}\|_{a\omega c} + \sum_{j=1}^{n}\Big(\|b_{ij}(s)\|_{\mathbb{B}}\,L_{f_j}$$
$$+\|e_{ij}(s)\|_{\mathbb{B}}\,L_{g_j}|c|^{-\tau_{ij}(s)/\omega} + \|d_{ij}(s)\|_{\mathbb{B}}\,L_{h_j}\int_{s-\delta_{ij}(s)}^{s}\Delta\nu\Big)\|x_j-y_j\|_{a\omega c}\Bigg)\Delta s$$
$$\le \frac{\bar{a}\bar{\eta}+n^2\left(\bar{b}\,\bar{L}_f+\bar{e}\,\bar{L}_g+\bar{d}\,\bar{L}_h\,\bar{\delta}\right)}{\underline{a}(1-\underline{a}\bar{\mu})}\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|.$$

In all the cases, we have

$$\|\psi(x)-\psi(y)\|_{a\omega c} \le \mathcal{C}\left(\frac{\left(1+\bar{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)}{\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\right)\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|,$$

which finally yield

$$\|\psi(x)-\psi(y)\|_{a\omega c} < \sup_{t\in\mathbb{T}_\omega^+}\|x-y\|,$$

by the Assumption **5**. Also,

$$\begin{aligned}
\|(\psi(x)-\psi(y))^{\Delta}\|_{a\omega c} &= \sum_{i=1}^{n}\|(\psi_i(x)-\psi_i(y))^{\Delta}\|_{a\omega c}\\
&= \sum_{i=1}^{n}\left\|c^{-t/\omega}\left(-a_i(t)(\psi_i(x)-\psi_i(y))(t)+((\Gamma_x)_i(t)-(\Gamma_y)_i(t))\right)\right\|_{\mathbb{B}}\\
&\le \bar{a}\|\psi(x)-\psi(y)\|_{a\omega c}+\sum_{i=1}^{n}\sup_{t\in\mathbb{T}_\omega^+}\Bigg\|c^{-t/\omega}\Bigg(a_i(t)\int_{t-\eta_i(t)}^{t}(x_i^{\Delta}(\nu)-y_i^{\Delta}(\nu))\,\Delta\nu+\\
&\quad \sum_{j=1}^{n}b_{ij}(t)(f_j(x_j(t))-f_j(y_j(t)))+\sum_{j=1}^{n}e_{ij}(t)(g_j(x_j(t-\tau_{ij}(t)))-g_j(y_j(t-\tau_{ij}(t))))\\
&\quad +\sum_{j=1}^{n}d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t}(h_j(x_j(\nu))-h_j(y_j(\nu)))\,\Delta\nu\Bigg)\Bigg\|_{\mathbb{B}}\\
&\le \frac{\bar{a}\left(1+\bar{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)\mathcal{C}}{\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|\\
&\quad +\sum_{i=1}^{n}\sup_{t\in\mathbb{T}_\omega^+}\Bigg(\bar{a}\int_{t-\eta_i(t)}^{t}|c|^{-(t-\nu)/\omega}\|c^{-\nu/\omega}(x_i^{\Delta}(\nu)-y_i^{\Delta}(\nu))\|_{\mathbb{B}}\,\Delta\nu
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\bar{b}\bar{L}_f\|c^{-t/\omega}(x_j(t)-y_j(t))\|_{\mathbb{B}}+\sum_{j=1}^{n}\bar{e}\bar{L}_g|c|^{-\tau_{ij}(t)/\omega}\|c^{-(t-\tau_{ij}(t))/\omega}(x_j(t-\tau_{ij}(t))\\
&-y_j(t-\tau_{ij}(t)))\|_{\mathbb{B}}+\sum_{j=1}^{n}\bar{d}\bar{L}_h\int_{t-\delta_{ij}(t)}^{t}|c|^{-(t-\nu)/\omega}\|c^{-\nu/\omega}(x_j(\nu)-y_j(\nu))\|_{\mathbb{B}}\,\Delta\nu\Bigg)\\
&\le \mathcal{C}\left(\frac{\bar{a}\left(1+\bar{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)}{\left(1+\underline{\mu}\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)\left(\frac{\ln|c|}{\omega}+\underline{a}\right)(1-\underline{a}\bar{\mu})}+1\right)\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|\\
&<\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|.
\end{aligned}
$$

Thus, we have

$$
\sup_{t\in\mathbb{T}_\omega^+}\|(\psi(x)-\psi(y))\|\le\max\{\|(\psi(x)-\psi(y))\|_{a\omega c},\|(\psi(x)-\psi(y))^{\Delta}\|_{a\omega c}\}<\sup_{t\in\mathbb{T}_\omega^+}\|x-y\|,
$$

which implies that ψ is a contraction mapping. Therefore, according to the Banach fixed point theorem, the operator ψ has a unique fixed point in the space $\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^+,\mathbb{B})$. As a result, the system (4.3.1) has a unique (ω, c)-asymptotically periodic solution. The proof is complete. □

4.3.3 Global exponential stability of the solution

In this subsection, we discuss the global exponential stability of the unique (ω, c)-asymptotically periodic solution of equation (4.3.1). With a restriction on the range of values for $|c|$, no additional assumptions are considered for the solution to be stable. Our result is based on the method of contradiction, as described in [177, 160, 159].

Theorem 4.3.6. *Under assumptions **1**–**5** with $|c|>e^{-\underline{a}\omega}$, let x be the unique (ω, c)-asymptotically periodic solution of system (4.3.1) with initial value φ, and let $\widetilde{x}$ be any arbitrary solution of system (4.3.1) with perturbed initial value $\widetilde{\varphi}$. Then, there exist positive constants λ and $\mathcal{M}$ (≥ 1) such that*

$$
\|x(t)-\widetilde{x}(t)\|\le\mathcal{M}e_{\ominus\lambda}(t,t_0)\sup_{t\in[-\theta,0]_{\mathbb{T}}}\|\varphi(t)-\widetilde{\varphi}(t)\|\quad\text{for all}\quad t\ge 0, \tag{4.3.6}
$$

where $\|\cdot\|$ is defined in (4.3.3). In other words, the unique (ω, c)-asymptotically periodic solution of the CNN system (4.3.1) is globally exponentially stable, with an exponential convergence rate λ.

Proof. For $i = 1, 2, ..., n$, let

$$
x^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))^T,
$$

where $x_i^*(t)=x_i(t)-\widetilde{x}_i(t)$. The initial condition is given by $\varphi^*(s)=\varphi(s)-\widetilde{\varphi}(s)$, $s\in[-\theta,0]_{\mathbb{T}_\omega}$. Being a solution of (4.3.1), x^* satisfies (4.3.4), and we have

$$
x_i^{*\Delta}(t)=-a_i(t)x_i^*(t)+\Bigg(a_i(t)\int_{t-\eta_i(t)}^{t}(x_i^{\Delta}(s)-\widetilde{x}_i^{\Delta}(s))\,\Delta s+\sum_{j=1}^{n}b_{ij}(t)(f_j(x_j(t))-f_j(\widetilde{x}_j(t)))+
$$

$$\sum_{j=1}^{n} e_{ij}(t)(g_j(x_j(t-\tau_{ij}(t))) - g_j(\widetilde{x}_j(t-\tau_{ij}(t)))) + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t}(h_j(x_j(s)) - h_j(\widetilde{x}_j(s))\,\Delta s\Bigg).$$

Consequently, for $t_0 \in [-\theta, 0]_{\mathbb{T}_\omega}$, we have

$$\begin{aligned} x_i^*(t) = x_i^*(t_0)e_{-a_i}(t,t_0) + \int_{t_0}^{t} e_{-a_i}(t,\sigma(s))\Bigg(a_i(s)\int_{t-\eta_i(s)}^{s}(x_i^{\Delta}(\nu) - \widetilde{x}_i^{\Delta}(\nu))\,\Delta\nu \\ + \sum_{j=1}^{n} b_{ij}(s)(f_j(x_j(s)) - f_j(\widetilde{x}_j(s))) + \sum_{j=1}^{n} e_{ij}(s)(g_j(x_j(s-\tau_{ij}(s))) - g_j(\widetilde{x}_j(s-\tau_{ij}(s)))) \\ + \sum_{j=1}^{n} d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s}(h_j(x_j(\nu)) - h_j(\widetilde{x}_j(\nu))\,\Delta\nu\Bigg)\,\Delta s. \end{aligned}$$

Taking norm on both sides, we get

$$\begin{aligned} \|x_i^*(t)\|_{\omega,c} \leq\ & \|x_i^*(t_0)\|_{\omega,c}|c|^{-(t-t_0)/\omega}e_{-a_i}(t,t_0) + \int_{t_0}^{t}|c|^{-(t-s)/\omega}e_{-a_i}(t,\sigma(s))\Bigg(\bar{a}\int_{t-\eta_i(s)}^{s}|c|^{-(s-\nu)/\omega} \\ & \|(x_i^{\Delta} - \widetilde{x}_i^{\Delta})\|_{\omega,c}\,\Delta\nu + \sum_{j=1}^{n}\bar{b}\|f_j(x_j) - f_j(\widetilde{x}_j)\|_{\omega,c} + \sum_{j=1}^{n}\bar{e}|c|^{-\tau_{ij}(s)/\omega}\|g_j(x_j) - g_j(\widetilde{x}_j)\|_{\omega,c} \\ & + \sum_{j=1}^{n}\bar{d}\int_{s-\delta_{ij}(s)}^{s}|c|^{-(s-\nu)/\omega}\|h_j(x_j) - h_j(\widetilde{x}_j)\|_{\omega,c}\,\Delta\nu\Bigg)\,\Delta s, \end{aligned}$$

which implies

$$\|x^*(t)\|_{\omega,c} \leq e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t,t_0)\|x^*(t_0)\|_{a\omega c} + \mathcal{C}\int_{t_0}^{t}\frac{e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t,s)}{1-\underline{a}\bar{\mu}}\|x^*(s)\|\Delta s. \tag{4.3.7}$$

Also,

$$\begin{aligned} \|x^{*\Delta}(t)\|_{\omega,c} \leq\ & \bar{a}\|x^*(t)\|_{\omega,c} + \Bigg(\bar{a}\int_{t-\eta_i(t)}^{t}|c|^{-(t-\nu)/\omega}\|(x_i^{\Delta} - \widetilde{x}_i^{\Delta})\|_{\omega,c}\,\Delta\nu + \\ & \sum_{j=1}^{n}\bar{b}\|f_j(x_j) - f_j(\widetilde{x}_j)\|_{\omega,c} + \sum_{j=1}^{n}\bar{e}|c|^{-\tau_{ij}(t)/\omega}\|g_j(x_j) - g_j(\widetilde{x}_j)\|_{\omega,c} \\ & + \sum_{j=1}^{n}\bar{d}\int_{t-\delta_{ij}(t)}^{t}|c|^{-(t-\nu)/\omega}\|h_j(x_j) - h_j(\widetilde{x}_j)\|_{\omega,c}\,\Delta\nu\Bigg) \\ \leq\ & \bar{a}\|x^*(t)\|_{a\omega c} + \mathcal{C}\sup_{t\in\mathbb{T}_\omega^+}\|x^*(t)\|. \end{aligned} \tag{4.3.8}$$

Now, define the functions $\mathcal{F}_1, \mathcal{F}_2 : [0,\infty) \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{F}_1(x) &= \frac{\ln|c|}{\omega} + \underline{a} - x - \frac{\mathcal{C}(1+\bar{\mu}x)}{1-\underline{a}\bar{\mu}}, \\ & \& \\ \mathcal{F}_2(x) &= \frac{\ln|c|}{\omega} + \underline{a} - x - \left(\frac{\bar{a}(1+\bar{\mu}x)}{1-\underline{a}\bar{\mu}} + \frac{\ln|c|}{\omega} + \underline{a} - x\right)\mathcal{C}. \end{aligned} \tag{4.3.9}$$

By Assumption **5**, it is evident that $\mathcal{F}_1(0), \mathcal{F}_2(0) > 0$, provided that $\frac{\ln|c|}{\omega} + \underline{a} > 0$. Furthermore, since $0 < \mathcal{C} < 1$, it implies $\lim\limits_{x\to\infty} \mathcal{F}_1(x) = -\infty = \lim\limits_{x\to\infty} \mathcal{F}_2(x)$. Considering the continuity of the functions $\mathcal{F}_1(x)$ and $\mathcal{F}_2(x)$, we conclude the existence of positive constants γ_1 and γ_2 such that $\mathcal{F}(\gamma_1) = 0 = \mathcal{F}(\gamma_2)$, $\mathcal{F}_1(x) > 0$ for all $x \in (0, \gamma_1)$, and $\mathcal{F}_2(x) > 0$ for all $x \in (0, \gamma_2)$. Therefore, for a constant $0 < \lambda < \min\left\{\gamma_1, \gamma_2, \underline{a} + \frac{\ln|c|}{\omega}\right\}$, we have $\mathcal{F}_1(\lambda), \mathcal{F}_2(\lambda) > 0$. Consequently,

$$\frac{(1+\bar{\mu}\lambda)\mathcal{C}}{1-\underline{a}\bar{\mu}}, \left(\frac{\bar{a}(1+\bar{\mu}\lambda)}{1-\underline{a}\bar{\mu}} + \frac{\ln|c|}{\omega} + \underline{a} - \lambda\right)\mathcal{C} < \frac{\ln|c|}{\omega} + \underline{a} - \lambda.$$

Now, consider the constant $\mathcal{M}$ as

$$\mathcal{M} = \min\left\{\frac{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega} + \underline{a} - \lambda\right)}{(1+\bar{\mu}\lambda)\mathcal{C}}, \left(1 + \frac{\bar{a}(1+\bar{\mu}\lambda)}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega} + \underline{a} - \lambda\right)}\right)^{-1} \frac{1}{\mathcal{C}}\right\}. \tag{4.3.10}$$

It is noteworthy that the inequality

$$\frac{1}{\mathcal{M}} - \frac{(1+\bar{\mu}\lambda)\mathcal{C}}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega} + \underline{a} - \lambda\right)} \leq 0 \tag{4.3.11}$$

holds. This observation is important for our subsequent analysis. Considering (4.3.7) and (4.3.8) and noting that $\mathcal{M} > 1$ and $e_{\ominus\lambda}(t, t_0) > 1$ for $t \in [-\theta, t_0]_{\mathbb{T}_\omega}$, we observe that

$$\|x^*(t)\| = \|\varphi^*\| = \max\{\|\varphi^*(t)\|_{\omega,c}, \|{\varphi^*}^{\Delta}(t)\|_{\omega,c}\} \leq \mathcal{M}e_{\ominus\lambda}(t,t_0)\sup_t\|\varphi^*(t)\| \quad \text{for all} \quad t \in [-\theta, t_0]_{\mathbb{T}_\omega}.$$

Next, we claim that for all $t \in (t_0, \infty)_{\mathbb{T}_\omega}$,

$$\|x^*(t)\| = \max\{\|x^*(t)\|_{\omega,c}, \|{x^*}^{\Delta}(t)\|_{\omega,c}\} \leq \mathcal{M}e_{\ominus\lambda}(t,t_0)\sup_t\|\varphi^*(t)\|. \tag{4.3.12}$$

This is equivalent to saying that for any $\alpha > 1$, the following inequality holds:

$$\|x^*(t)\| < \alpha\mathcal{M}e_{\ominus\lambda}(t,t_0)\sup_t\|\varphi^*(t)\|, \quad t > t_0. \tag{4.3.13}$$

We will prove the above by using the method of contradiction. Suppose that (4.3.13) is not true. Then, there must exist some $t_1 > t_0$ and $K > 1$ such that

$$\|x^*(t_1)\| = K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|, \text{ and } \|x^*(t)\| \leq K\alpha\mathcal{M}e_{\ominus\lambda}(t,t_0)\sup_t\|\varphi^*(t)\|, \ \forall\, t \in (t_0, t_1)_{\mathbb{T}_\omega}. \tag{4.3.14}$$

Using (4.3.14) in (4.3.7), we obtain

$$\begin{aligned}
&\|x^*(t_1)\|_{\omega,c}\\
&\le e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)\sup_t\|\varphi^*(t)\| + \sup_t\|\varphi^*(t)\|\mathcal{C}\int_{t_0}^{t_1}\frac{e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,s)}{1-\underline{a}\bar{\mu}}K\alpha\mathcal{M}e_{\ominus\lambda}(s,t_0)\Delta s\\
&= K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\left(\frac{e_{\lambda}(t_1,t_0)e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)}{K\alpha\mathcal{M}}+\mathcal{C}\int_{t_0}^{t_1}\frac{e_{\lambda}(t_1,s)e_{\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,s)}{1-\underline{a}\bar{\mu}}\Delta s\right)\\
&= K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\left(\frac{e_{\lambda\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)}{K\alpha\mathcal{M}}+\mathcal{C}\int_{t_0}^{t_1}\frac{(1+\mu(s)\lambda)\,e_{\lambda\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,\sigma(s))}{(1-\underline{a}\bar{\mu})\left(1+\mu(s)\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)}\Delta s\right)\\
&\le K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\left(\frac{e_{\lambda\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)}{K\alpha\mathcal{M}}+\mathcal{C}\frac{(1+\bar{\mu}\lambda)\left(e_{\lambda\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)-1\right)}{(1-\underline{a}\bar{\mu})\left(\lambda-\left(\frac{\ln|c|}{\omega}+\underline{a}\right)\right)}\right)\\
&\le K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\left(e_{\lambda\ominus(\frac{\ln|c|}{\omega}+\underline{a})}(t_1,t_0)\left(\frac{1}{K\mathcal{M}}-\frac{(1+\bar{\mu}\lambda)\mathcal{C}}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega}+\underline{a}-\lambda\right)}\right)+\right.\\
&\qquad\left.\frac{(1+\bar{\mu}\lambda)\mathcal{C}}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega}+\underline{a}-\lambda\right)}\right)\\
&\le K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\frac{(1+\bar{\mu}\lambda)\mathcal{C}}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega}+\underline{a}-\lambda\right)}\\
&< K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|x^{*\Delta}(t_1)\|_{\omega,c} &\le \bar{a}\|x^*(t_1)\|_{a\omega c}+\mathcal{C}\|x^*(t_1)\|\\
&\le K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\frac{\bar{a}(1+\bar{\mu}\lambda)\mathcal{C}}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega}+\underline{a}-\lambda\right)}+K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\\
&= K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|\left(1+\frac{\bar{a}(1+\bar{\mu}\lambda)}{(1-\underline{a}\bar{\mu})\left(\frac{\ln|c|}{\omega}+\underline{a}-\lambda\right)}\right)\mathcal{C}\\
&< K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|.
\end{aligned}$$

Thus, we have

$$\|x^*(t_1)\| < K\alpha\mathcal{M}e_{\ominus\lambda}(t_1,t_0)\sup_t\|\varphi^*(t)\|,$$

which contradicts (4.3.14). Hence, (4.3.12) holds true for all $t\in(t_0,\infty)_{\mathbb{T}_\omega}$. ☐

As every (ω, c)-periodic function defined on a time scale is an (ω, c)-asymptotically periodic function, we can have the following result.

Corollary 4.3.7. *Consider that there exists $c \in \mathbb{F}_{\mathbb{B}} \setminus \{0\}$ such that for $1 \le i, j \le n$, $a_i, b_{ij}, e_{ij}, d_{ij} \in \mathrm{P}_\omega(\mathbb{T}_\omega, \mathbb{B})$, $\eta_i, \tau_{ij}, \delta_{ij} \in \mathrm{P}_\omega(\mathbb{T}_\omega, \Pi^{0+}_{\mathbb{T}_\omega})$, $U_i \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, and $f_i, g_j, h_j \in \mathrm{P}_{\omega,c}(\mathbb{B}, \mathbb{B})$. Furthermore, suppose that Assumptions **2**, 4, and **5** hold. Then, system (4.3.1) has a unique (ω, c)-periodic solution that is globally exponentially stable.*

4.4 Illustrative examples

In this section, we provide examples from various time scales to illustrate the application of the main results. Taking into account particular values of c, we investigated the existence of stable (ω, c)-asymptotically periodic solutions for three different DCNNs.

Consider the following system of DCNNs with two cells on a time scale $\mathbb{T}_\omega$:

$$\begin{aligned} x_i^\Delta(t) &= -a_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{2} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{2} e_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{2} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))\Delta s + U_i(t), \quad (i=1,2) \end{aligned} \tag{4.4.1}$$

supplemented with the initial condition $\varphi = (-1, 1)^T$.

Example 4.4.1. *For $\mathbb{T} = \mathbb{R}$, take the activation functions and the coefficients described by*

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3\sqrt{3}}\cos x + \frac{1}{25}\sin 5x \\ \frac{1}{4}x + \frac{\sin x}{2} + \frac{1}{4}e^{\cos x} \end{bmatrix}, \quad \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2e}\sin(x)e^{-\cos(x)} \\ \frac{1}{3}x + \frac{1}{3}\cos(\sin^2 x) \end{bmatrix},$$

$$\begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{\sin x}{2} + \frac{1}{5}e^{-\sin^2 x} \\ \frac{\sqrt{3}}{4}x + \frac{1}{2}\cos(\sin x) \end{bmatrix}, \quad \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)^{t/\pi}\cos(t) + te^{-t} \\ \left(\frac{3}{2}\right)^{t/\pi}\sin(t) + e^{-\sin 7t} \end{bmatrix},$$

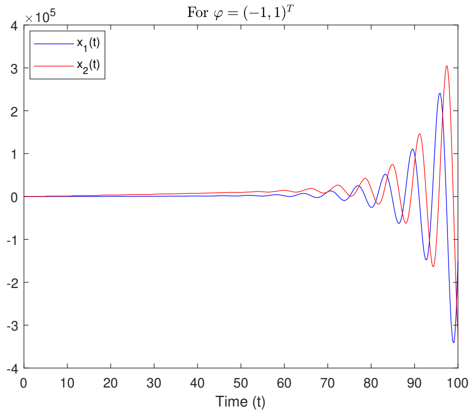
$$[a_i] = \begin{bmatrix} \frac{1}{5} + \frac{1}{10}e^{\sin 2t + \cos 2t} \\ \frac{3}{4} + \frac{1}{4}|\cos t| \end{bmatrix}, \quad [\eta_i] = \begin{bmatrix} \frac{2}{41}|\cos t| \\ \frac{1}{20}|\sin(\cos t)| \end{bmatrix},$$

$$[b_{ij}] = \begin{bmatrix} \frac{7}{6e^5}e^{\sin 2t} + \frac{1}{5e^5\pi^t}\cos\left(t + \frac{\pi}{6}\right) & \frac{1}{175\sqrt{7}}\cos 2t + \frac{1}{129}e^{-t^2} \\ \frac{1}{117}\sin 2t + \frac{1}{101}e^{-3t} & \frac{1}{75\pi^2}e^{\cos 2t} + \frac{1}{27e^2}\frac{\cos^3 5t}{5^t} \end{bmatrix},$$

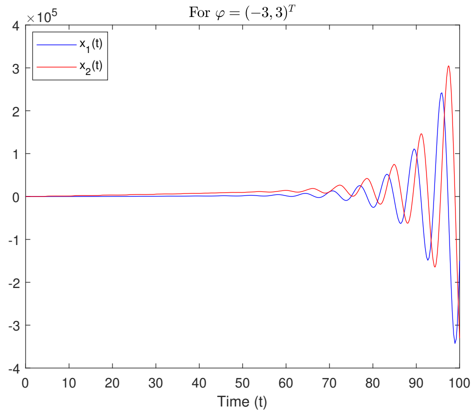
$$[e_{ij}] = \begin{bmatrix} \frac{\sqrt{2}}{77}\cos 2t + \frac{1}{17\sqrt{3}}te^{-7t} & \frac{1}{171}e^{\cos 2t} + \frac{1}{127\cdot 7^t}\sin\left(\frac{\pi}{4} + \sqrt{7}t\right) \\ \frac{1}{51}\cos(\sin 2t) + \frac{\sin 7t}{7^t} & \frac{1}{17\pi}\cos 2t + \frac{1}{7}t^2e^{-5t} \end{bmatrix},$$

$$[d_{ij}] = \begin{bmatrix} \frac{3}{2}\sin 2t + \frac{6}{5\cdot 4^t}\sin\left(t + \frac{\pi}{4}\right) & \frac{2}{5}e^{\sin(\cos 2t)} + e^{-3t} \\ \sin(\cos 2t) + 2^{-t} & \frac{7}{5}\cos 2t + \frac{1}{6^t}\sin\left(t + \frac{\pi}{6}\right) \end{bmatrix},$$

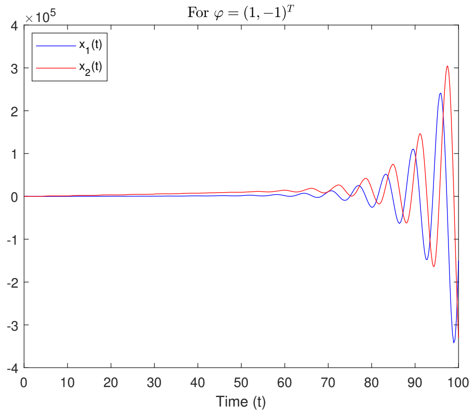
$$[\tau_{ij}] = \begin{bmatrix} \frac{3}{5} + \frac{1}{2}e^{|\sin\frac{\pi}{2}t|} & \frac{9}{5} - |\cos\frac{\pi}{2}t| \\ \frac{6}{5} + \frac{3}{5}|\cos t| & \frac{1}{2} + |\sin\frac{\pi}{2}t| \end{bmatrix}, \quad [\delta_{ij}] = \begin{bmatrix} \frac{1}{10}|\cos t| & \frac{1}{13}|\sin t| \\ \frac{1}{20} + \frac{1}{20}|\sin t| & \frac{1}{3\pi^2}e^{|\cos t|} \end{bmatrix}.$$



(a) Solution of (4.4.1) with the coefficients provided in Example 4.4.1, computed over the domain $[0, 100]$.



(b) Solution of (4.4.1) computed over $[0, 100]$, with the coefficients given in Example 4.4.1 and the perturbed initial condition $(-3, 3)^T$.



(c) Solution of (4.4.1) computed over $[0, 100]$, with the coefficients given in Example 4.4.1 and the perturbed initial condition $(1, -1)^T$.

Figure 4.1: Solution trajectories of the CNN system (4.4.1) on the time scale $\mathbb{R}$ with the coefficients given in Example 4.4.1 and the initial values, $(-1, 1)^T$, $(-3, 3)^T$, and $(1, -1)^T$.

Clearly, we have $\mu \equiv 0$, and for $\omega = \pi$ and $c = -\frac{3}{2}$, it is not difficult to verify that all conditions of Assumptions 1–4 *are satisfied. Furthermore, by computing all the bounds, we obtain*

$$\bar{a} = 1, \quad \underline{a} = 0.75, \quad \bar{b} = 0.01, \quad \bar{e} = 0.02, \quad \bar{d} = 2,$$

$$\bar{\tau} = 1.8, \quad \underline{\tau} = 0.8, \quad \bar{\eta} = 0.05 \quad \underline{\eta} = 0, \quad \bar{\delta} = 0.1, \quad \underline{\delta} = 0,$$

$$\bar{L}_f = 1, \qquad \bar{L}_g = 1, \qquad \bar{L}_h = 1.$$

This implies that

$$\mathcal{C} = \frac{\pi\bar{a}(1-(3/2)^{-\underline{\eta}/\pi})}{\ln(3/2)} + 4\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g(3/2)^{-\underline{\tau}/\pi} + \frac{\pi\bar{d}\bar{L}_h(1-(3/2)^{-\underline{\delta}/\pi})}{\ln(3/2)}\right) \leq 0.112152,$$

consequently,

$$\frac{\mathcal{C}}{\left(\frac{\ln(3/2)}{\pi} + \underline{a}\right)} < \frac{0.112152}{0.879063} = 0.1275813, \qquad \left(\frac{\bar{a}}{\left(\frac{\ln(3/2)}{\pi} + \underline{a}\right)} + 1\right)\mathcal{C} < 2.1375742 \times 0.112152 = 0.239733,$$

therefore, by Theorem 4.3.5 and Theorem 4.3.6, system (4.4.1) has a unique $(\pi, -\frac{3}{2})$-asymptotically periodic solution on $\mathbb{R}$ (evident from Figure (4.1a)) that is globally exponentially stable (can be graphically visualized by comparing Figures (4.1b) and (4.1c) with Figure (4.1a)). Also, for any solution $\widetilde{x}$ of (4.4.1) on $\mathbb{R}$ with initial condition $\widetilde{\varphi}$, (4.3.6) holds for $\mathcal{M} = \frac{0.87906-\lambda}{(1.87906-\lambda)\times 0.11215}$ and $0 < \lambda < 0.75273$ (calculated using (4.3.9) and (4.3.10)). Thus, if $\widetilde{x}$ is a solution of the considered example with the initial condition $(-3,3)^T$ or $(-1,1)^T$, in both cases, we have

$$\|x - \widetilde{x}\| \leq \frac{0.87906-\lambda}{(1.87906-\lambda)\times 0.11215} 4e^{-\lambda t} \quad \textit{for all} \quad t \geq 0, \quad \textit{and} \quad \lambda \in (0, 0.75273).$$

Example 4.4.2. *For $\mathbb{T} = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1]$, take the activation functions and the coefficients described by*

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} \frac{\cos\pi x}{4} + \frac{\sin 5x}{6} \\ \frac{1}{3}\sin\pi x + \frac{1}{5}e^{-\cos 2x} \end{bmatrix}, \qquad \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{25}x + \frac{1}{\sqrt{5}}\sin(\pi x) \\ \frac{\cos(\pi x)}{5} + \frac{\cos(\sin^2 x)}{7} \end{bmatrix},$$

$$\begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} \frac{\sin\pi x}{3} + \frac{1}{\sqrt{7}}x \\ \frac{\cos(\pi x)}{6} + e^{-sin(x)} \cdot \frac{\cos(\sin x)}{6} \end{bmatrix}, \qquad \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} \left(\sqrt{\frac{3}{2}}\right)^t \cos(\pi t) + \left(\frac{\sqrt{3}}{5}\right)^t \cos^2 t \\ \left(\sqrt{\frac{3}{2}}\right)^t \sin(\pi t) + \left(\frac{2}{3}\right)^t e^{\sin 7t} \end{bmatrix},$$

$$[a_i] = \begin{bmatrix} \frac{29}{50} + \frac{3}{20}e^{\sin\pi t + \cos\pi t} \\ \frac{41}{50} + \frac{17}{100}\cos\pi t \end{bmatrix}, \quad [\eta_i] = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad [\tau_{ij}] = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}, \quad [\delta_{ij}] = \begin{bmatrix} 2 & 8 \\ 0 & 10 \end{bmatrix},$$

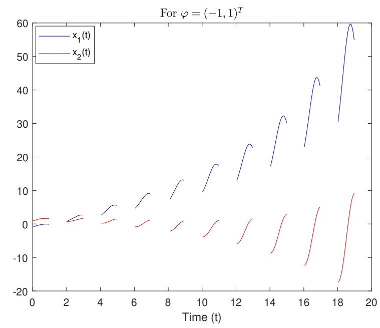
$$[b_{ij}] = \begin{bmatrix} \frac{197}{10^5}e^{\sin\pi t} + \frac{69}{10^4\cdot\pi^t}\cos\left(t+\frac{\pi}{6}\right) & -\frac{1}{100}\cos\pi t + \frac{1}{4}e^{-t^2} \\ \frac{1}{5}\sin\pi t + \frac{87}{40\cdot e^4}e^{-3t} & \frac{25}{2298}e^{\cos\pi t} + \frac{5}{1933}\frac{\cos^3(5t)}{5^t} \end{bmatrix},$$

$$[e_{ij}] = \begin{bmatrix} \frac{1}{20}\cos\pi t + \frac{57}{2}te^{-7t} & \frac{1}{40}e^{\cos\pi t} + \frac{33}{10^4\cdot 7^t}\sin\left(\frac{\pi}{4} + \sqrt{7}t\right) \\ \frac{7}{100}\cos(\sin\pi t) + \frac{1}{10}\frac{\sin 7t}{7^t} & \frac{3}{40}\cos\pi t + 23t^2e^{-5t} \end{bmatrix},$$

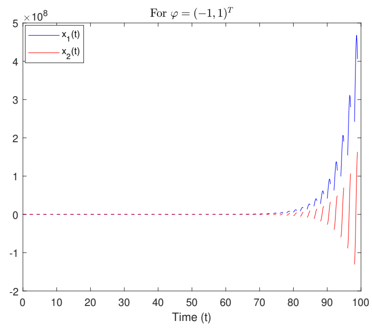
$$[d_{ij}] = \begin{bmatrix} \sin\pi t + \frac{49}{10^3\cdot 4^t}\sin\left(t+\frac{\pi}{4}\right) & \frac{1}{100}e^{\sin(\cos\pi t)} + \frac{23}{500}\frac{1}{(t+2)^2} \\ \frac{1}{50}\sin(\cos\pi t) + \frac{9}{500}\frac{1}{\sqrt{t+1}} & \frac{13}{10^3}\cos\pi t + \frac{1}{40\cdot 6^t}\sin\left(t+\frac{\pi}{6}\right) \end{bmatrix}.$$

We have $(\omega, c) = (2, 3/2)$, and $\mu = 0$ for all $t \in [2n, 2n+1)$, $\mu = 1$ for all $t \in \{2n+1\}$, $n \in \mathbb{Z}$. Also, by computing the bounds, we get

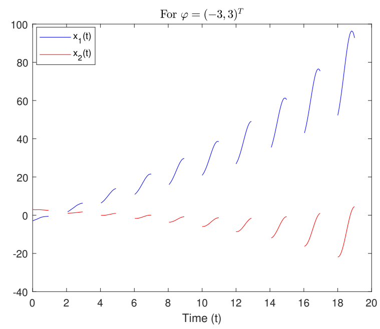
$$\bar{a} = 1, \quad \underline{a} = 0.65, \quad \bar{b} = .04, \quad \bar{e} = .08, \quad \bar{d} = 0.035,$$



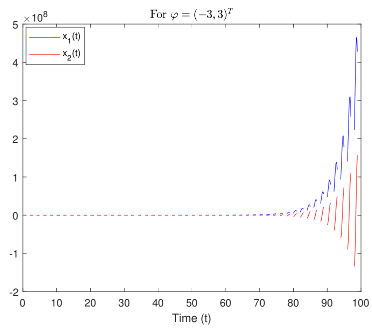
(a) Solution of (4.4.1) computed over the domain $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 20]$, with the coefficients given in Example 6.6.3.



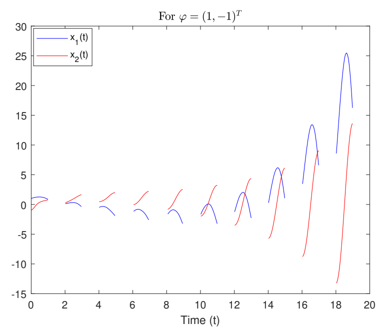
(b) Solution of (4.4.1) computed over the domain $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 100]$, with the coefficients given in Example 6.6.3.



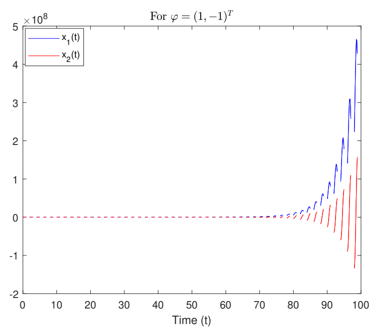
(c) Solution of (4.4.1) computed over $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 20]$, with the coefficients given in Example 6.6.3 and the perturbed initial condition $(-3, 3)^T$.



(d) Solution of (4.4.1) computed over $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 100]$, with the coefficients given in Example 6.6.3 and the perturbed initial condition $(-3, 3)^T$.



(e) Solution of (4.4.1) computed over $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 20]$, with the coefficients given in Example 4.4.2 and the perturbed initial condition $(1, -1)^T$.



(f) Solution of (4.4.1) computed over $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]\cap[0, 100]$, with the coefficients given in Example 6.6.3 and the perturbed initial condition $(1, -1)^T$.

Figure 4.2: The solution trajectories of the CNN system (4.4.1) on the time scale $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]$ with the coefficients given in Example 4.4.2 and the initial values $(-1, 1)^T$, $(-3, 3)^T$, and $(1, -1)^T$, respectively, are shown in Figures (4.2b), (4.2d), and (4.2f). Figures (4.2a), (4.2c), and (4.2e) provide a clearer view of the behavior of the respective solutions on initial time phases.

$$\underline{\tau} = 2, \quad \underline{\eta} = 0, \quad \underline{\delta} = 0,$$

$$\bar{L}_f = \frac{1}{2}, \quad \bar{L}_g = \frac{1}{2}, \quad \bar{L}_h = \frac{1}{2}.$$

Thus, we obtain

$$\mathcal{C} = \frac{2\bar{a}(1-(3/2)^{-\underline{\eta}/2})}{\ln(3/2)} + 4\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g(3/2)^{-\underline{\tau}/2} + \frac{2\bar{d}\bar{L}_h(1-(3/2)^{-\underline{\delta}/2})}{\ln(3/2)}\right) < 0.1867,$$

which further implies

$$\frac{\mathcal{C}}{\left(\frac{\ln(3/2)}{2} + \underline{a}\right)(1-\underline{a})} < 3.3506 \times 0.1867 < 0.6256, \quad \textit{and}$$

$$\left(\frac{\bar{a}}{\left(\frac{\ln(3/2)}{2} + \underline{a}\right)(1-\underline{a})} + 1\right)\mathcal{C} < 4.3506 \times 0.1867 < 0.8123.$$

Since all the conditions of Assumptions **1**–**5** *are satisfied, system 4.4.1 has a unique asymptotically (2,3/2)-periodic solution on* $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]$ *(evident from Figure (4.2b)) that is globally exponentially stable (can be graphically visualized by comparing Figures (4.2d) and (4.2f) with Figure (4.2b)). Note that for any arbitrary solution* $\tilde{x}$ *of* (4.4.1) *on* $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]$ *with the initial condition* $\tilde{\varphi}$*,* (4.3.6) *holds for* $\mathcal{M} = \frac{0.85273-\lambda}{1.2985-0.35\lambda} \times 1.8746$ *and* $0 < \lambda < 0.09305$ *(calculated using* (4.3.9) *and* (4.3.10)*). Thus, if* $\tilde{x}$ *is a solution to the considered example with the initial condition* $(-3,3)^T$ *or* $(-1,1)^T$*, we have*

$$\|x - \tilde{x}\| \leq \frac{0.85273-\lambda}{1.2985-0.35\lambda} \times 1.8746 \times 4e_{\ominus\lambda(t,0)} \quad \textit{for all} \quad t \geq 0, \quad \textit{and} \quad \lambda \in (0, 0.09305).$$

Example 4.4.3. *For* $\mathbb{T} = \mathbb{Z}$*, take the activation functions and the coefficients described by*

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+\cos(x)) \\ \sin(x) \end{bmatrix}, \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \sin(x)\cos(x) \\ \frac{1}{4}x + \frac{3}{4}\sin^2(x) \end{bmatrix}, \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{1}{2}e^{\cos(x)} \\ \frac{2}{11}x + \frac{1}{6}e^{\sin(2x)} \end{bmatrix},$$

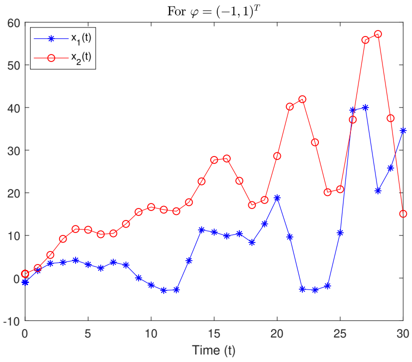
$$\begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} 2^{t/6}\cos\frac{\pi}{3}t + \cos^2 t \\ 2^{t/6}\sin\frac{\pi}{3}t + e^{\sin 7t} \end{bmatrix},$$

$$[a_i] = \begin{bmatrix} \frac{1}{2} + \frac{1}{16}e^{\sin\frac{\pi}{3}t + \cos\frac{\pi}{3}t} \\ \frac{1}{2} + \frac{1}{4}|\cos\frac{\pi}{6}t| \end{bmatrix}, [\eta_i] = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, [\tau_{ij}] = \begin{bmatrix} 12 & 5 \\ 3 & 7 \end{bmatrix}, [\delta_{ij}] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

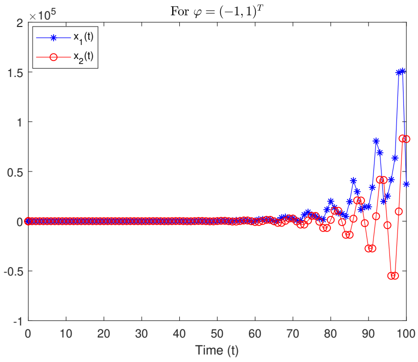
$$[b_{ij}] = \begin{bmatrix} 0.01e^{\sin\frac{\pi}{3}t} + \frac{0.034}{\pi^t}\cos(t+\frac{\pi}{6}) & 0.02\cos\frac{\pi}{3}t + 0.02e^{-t^2} \\ 0.03\sin\frac{\pi}{3}t + 0.04e^{-3t} & 0.11e^{\cos\frac{\pi}{3}t} + 0.1\frac{\cos^3 5t}{5^t} \end{bmatrix},$$

$$[e_{ij}] = \begin{bmatrix} 0.0015\cos\frac{\pi}{3}t + 9te^{-7t} & 0.0003e^{\cos\frac{\pi}{3}t} + \frac{0.0008}{7^t}\sin(\frac{\pi}{4} + \sqrt{7}t) \\ 0.0013\cos(\sin\frac{\pi}{3}t) + 0.0059\frac{\sin 7t}{7^t} & 0.01\cos\frac{\pi}{3}t + 0.133\ t^2e^{-5t} \end{bmatrix},$$

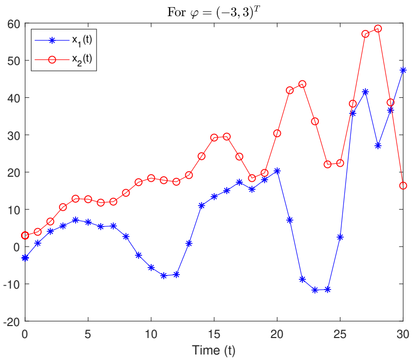
$$[d_{ij}] = \begin{bmatrix} 0.01\sin\frac{\pi}{3}t + \frac{0.031}{4^t}\sin(t+\frac{\pi}{4}) & 0.05e^{\sin(\cos\frac{\pi}{3}t)} + 0.01e^{-3t} \\ 0.013\sin(\cos\frac{\pi}{3}t) + \frac{0.011}{2^t} & 0.011\cos\frac{\pi}{3}t + \frac{0.012}{6^t}\sin(t+\frac{\pi}{6}) \end{bmatrix}.$$



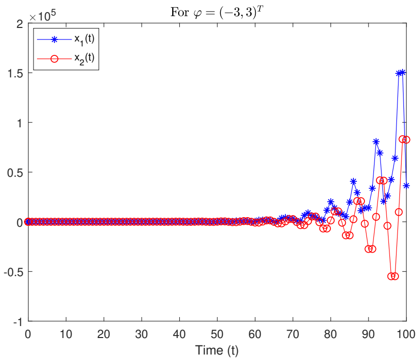
(a) Solution of (4.4.1) computed over the domain $[0, 30]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3.



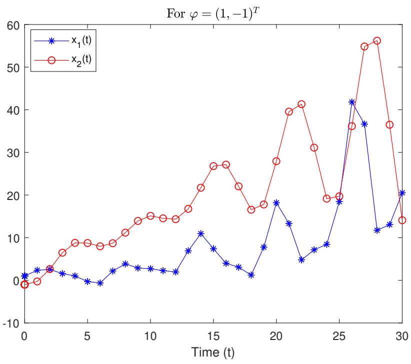
(b) Solution of (4.4.1) computed over the domain $[0, 100]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3.



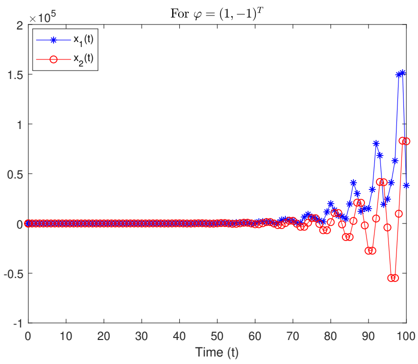
(c) Solution of (4.4.1) over the domain $[0, 30]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3 and the perturbed initial condition $(-3, 3)^T$.



(d) Solution of (4.4.1) over the domain $[0, 100]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3 and the perturbed initial condition $(-3, 3)^T$.



(e) Solution of (4.4.1) over the domain $[0, 30]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3 and the perturbed initial condition $(1, -1)^T$.



(f) Solution of (4.4.1) over the domain $[0, 100]_{\mathbb{Z}}$, with the coefficients given in Example 4.4.3 and the perturbed initial condition $(1, -1)^T$

Figure 4.3: Figures (4.3b), (4.3d), and (4.3f) show the solution trajectories of the CNN system (4.4.1) on the time scale $\mathbb{Z}$ with the coefficients given in Example 4.4.3 and the initial values $(-1, 1)^T$, $(-3, 3)^T$, and $(1, -1)^T$, respectively. Figures (4.3a), (4.3c), and (4.3e) provide a more precise depiction of the behaviour of the respective solutions during the initial time phases.

As, $(\omega, c) = (6, 2)$, $\mu \equiv 1$, *and*

$$\bar{a} = 0.75, \quad \underline{a} = 0.5, \quad \bar{b} = 0.04, \quad \bar{e} = 0.014, \quad \bar{d} = 0.022\ ,$$

$$\bar{\tau} = 15, \quad \underline{\tau} = 3, \quad \bar{\eta} = 4, \quad \underline{\eta} = 0, \quad \bar{\delta} = 2, \quad \underline{\delta} = 1,$$

$$\bar{L}_f = 1, \quad \bar{L}_g = 1, \quad \bar{L}_h = 1,$$

we have

$$\mathcal{C} = \frac{\bar{a}(1 - 2^{-\underline{\eta}/6})}{2^{1/6} - 1} + 4\left(\bar{b}\,\bar{L}_f + \bar{e}\,\bar{L}_g 2^{-\underline{\tau}/6} + \frac{\bar{d}\bar{L}_h(1 - 2^{-\underline{\delta}/6})}{2^{1/6} - 1}\right) < 0.278,$$

which implies

$$\frac{\mathcal{C}}{\left(\frac{\ln 2}{6} + \underline{a}\right)(1 - \underline{a})} < 3.2493 \times 0.278 < 0.90331, \quad \textit{and}$$

$$\left(\frac{\bar{a}}{\left(\frac{\ln 2}{6} + \underline{a}\right)(1 - \underline{a})} + 1\right)\mathcal{C} < 3.43695 \times 0.278 < 0.9555.$$

Therefore, system 4.4.1 has a unique $(6, 2)$*-asymptotically periodic solution on* $\mathbb{Z}$ *(evident from Figure (4.3b)) that is also globally exponentially stable (can be graphically visualized by comparing Figures (4.3d) and (4.3f) with Figure (4.3b)). Furthermore, for any solution* $\widetilde{x}$ *of* (4.4.1) *on* $\mathbb{Z}$ *with initial condition* $\widetilde{\varphi}$*,* (4.3.6) *holds for* $\mathcal{M} = \frac{0.61552-\lambda}{2.11552-\lambda} \times 3.5971$ *and* $0 < \lambda < 0.02406$ *(calculated using* (4.3.9) *and* (4.3.10)*). Thus, if* $\widetilde{x}$ *is a solution of the considered example with the initial condition* $(-3, 3)^T$ *or* $(-1, 1)^T$*, we have*

$$\|x - \widetilde{x}\| \leq \frac{4(0.61552 - \lambda)}{0.278 \times (2.11552 - \lambda)(1 + \lambda)^t} \quad \textit{for all} \quad t \geq 0, \quad \textit{and} \quad \lambda \in (0, 0.02406).$$

4.5 Conclusion

As many functions do not satisfy strict (ω, c)-periodicity from the initial phase, the concept of (ω, c)-asymptotic periodicity naturally arises to describe functions that attain (ω, c)-periodic behavior asymptotically after a transient phase. In this chapter, (ω, c)-asymptotic periodicity has been introduced within the framework of time scales, providing a unified approach for studying asymptotic periodicity, antiperiodicity, Bloch periodicity, and related bounded and unbounded functions. Motivated by the importance of qualitative analysis in time-delayed systems, a class of delayed cellular neural networks has been considered, and under appropriate conditions on system coefficients and activation functions, sufficient conditions for the existence and stability of (ω, c)-asymptotically periodic solutions have been established. The theoretical results have been illustrated through examples and numerical simulations, where stability has been confirmed by verifying that the norms of solution differences between original and perturbed systems remain within the prescribed bounds.

CHAPTER 5

Exponential Synchronization of Weighted Pseudo (ω, c)-Periodic Solutions for Neutral-type Impulsive Evolution Systems on Time Scales [1]

This chapter introduces the concept of weighted pseudo (ω, c)-periodicity on ω-periodic time scales. This notion generalizes (ω, c)-asymptotic periodicity and pseudo Bloch periodicity, and encompasses several classes of bounded and unbounded functions defined on time scales. We derive existence results for weighted pseudo (ω, c)-periodic solutions for a class of neutral-type evolution systems on time scales with piecewise impulses and time-varying delays. Several fundamental properties of pseudo (ω, c)-periodic functions on time scales are established, which play a crucial role in the qualitative analysis of the considered nonlinear dynamic equations. Furthermore, by designing an appropriate state feedback controller and constructing a suitable Lyapunov function, we obtain sufficient conditions ensuring global exponential synchronization for the considered system with weighted pseudo (ω, c)-periodic solutions.

[1] The content of this chapter is published with title "*Exponential synchronization of the weighted pseudo (ω, c)-periodic solution for neutral-type impulsive evolution system on time scales*" in Communication in Communication in Nonlinear Science and Numerical Simulation 161 (2026) 110177. https://www.sciencedirect.com/science/article/pii/S1007570426005344.

5.1 Introduction

Dynamic equations on time scales have emerged as a powerful framework for modeling systems that evolve over both continuous and discrete domains. Introducing delays into these equations enhances their realism, particularly for systems with memory or dependence on past states. Delay dynamic equations (DDEs) form a well-studied class, in which the delta derivative of the unknown function at a given time depends on its values at earlier times. DDEs have been widely applied in various fields, including population dynamics with delayed reproduction [189], the spread of infectious diseases with incubation periods [190], control systems, and neural networks or coupled oscillators exhibiting delayed interactions [191, 192, 193]. A further generalization, known as neutral-type delay dynamic equations (NDDEs), involve delays not only in the function itself but also in its derivative, that is, the present rate of change of the system depends on both current and past values of the state and its rate of change. NDDEs have gained significant attention in recent years, especially for modeling control systems where such dependencies are critical [194, 195, 196, 197, 198].

Many real-world problems in science, technology, and economics involve evolutionary processes subject to abrupt changes in state due to instantaneous external disturbances, known as impulses. Such impulsive behavior is inherent in numerous systems, including bursting rhythm models in biology, blood flow, heartbeats, certain satellite motions, and population dynamics. Impulsive models are typically represented by dynamical systems that evolve continuously in time, except at specific moments when sudden perturbations cause discrete changes. These models are particularly useful for accurately capturing how abrupt changes influence the behavior of a system and have broad applications in different fields. Impulsive effects can be classified into two types: instantaneous, which occur abruptly over negligible durations, and non-instantaneous, which begin at a defined moment and persist over a finite interval. Real-world phenomena such as harvesting or natural disasters often involve such sudden transitions, leading to piecewise-continuous system trajectories. To model these processes, researchers have developed various impulsive dynamic systems on time scales and have extensively studied properties such as the existence of solutions, stability, controllability, and synchronization [4, 196, 199, 200, 201].

It is often observed in biomathematics that periodic and impulsive phenomena coexist within the same system, which has attracted considerable research interest in periodic dynamic systems on time scales with impulsive effects. Although periodicity is a foundational concept in the qualitative analysis of dynamic equations, not all real-world processes exhibit strict periodic behavior. Instead, some demonstrate more complex recurrence patterns, such as antiperiodicity, Bloch periodicity, almost periodicity, asymptotic periodicity, or pseudoperiodicity [92, 169, 202, 203, 204]. Among these, Bloch periodic functions, which generalize the classical periodic and antiperiodic functions, preserve their magnitude across each interval of the Bloch periodicity. Another extension known as (ω, c)-periodicity has been introduced in [14, 98, 115], which encompasses functions that increase or decrease proportionally after each interval of a fixed period. Depending on the value of c, the class of (ω, c)-periodic functions unifies several well-known types of recurrence, including periodic, Bloch periodic, and antiperiodic, and also includes certain unbounded functions with structured behavior. This versatility makes it a powerful tool for analysis, as discussed in Chapter 2. Some important extensions of (ω, c)-periodic functions, such as (ω, c)-asymptotically periodic functions and (ω, c)-pseudo periodic functions, have been developed in continuous-time setting to study perturbed (ω, c)-periodic behaviors. In particular, every (ω, c)-asymptotically periodic function belongs to the space of (ω, c)-pseudoperiodic functions. To capture even more nuanced dynamics, this framework has been further extended to weighted (ω, c)-pseudoperiodic functions. A comprehensive discussion on (ω, c)-periodicity and its variants, including theoretical developments and applications, can be found in [103, 205, 206, 207, 208, 209] and the references therein.

In Chapter 4, the time scale version of (ω, c)-asymptotically periodic functions and their applications to cellular neural networks are discussed. While asymptotically periodic oscillations require solutions to converge to a periodic function, pseudoperiodic oscillations relax this requirement by allowing a persistent nonperiodic component. Weighted pseudoperiodic oscillations further generalize this notion by replacing pointwise convergence with the vanishing of weighted averages of the nonperiodic component. Doubly weighted pseudoperiodic oscillations constitute a subsequent extension, employing two weight functions to capture more complex persistent perturbations beyond both standard and singly weighted pseudoperiodic oscillations. This chapter introduces and investigates the concept, fundamental properties, and applications of doubly-weighted pseudo (ω, c)-periodic functions on ω-periodic time scales, from which other cases follow as special cases. In recent years, there has been renewed interest in establishing criteria for the oscillatory behavior of various classes of dynamic equations on time scales. Motivated by this, we focus on (doubly) weighted pseudo (ω, c)-periodic oscillation of solutions to neutral-type evolution equations on time scales with piecewise impulsive effects and time-varying delays. We explore several properties of weighted (ω, c)-pseudo periodic functions, including their characterization via weighted pseudoperiodic functions on time scales and the uniqueness of their decomposition as a sum of an (ω, c)-periodic function and a weighted (ω, c)-ergodic function on time scales (defined later in the paper), along with their algebraic properties. We show that the space of weighted pseudo (ω, c)-periodic functions, endowed with a suitable norm, forms a Banach space. We also present illustrative examples and a composition result. In contrast to pseudo Bloch periodic (and thus pseudoperiodic and pseudo antiperiodic) functions, the results in this chapter admit unbounded behavior on time scales in both the periodic and ergodic components. These findings lead to necessary and sufficient criteria for the existence of a unique weighted pseudo (ω, c)-periodic solution to the neutral-type impulsive evolution system defined in (5.3.1).

Synchronization is a widespread and intriguing phenomenon that is observed in various disciplines, including science, engineering, and social behavior. In general, synchronization refers to the adjustment of rhythms among oscillating systems due to weak interactions that enable them to evolve in a coordinated manner over time. Mathematically, it typically implies that the trajectories of two or more coupled systems converge as time approaches infinity. In recent years, the synchronization of almost-periodic solutions in dynamic systems has received considerable attention [210, 211, 212, 213, 214, 215], underscoring its importance in both theoretical and applied contexts. Various types of synchronization have been studied, including complete synchronization, exponential synchronization, finite-time synchronization, and adaptive synchronization. Among these, exponential synchronization is particularly important, as it quantifies the rate at which systems synchronize. For instance, the global asymptotic synchronization of fractional-order neural networks with impulsive effects has been demonstrated in [212]. Techniques based on contradiction and inequalities have been employed in [211, 213], while in [215], the authors have investigated the exponential synchronization of almost periodic solutions for delayed neural networks using a Lyapunov functional. In [216], criteria for global exponential synchronization of weighted pseudo almost periodic solutions in delayed cellular neural networks have been established. Adaptive event-triggered control has been applied in [217] to study the exponential synchronization of complex delayed dynamical networks. Antiperiodic synchronization for cellular neural networks of neutral-type has been explored in [218], and exponential synchronization of periodic solutions of coupled DDEs on time scales has been addressed in [87]. In [219], the authors have investigated the asymptotic synchronization of complex networks with time-varying topology on time scales. Furthermore, synchronization criteria for almost periodic solutions in delayed neural networks with impulses on time scales have been developed in [220], while pseudo almost periodic synchronization in delayed cellular neural networks on time scales has been analyzed in [183]. Motivated by these studies, this article further investigates the problem of exponential synchronization of weighted pseudo (ω, c)-periodic solutions to (5.3.1).

A brief outline of this chapter is as follows: In Section 5.2, we introduce the concept of (doubly) weighted pseudo (ω, c)-periodicity on time scales and present several fundamental results for this class of functions. Section 5.3 includes a brief description of the evolution model considered for the study. In Section 5.4, we establish sufficient conditions for the existence of weighted pseudo (ω, c)-periodic solutions to the considered model and investigate the global exponential synchronization of the evolution system admitting such solutions. In Section 5.5, numerical examples are presented to demonstrate the effectiveness of the theoretical results. Finally, Section 5.6 concludes the chapter with a brief summary of findings.

5.2 Weighted pseudo (ω, c)-periodicity in time scales

In this section, we introduce a class of functions, referred to as (doubly) weighted pseudo (ω, c)-periodic functions defined on ω-periodic time scales, and we investigate the properties of these functions. Throughout this chapter, we consider $\mathbb{T}_\omega$ to be an ω-periodic time scale, and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ denote a Banach space over the field $\mathbb{F}_{\mathbb{B}} = \mathbb{R}$ (or $\mathbb{C}$), and let $(\mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}, \mathbb{B}), \|\cdot\|_{\omega,\mathrm{c}})$ represents the space of all (ω, c)-periodic functions that share the same c-period ω, where $\|\cdot\|_{\omega,c}$ is defined in (4.2.1).

Let $\mathfrak{U}$ denote the collection of positive weights $\nu : \mathbb{T}_\omega \to \mathbb{R}^+$ that are Δ-integrable (locally) on $\mathbb{T}_\omega$. For any $l \in \Pi^+_{\mathbb{T}_\omega}$ we define the sets

$$\mathfrak{U}_\infty := \{\nu \in \mathfrak{U} : \lim_{l\to\infty} \mathfrak{m}(\nu, l, t_0) = \infty\}, \quad \text{and}$$

$$\mathfrak{U}_B := \{\nu \in \mathfrak{U}_\infty : \nu \text{ is bounded and } \inf_{t\in\mathbb{T}_\omega} \nu(t) > 0\},$$

where $\mathfrak{m}(\nu, l, t_0) := \int_{t_0-l}^{t_0+l} \nu(t)\Delta t$. Furthermore, let

$$(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) := \{f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B}) : \sup_{t\in\mathbb{T}_\omega} \|f(t)\|_{\omega,c} < \infty\},$$

and for $\nu \in \mathfrak{U}_\infty$, define

$$\mathcal{WE}^{\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) := \{\varphi \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) : \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\nu, l, t_0)} \int_{t_0-l}^{t_0+l} \|\varphi(t)\|_{\omega,c}\nu(t)\Delta t = 0\}.$$

In general, if $\varrho, \nu \in \mathfrak{U}_\infty$, we define the space of weighted (ω, c)-ergodic functions defined over a time scale $\mathbb{T}_\omega$ and having weights (ϱ, ν) as

$$\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) := \{\varphi \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) : \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|\varphi(t)\|_{\omega,c}\nu(t)\Delta t = 0\}.$$

Consider $\varrho, \nu \in \mathfrak{U}_\infty$. If $(\varrho/\nu) \in \mathfrak{U}_B$, then ϱ is equivalent to ν, that is, $\varrho \sim \nu$. Clearly, when $\varrho \sim \nu$, there exists some $a_1, a_2, b_1, b_2 > 0$ such that $a_1\varrho \le \nu \le b_1\varrho$ and $a_2\nu \le \varrho \le b_2\nu$. So that $a_1\mathfrak{m}(\varrho, l, t_0) \le \mathfrak{m}(\nu, l, t_0) \le b_1\mathfrak{m}(\varrho, l, t_0)$ and $a_2\mathfrak{m}(\nu, l, t_0) \le \mathfrak{m}(\varrho, l, t_0) \le b_2\mathfrak{m}(\nu, l, t_0)$, and we have

$$\mathcal{WE}^{\varrho}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) = \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) = \mathcal{WE}^{\nu,\varrho}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) = \mathcal{WE}^{\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}).$$

Thus, we can conclude that when $\varrho \sim \nu$, the weighted (ω, c)-ergodic space $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ coincides with $\mathcal{WE}^{\varrho}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. This suggests that the class $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ is more general and richer than $\mathcal{WE}^{\varrho}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, especially when ϱ and ν are not equivalent.

Next, we present the definition of weighted pseudo (ω, c)-periodic functions defined on time scales.

Definition 5.2.1. *Let $\varrho, \nu \in \mathfrak{U}_\infty$. A function $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is said to be a weighted pseudo (ω, c)-periodic with weights (ϱ, ν) if it can be decomposed as $f = f_{op} + f_\varphi$, where $f_{op} \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ and $f_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. We represent the collection of such functions by $\mathrm{WPP}^{\varrho,\nu}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$.*

Definition 5.2.1 coincides with the standard classical definition of weighted pseudo (ω, c)-periodic functions when $\mathbb{T}_\omega = \mathbb{R}$ and $c = 1$. Clearly, the class of functions defined in Definition 5.2.1 encompasses the class of singly weighted pseudo (ω, c)-periodic functions on time scales.

Remark 5.2.2. *When $c = 1$, the class $\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ represents the space of (doubly) weighted pseudoperiodic functions on $\mathbb{T}$ with weights (ϱ, ν). When $c = -1$, it becomes the space of (doubly) weighted pseudo antiperiodic functions. For $c = e^{ik\omega}$, $k \in \mathbb{R}$, the class represents the space of (doubly) weighted pseudo Bloch periodic functions in $\mathbb{T}_\omega$.*

We can consider the weights $(\varrho, \nu) = (1, 1)$ to obtain a subclass of $\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ as $\mathrm{PP}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, which is a collection of pseudo (ω, c)-periodic functions defined in $\mathbb{T}_\omega$. This consideration leads to the following definition of pseudo (ω, c)-periodic function, illustrating the generality of the proposed notion. The continuous analog of this definition can be found in [103].

Definition 5.2.3. *A function $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is said to be pseudo (ω, c)-periodic if it can be decomposed as $f_{op} + f_\varphi$, where $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_\varphi \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ with $\lim_{t\to\infty} \frac{1}{2l} \int_{t_0-l}^{t_0+l} \|f_\varphi(t)\|_{\omega,c} \Delta t = 0$, $l \in \Pi^+_{\mathbb{T}_\omega}$.*

Example 5.2.4. *Consider $\mathbb{T} = \mathbb{R}$, $\varrho(t) = e^{|t|}$ and $\nu = 1$. Then the function*

$$f(t) = c^{t/\pi} \sin(2t) + c^{t/\pi} \arctan(t)$$

is $(2, c)$-pseudoperiodic in $\mathbb{R}$ under the weights (ϱ, ν) but it is not the usual $(2, c)$-pseudoperiodic function.

Remark 5.2.5. *If we consider $\mathbb{T} = \mathbb{R}$, then $\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{R}, \mathbb{B})$ gives rise to an enlarged space of pseudo (ω, c)-periodic functions defined in [103].*

We next propose an important characterization of weighted pseudo (ω, c)-periodic functions on time scales.

Proposition 5.2.6. *An rd-continuous function $f : \mathbb{T}_\omega \to \mathbb{B}$ is (doubly) weighted pseudo (ω, c)-periodic if and only if*

$$f(t) = c^{t/\omega} \tilde{f}(t)$$

where $\tilde{f}$ is a (doubly) weighted ω-pseudoperiodic function with the same weights.

Proof. Let us first assume that $f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Then we have $f = f_{op} + f_\varphi$ with $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Define $\tilde{f}(t) := c^{-t/\omega} f(t)$. It clearly follows from the definitions that $c^{-t/\omega} f_{op}(t) \in \mathcal{P}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$ and $c^{-t/\omega} f_\varphi(t) \in \mathcal{WE}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, which implies $\tilde{f}(t) \in \mathrm{WPP}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$.

Furthermore, for some $\tilde{f} \in \mathrm{WPP}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, if we consider $f(t) = c^{t/\omega}\tilde{f}(t)$, then we have $f(t) = c^{t/\omega}\tilde{f}_p(t) + c^{t/\omega}\tilde{f}_\varphi(t)$ with $\tilde{f}_p \in \mathcal{P}_{(\omega,1)}(\mathbb{T}_\omega, \mathbb{B})$ and $\tilde{f}_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega, \mathbb{B})$, which implies that $f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. $\square$

It is important to note that the decomposition in Definition 5.2.3 is unique; i.e., there exist a unique $f_g \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and a unique $f_\varphi \in \mathcal{WE}^1_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ such that $f = f_g + f_\varphi \in \mathrm{WPP}^1_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. One can easily verify this using the unique representation of functions in the space of pseudoperiodic functions on time scales and Proposition 5.2.6. The question arises whether this is also true for weighted pseudo (ω, c)-periodic functions. The answer is: Not always. To illustrate this, we present the following example.

Example 5.2.7. *Consider* $\mathbb{T} = \bigcup_{k=-\infty}^{\infty} [2k, 2k+1]$. *Clearly,* $\mathbb{T}$ *is a periodic time scale. Let* $\varrho = e^{e^{|t|}}$. *Evidently,* $\inf_{t\in\mathbb{T}} \varrho(t) = e > 0$. *Also,*

$$
\begin{aligned}
\lim_{l\to\infty} \mathfrak{m}(\varrho, l, 0) &= \lim_{k\to\infty} \int_{-2k}^{2k} e^{e^{|t|}} \Delta t \\
&= \lim_{k\to\infty} \sum_{m=1}^{k} \left(\int_{-2m}^{-2m+1} e^{e^{-t}} dt + \int_{-2m+1}^{\sigma(-2m+1)} e^{e^{-t}} \Delta t + \int_{2m-2}^{2m-1} e^{e^{t}} dt + \int_{2m-1}^{\sigma(2m-1)} e^{e^{t}} \Delta t \right) \\
&= \lim_{k\to\infty} \sum_{m=1}^{k} \left(\int_{e^{2m-2}}^{e^{2m}} \frac{e^t}{t} dt + 2e^{e^{2m-1}} \right) \\
&= \lim_{k\to\infty} \left(\int_{1}^{e^{2k}} \frac{e^t}{t} dt + 2\left(e^e + e^{e^3} + \ldots + e^{e^{2k-1}} \right) \right) \\
&\sim \lim_{k\to\infty} \left(\frac{e^{e^{2k}}}{e^{2k}} + 2\left(e^e + e^{e^3} + \ldots + e^{e^{2k-1}} \right) \right) = \infty,
\end{aligned}
$$

imply $\varrho \in \mathfrak{U}_\infty$. *Consider* $f(t) = c^{t/2} sin^2(2\pi t)$. *Clearly* $f \in \mathcal{P}_{2,c}(\mathbb{T}, \mathbb{B})$. *Moreover, since*

$$
\begin{aligned}
\int_{-2k}^{2k} \sin^2(2\pi t) e^{e^{|t|}} \Delta t &= \sum_{m=1}^{k} \Big(\int_{-2m}^{-2m+1} \sin^2(2\pi t) e^{e^{-t}} dt + \int_{2m-2}^{2m-1} \sin^2(2\pi t) e^{e^{t}} dt \Big) \\
&\sim 8\pi^2 \left(\frac{e^{e^{2k}}}{e^{6k}} + \frac{e^{e^{2k-1}}}{e^{6k-3}} \right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, 0)} \int_{-l}^{l} \|f(t)\|_{\omega,c} \varrho(t) \Delta t &= \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, 0)} \int_{-l}^{l} \|c^{t/2} \sin^2(2\pi t)\|_{\omega,c} e^{e^{|t|}} \Delta t \\
&= \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, 0)} \int_{-l}^{l} |\sin^2(2\pi t)| e^{e^{|t|}} \Delta t \; = 0,
\end{aligned}
$$

that is, $f(t) \in \mathcal{P}_{2,c}(\mathbb{T}, \mathbb{B}) \bigcap \mathcal{WE}^{\varrho}_{2,c}(\mathbb{T}, \mathbb{B})$.

It is clear from Example 5.2.7 that the decomposition of a weighted pseudo (ω, c)-periodic function as a sum of (ω, c)-periodic function and weighted (ω, c)-ergodic function is not necessarily unique. However, the uniqueness of such a decomposition relies on the translation invariance of the space $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. By considering functions with respect to the (ω, c)-norm, the class of weighted pseudo (ω, c)-periodic functions can be interpreted as weighted pseudo periodic functions, allowing classical arguments to extend naturally to this framework and leading to the following result.

Theorem 5.2.8. *Suppose that $\varrho, \nu \in \mathfrak{U}_\infty$ and for any $\tau \in \Pi_{\mathbb{T}_\omega}$, $\limsup\limits_{|l|\to\infty} \dfrac{\varrho(l+\tau)}{\varrho(l)}$, $\limsup\limits_{|l|\to\infty} \dfrac{\nu(l+\tau)}{\nu(l)} < \infty$, then $\limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\varrho, l+\tau, t_0)}{\mathfrak{m}(\varrho, l, t_0)}$, $\limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\nu, l+\tau, t_0)}{\mathfrak{m}(\nu, l, t_0)} < \infty$, and the space $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ is translation invariant.*

Proof. The first part of the theorem is obvious for $\tau < 0$. We need only to prove this for $\tau > 0$. Note that $\limsup\limits_{|l|\to\infty} \dfrac{\varrho(l+\tau)}{\varrho(l)} > 0$; otherwise, $\limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\varrho, l-\tau, t_0)}{\mathfrak{m}(\varrho, l, t_0)} = \limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\varrho, l, t_0)}{\mathfrak{m}(\varrho, l+\tau, t_0)} = \infty$. Thus, there exists a $l_0 > t_0$ such that for any $|t| > l_0$, $\frac{\varrho(t+\tau)}{\varrho(t)} < 2\limsup\limits_{|l|\to\infty} \dfrac{\varrho(l+\tau)}{\varrho(l)}$ and $\frac{\nu(t+\tau)}{\nu(t)} < 2\limsup\limits_{|l|\to\infty} \dfrac{\nu(l+\tau)}{\nu(l)}$. Now, for $l > l_0 + \tau + t_0$, we can have

$$
\begin{aligned}
\mathfrak{m}(\varrho, l+\tau, t_0) &= \int_{t_0-l-\tau}^{t_0+l+\tau} \varrho(t)\Delta t = \int_{t_0-l}^{t_0+l} \varrho(t)\Delta t + \int_{t_0-l-\tau}^{t_0-l} \varrho(t)\Delta t + \int_{t_0+l}^{t_0+l+\tau} \varrho(t)\Delta t \\
&= \int_{t_0-l}^{t_0+l} \varrho(t)\Delta t + \int_{t_0-l}^{t_0-l+\tau} \varrho(t-\tau)\Delta t + \int_{t_0+l-\tau}^{t_0+l} \varrho(t+\tau)\Delta t \\
&< \int_{t_0-l}^{t_0+l} \varrho(t)\Delta t + 2\limsup_{|l|\to\infty} \frac{\varrho(l-\tau)}{\varrho(l)} \int_{t_0-l}^{t_0-l+\tau} \varrho(t)\Delta t + 2\limsup_{|l|\to\infty} \frac{\varrho(l+\tau)}{\varrho(l)} \int_{t_0+l-\tau}^{t_0+l} \varrho(t)\Delta t \\
&\le \left(1 + 2\limsup_{|l|\to\infty} \frac{\varrho(l+\tau)}{\varrho(l)}\right) \mathfrak{m}(\varrho, l, t_0).
\end{aligned}
$$

Thus, $\limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\varrho, l+\tau, t_0)}{\mathfrak{m}(\varrho, l, t_0)} \le 1 + 2\limsup\limits_{|l|\to\infty} \dfrac{\varrho(l+\tau)}{\varrho(l)} < \infty$, and similarly $\limsup\limits_{|l|\to\infty} \dfrac{\mathfrak{m}(\nu, l+\tau, t_0)}{\mathfrak{m}(\nu, l, t_0)} \le 1 + 2\limsup\limits_{|l|\to\infty} \dfrac{\nu(l+\tau)}{\nu(l)} < \infty$.

Next, we prove that, under the given conditions, $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ is translation invariant. Suppose $f \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, that is, for any given $\varepsilon > 0$, there exists $l_1 > t_0$ such that for any l with $|l| > l_1$, we have

$$\frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f(t)\|_{\omega,c}\nu(t)\Delta t < \varepsilon.$$

We can choose $l_1 > l_0$ so that when $|t| > l_1+\tau+t_0$, $\frac{\varrho(t+\tau)}{\varrho(t)} < 2\limsup\limits_{|l|\to\infty} \dfrac{\varrho(l+\tau)}{\varrho(l)}$ and $\frac{\nu(t+\tau)}{\nu(t)} < 2\limsup\limits_{|l|\to\infty} \dfrac{\nu(l+\tau)}{\nu(l)}$. Additionally, since ν is locally integrable, there exists $l_2 = l_2(l_1, \tau) > l_1$ such that for any $l > l_2$, $\frac{\mathfrak{m}(\nu, l_1, t_0+\tau)}{\mathfrak{m}(\varrho, l, t_0)} < \varepsilon$.
If $l > l_2$, then

$$\frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f(t-\tau)\|_{\omega,c}\nu(t)\Delta t$$

$$
\begin{aligned}
&= \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l-\tau}^{t_0+l-\tau} \|f(t)\|_{\omega,c}\nu(t+\tau)\Delta t \\
&\leq \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \left(\int_{t_0-l_1}^{t_0+l_1} \|f(t)\|_{\omega,c}\nu(t+\tau)\Delta t + \int_{t_0-l-\tau}^{t_0-l_1} \|f(t)\|_{\omega,c}\nu(t+\tau)\Delta t + \int_{t_0+l_1}^{t_0+l-\tau} \|f(t)\|_{\omega,c}\nu(t+\tau)\Delta t \right) \\
&< \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \frac{\mathfrak{m}(\varrho, l+\tau, t_0)}{\mathfrak{m}(\varrho, l, t_0)} \frac{1}{\mathfrak{m}(\varrho, l+\tau, t_0)} \int_{t_0-l-\tau}^{t_0+l+\tau} \|f(t)\|_{\omega,c}\nu(t)\Delta t \\
&\quad + \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \left(\int_{t_0-l_1}^{t_0+l_1} \|f(t)\|_{\omega,c}\nu(t+\tau)\Delta t - \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \int_{t_0-l_1}^{t_0+l_1} \|f(t)\|_{\omega,c}\nu(t)\Delta t \right. \\
&\quad \left. - \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \int_{t_0+l-\tau}^{t_0+l+\tau} \|f(t)\|_{\omega,c}\nu(t)\Delta t \right) \\
&< \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \frac{\mathfrak{m}(\varrho, l+\tau, t_0)}{\mathfrak{m}(\varrho, l, t_0)} \varepsilon + \sup_{t\in\mathbb{T}} \|f(t)\|_{\omega,c} \frac{\mathfrak{m}(\nu, l_1, t_0+\tau)}{\mathfrak{m}(\varrho, l, t_0)} \\
&\quad - \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \sup_{t\in\mathbb{T}} \|f(t)\|_{\omega,c} \frac{\mathfrak{m}(\nu, l_1, t_0)}{\mathfrak{m}(\varrho, l, t_0)} - \left(2 \limsup_{|l|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \sup_{t\in\mathbb{T}} \|f(t)\|_{\omega,c} \frac{\mathfrak{m}(\nu, \tau, t_0+l)}{\mathfrak{m}(\varrho, l, t_0)} \\
&< \left(2 \limsup_{|l_1|\to\infty} \frac{\nu(l+\tau)}{\nu(l)} \right) \left(1 + 2 \limsup_{|l|\to\infty} \frac{\varrho(l+\tau)}{\varrho(l)} \right) \varepsilon + \sup_{t\in\mathbb{T}} \|f(t)\|_{\omega,c}\varepsilon.
\end{aligned}
$$

Thus, $f(\cdot - \tau) \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ whenever $f \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. The proof is complete. $\square$

Drawing from Theorem 5.2.8, here we present a collection of weights that makes the space of (single) weighted pseudo (ω, c)-periodic functions invariant under the effect of translations:

$$
\mathfrak{U}^{inv}_\infty := \left\{ \varrho \in \mathfrak{U}_\infty : \text{for all } \tau \in \Pi_{\mathbb{T}_\omega},\ \limsup_{|l|\to\infty} \frac{\varrho(l+\tau)}{\varrho(l)} < \infty \right\}.
$$

Theorem 5.2.9. *Let $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$ such that $\lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} > 0$. If $f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, then there exist unique $g \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$ and $h \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ such that $f = g + h$.*

Proof. Consider $f \not\equiv 0$ and $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}) \cap \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Then there exists a $t_1 \in \mathbb{T}_\omega$ such that $f(t_1) \neq 0$. Thus, we can have $\delta > 0$ such that $\|f(t_1)\|_{\omega,c} \geq 2\delta$. Define a set

$$
\mathcal{B}_\delta := \{\tau \in \Pi_{\mathbb{T}_\omega} : \|f(t_1+\tau) - f(t_1)\|_{\omega,c} \leq \delta\},
$$

so that for all $t \in \mathcal{B}_\delta$,

$$
\|f(t_1+t)\|_{\omega,c} \geq \|f(t_1)\|_{\omega,c} - \|f(t_1+t) - f(t_1)\|_{\omega,c} \geq \delta.
$$

Let $l \in \Pi^+_{\mathbb{T}_\omega}$. Since $f \in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$, for any $t \in [t_0 - l, t_0 + l]_{\mathbb{T}_\omega}$, there exist $l_\delta > 0$ and $\tau \in [t - l_\delta, t]_{\mathbb{T}_\omega} \cap \mathcal{B}_\delta$, such that

$$
t = \tau + (t - \tau) \in (t - \tau) + \mathcal{B}_\delta \subset \bigcup_{s\in\mathbb{T}_\omega} (s + \mathcal{B}_\delta).
$$

Additionally, we can find $s_1, \ldots, s_k \in \mathbb{T}_\omega$ such that

$$[t_0 - l, t_0 + l]_{\mathbb{T}_\omega} \subset \bigcup_{k=1}^{m} (s_k + \mathcal{B}_\delta).$$

Also, for every $t \in [t_0 - l, t_0 + l]_{\mathbb{T}_\omega}$, there exists an $i \in \{1, \ldots, m\}$ such that $t - s_i \in \mathcal{B}_\delta$, hence $\|f(t - s_i + t_1)\|_{\omega,c} \geq \delta$. Consider

$$f(t) = \|f(t + t_1)\|_{\omega,c} + \|f(t + t_1 - s_1)\|_{\omega,c} + \ldots + \|f(t + t_1 - s_m)\|_{\omega,c}.$$

Clearly, $f(t) \geq \delta$ for all $t \in [t_0 - l, t_0 + l]_{\mathbb{T}_\omega}$. This implies,

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} f(t)\nu(t)\Delta t \geq \delta \lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} > 0. \tag{5.2.1}$$

Moreover $f \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and the space $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ is translation-invariant. Consequently, $f(t + t_1), f(t + t_1 - s_k) \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ for $(k = 1, \ldots, m)$, i.e.,

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f(t + t_1)\|_{\omega,c}\nu(t)\Delta t = 0.$$

and

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f(t + t_1 - s_k)\|_{\omega,c}\nu(t)\Delta t = 0, \quad k = 1, \ldots, m,$$

which contradicts (5.2.1). Therefore, $\mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) \cap \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) = \{0\}$. The proof is complete. $\square$

As a direct result of Theorem 5.2.9, we have the following basic property of this class of functions:

Theorem 5.2.10. *Let $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$, $\lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} > 0$. Then, $(f_1 + f_2)$, $af \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ whenever $f_1, f_2, f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $a \in \mathbb{F}_\mathbb{B}$.*

Lemma 5.2.11. *If $f = f_{op} + f_\varphi \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ is a unique decomposition of f so that $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, then $\sup_{\mathbb{T}_\omega} \|f_{op}\|_{\omega,c} \leq \sup_{\mathbb{T}_\omega} \|f\|_{\omega,c}$.*

Theorem 5.2.12. *Let $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$ with $0 < \lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} < \infty$. Then, the space $\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ equipped with the norm*

$$\|f\|_{p\omega c} := \sup_{t\in\mathbb{T}_\omega} \|f(t)\|_{\omega,c} \tag{5.2.2}$$

is a Banach space.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ be such that $\lim_{n\to\infty} \|f_n - f\|_{p\omega c} = 0$. By Theorem 5.2.9, for each $n \in \mathbb{N}$, we can write $f_n = f_{op_n} + f_{\varphi_n}$, where $f_{op_n} \in \mathcal{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_{\varphi_n} \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Also, using Theorem 5.2.10, for each $m, n \in \mathbb{N}$, we can have $f_m - f_n = \left(f_{op_m} - f_{op_n}\right) + \left(f_{\varphi_m} - f_{\varphi_n}\right)$, with $\left(f_{op_m} - f_{op_n}\right) \in \mathcal{P}_{\omega,c}(\mathbb{T}, \mathbb{B})$ and $\left(f_{\varphi_m} - f_{\varphi_n}\right) \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. From Lemma 5.2.11, it follows that $\|f_{op_m} - f_{op_n}\|_{p\omega c} \leq$

$\|f_m - f_n\|_{p\omega c}$. Being a convergent sequence (f_n) is a Cauchy sequence, and hence (f_{op_n}) is Cauchy. Consequently, following Theorem 2.2.16, there exists $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ such that $\lim_{n\to\infty} \|f_{op_n} - f_{op}\|_{p\omega c} = 0$. Next, consider $f_\varphi = f - f_{op}$. It is very obvious that $\lim_{n\to\infty} \|f_{\varphi_n} - f_\varphi\|_{p\omega c} = 0$, which implies $f_\varphi \in \mathrm{BC}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$. Additionally,

$$\begin{aligned}
\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_\varphi(t)\|_{\omega,c} \nu(t)\Delta t \quad &\leq \quad \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_{\varphi_n}(t) - f_\varphi(t)\|_{\omega,c} \nu(t)\Delta t \\
&\quad + \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_{\varphi_n}(t)\|_{\omega,c} \nu(t)\Delta t \\
&\leq \quad \|f_{\varphi_n}(t) - f_\varphi(t)\|_{p\omega c} \lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} \ \to\ 0 \text{ (as } n \to \infty),
\end{aligned}$$

that is, $f_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. It follows that $f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Thus, being a closed subspace of Banach space $(\mathrm{BC}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B}), \|\cdot\|_{\mathrm{p}\omega\mathrm{c}})$, $\left(\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}), \|\cdot\|_{\mathrm{p}\omega\mathrm{c}}\right)$ is a Banach space. The theorem is proved. $\square$

Theorem 5.2.13. *Let $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$, $\lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} > 0$. Then, $f_1 \cdot f_2 \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$ whenever $f_1 \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c_1}(\mathbb{T}_\omega, \mathbb{B})$ and $f_2 \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c_2}(\mathbb{T}_\omega, \mathbb{B})$, provided that $\mathbb{B}$ is a Banach algebra.*

Proof. We can write $f_1 = f_{1op} + f_{1\phi}$ and $f_2 = f_{2op} + f_{2\phi}$, where $f_{1op} \in \mathcal{P}_{\omega,c_1}(\mathbb{T}_\omega, \mathbb{B})$, $f_{2op} \in \mathcal{P}_{\omega,c_2}(\mathbb{T}_\omega, \mathbb{B})$, $f_{1\phi} \in \mathcal{WE}^{\varrho,\nu}_{\omega,c_1}(\mathbb{T}_\omega, \mathbb{B})$, and $f_{2\phi} \in \mathcal{WE}^{\varrho,\nu}_{\omega,c_2}(\mathbb{T}_\omega, \mathbb{B})$. Let,

$$f := f_1 \cdot f_2 = f_{op} + f_\varphi,$$

where $f_{op} = f_{1op} \cdot f_{2op}$ and $f_\varphi = f_{1\phi} \cdot f_{2op} + f_{1op} \cdot f_{2\phi} + f_{1\phi} \cdot f_{2\phi}$. Obviously, for all $t \in \mathbb{T}_\omega$ we have

$$f_{op}(t + \omega) = c_1 \cdot c_2 f_{op}(t).$$

Also, for periodic functions f_{1p} and f_{2p} associated with f_{1op} and f_{2op}, respectively, we can write

$$\begin{aligned}
\|f_\varphi\|_{\omega,c} &= \|f_{2p}(t)c_1^{-t/\omega} f_{1\phi}(t) + f_{1p}(t)c_2^{-t/\omega} f_{2\phi}(t) + c_1^{-t/\omega} f_{1\phi}(t)c_2^{-t/\omega} f_{2\phi}(t)\|_{\mathbb{B}} \\
&\leq \|f_{2p}\|_{\omega,1}\|f_{1\phi}\|_{\omega,c_1} + \|f_{1p}\|_{\omega,1}\|f_{2\phi}\|_{\omega,c_2} + \|f_{1\phi}\|_{\omega,c_1}\|f_{2\phi}\|_{\omega,c_2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_\varphi(t)\|_{\omega,c} \nu(t)\Delta t \quad &\leq \quad \sup_{\mathbb{T}_\omega} \|f_{2p}\|_{\omega,1} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_{1\phi}\|_{\omega,c_1} \nu(t)\Delta t \\
&\quad + \sup_{\mathbb{T}} \|f_{1p}\|_{\omega,1} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_{2\phi}\|_{\omega,c_2} \nu(t)\Delta t \\
&\quad + \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|f_{1\phi}\|_{\omega,c_1}\|f_{2\phi}\|_{\omega,c_2} \nu(t)\Delta t \\
&= \quad 0.
\end{aligned}$$

Hence, $f_1 \cdot f_2 \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c_1\cdot c_2}(\mathbb{T}_\omega, \mathbb{B})$. $\square$

The next theorem shows the characterization of the translation invariant linear space $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.

Theorem 5.2.14. *Let* $\varphi \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B})$*,* $l \in \Pi_{\mathbb{T}_\omega}$ *and* $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$*, with* $0 < \lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} < \infty$*. Then* $\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$*, if and only if, for every* $\varepsilon > 0$*,*

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{\mathcal{F}(\varepsilon,l,\varphi)} \nu(t)\Delta t = 0, \tag{5.2.3}$$

where $\mathcal{F}(\varepsilon, l, \varphi) := \{t \in [t_0 - l, t_0 + l]_{\mathbb{T}} : \|\varphi(t)\|_{\omega,c} \geq \varepsilon\}$*.*

Proof. Let $\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. If possible, consider that there exists a positive ε_0 such that

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{\mathcal{F}(\varepsilon_0,l,\varphi)} \nu(t)\Delta t \neq 0.$$

This implies that there exists $\delta > 0$ such that for each $n \in \mathbb{N}$,

$$\frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{\mathcal{F}(\varepsilon_0,l,\varphi)} \nu(t)\Delta t \geq \delta,$$

for some l_n $(\in \Pi_{\mathbb{T}_\omega}) > N$. So we get

$$\begin{aligned}\frac{1}{\mathfrak{m}(\varrho, l_n, t_0)} \int_{t_0-l_n}^{t_0+l_n} \|\varphi(s)\|_{\omega,c}\nu(s)\Delta s &\geq \frac{\varepsilon_0}{\mathfrak{m}(\varrho, l_n, t_0)} \int_{\mathcal{F}(\epsilon_0,l_n,\varphi)} \nu(s)\Delta s \\ &\quad + \frac{1}{\mathfrak{m}(\varrho, l_n, t_0)} \int_{[t_0-l,t_0+l]_{\mathbb{T}}\setminus\mathcal{F}(\varepsilon,l,\varphi)} \|\varphi(s)\|_{\omega,c}\nu(s)\Delta s \\ &\geq \varepsilon_0\delta,\end{aligned}$$

which contradicts $\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Thus, (5.2.3) holds.

Conversely, let us consider that (5.2.3) holds. This implies that for any given $\varepsilon > 0$, there exists a positive l_0 such that, for $l > l_0$,

$$\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{\mathcal{F}(\varepsilon,l,\varphi)} \nu(t)\Delta t < \varepsilon.$$

Then we have

$$\begin{aligned}\frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|\varphi(t)\|_{\omega,c}\nu(t)\Delta t &= \varepsilon \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{[t_0-l,t_0+l]_{\mathbb{T}}\setminus\mathcal{F}(\varepsilon,l,\varphi)} \nu(t)\Delta t \\ &\quad + \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{\mathcal{F}(\varepsilon,l,\varphi)} \|\varphi(t)\|_{\omega,c}\nu(t)\Delta t \\ &\leq \varepsilon \lim_{l\to\infty} \frac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} + \|\varphi\|_{p\omega c}\, \varepsilon,\end{aligned}$$

i.e., $\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Hence, the proof is complete. □

Corollary 5.2.15. *Let* $\mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ *be the set of* (ω, c)*-asymptotic functions as defined in Chapter 4, i.e.,*

$$\mathcal{A}_{\omega,c} := \{f \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega, \mathbb{B}) : \lim_{t\to\infty} \|f\|_{\omega,c} = 0\}.$$

Then, $f \in \mathcal{A}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ *implies* $f \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$*.*

Example 5.2.16. *Consider $\mathbb{T} = \mathbb{Z}$, and $\varrho = \nu = 1$. Then, the function defined as*

$$f(t) = \begin{cases} c^{t/\omega} & t = \pm n^2, \quad n \in \mathbb{N}, \\ 0 & \textit{otherwise}, \end{cases}$$

is in $\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{Z}, \mathbb{B})$, but does not belong to the space $\mathcal{A}_{\omega,c}(\mathbb{Z}, \mathbb{B})$.

Remark 5.2.17. *From Corollary 5.2.15 we can conclude that an (ω, c)-asymptotically periodic function is also pseudo (ω, c)-periodic, but the converse does not always follow. Thus, the results in this chapter generalize the results of Chapter 4.*

Next, for $\mathcal{S} \subset \mathbb{B}$, we define the set

$$\mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B}) := \{f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B}) | f = f_{op} + \varphi \text{ with } f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega \times \Omega, \mathbb{B}) \text{ and } \varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B})\},$$

where

$$\mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B}) := \{\varphi \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B}) | \lim_{\mathrm{l}\to\infty} \frac{1}{\mathfrak{m}(\varrho, \mathrm{l}, \mathrm{t}_0)} \int_{\mathrm{t}_0-1}^{\mathrm{t}_0+1} \|\varphi(\mathrm{t}, \mathrm{x})\|_{\omega,\mathrm{c}} \nu(\mathrm{t}) \Delta \mathrm{t} = 0\},$$

for all x in any compact subset of $\mathcal{S}$, where $l \in \Pi_{\mathbb{T}_\omega}$ and

$$(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B}) = \{f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega \times \mathcal{S}, \mathbb{B}) : \sup_{t \in \mathbb{T}_\omega} \|f(t, x)\|_{\omega,c} < \infty \text{ for all } x \text{ in any compact subset of } \mathcal{S}\}.$$

Theorem 5.2.18 (Composition Result)**.** *Assume that $\varrho, \nu \in \mathfrak{U}^{inv}_\infty$, $0 < \lim_{l\to\infty} \dfrac{\mathfrak{m}(\nu, l, t_0)}{\mathfrak{m}(\varrho, l, t_0)} = k < \infty$. Suppose further that*

1. *$f = f_{op} + f_\varphi$ such that $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega \times \mathbb{B}, \mathbb{B})$ and for z in any bounded subset of $\mathbb{B}$, $\tilde{f}_\varphi(t, z) \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ uniformly in $t \in \mathbb{T}_\omega$, where $\tilde{f}_\varphi(t, z) := f_\varphi(t, c^{t/\omega} z)$.*
2. *f and f_{op} satisfy the Lipschitz condition w.r.t. norm $\|\cdot\|_{\mathbb{B}}$ in the second variable with Lipschitz constants L_f and $L_{f_{op}}$ respectively.*

Then, $f(\cdot, x(\cdot)) \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega \times \mathbb{T}_\omega, \mathbb{B})$ provided $x = x_{op} + x_\varphi$ with $x_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $x_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$.

Proof. We can write the decomposition of the function f as

$$f(t, x(t)) = f(t, x(t)) + f_{op}(t, x_{op}(t)) + f_\varphi(t, x_{op}(t)) - f(t, x_{op}(t)).$$

Let us denote

$$\mathcal{J}_1(t) = f_{op}(t, x_{op}(t)), \quad \mathcal{J}_2(t) = f(t, x(t)) - f(t, x_{op}(t)), \quad \text{and} \quad \mathcal{J}_3(t) = f_\varphi(t, x_{op}(t)).$$

By Theorem 2.2.17, we have $\mathcal{J}_1(\cdot)\in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega\times\mathbb{T}_\omega,\mathbb{B})$. We only need to show that $\mathcal{J}_2(\cdot)$, $\mathcal{J}_3(\cdot)\in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega\times\mathbb{T}_\omega,\mathbb{B})$. As we have

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|\mathcal{J}_2(t)\|_{\omega,c}\nu(t)\Delta t &= \lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|f(t,x(t))-f(t,x_{op}(t))\|_{\omega,c}\nu(t)\Delta t\\ &\le \lim_{l\to\infty}\frac{L_f}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|x_\varphi(t)\|_{\omega,c}\nu(t)\Delta\\ &= 0,\end{aligned}$$

we only need to show that $\lim\limits_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|\mathcal{J}_3(t)\|_{\omega,c}\nu(t)\Delta t=0$. Notably, for any $\varepsilon>0$, there exists a $l_0>0$ such that for any $l>l_0$, we have

$$\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|f_\varphi(t,c^{t/\omega}x)\|_{\omega,c}\nu(t)\Delta t<\varepsilon \ \text{ for all } \ x\in\mathcal{B}, \text{ any bounded set of } \mathbb{B}.$$

Since $\mathcal{H}:=\{c^{-t/\omega}x_{op}(t):t\in\mathbb{T}_\omega\}$ consists of values of a periodic function taking values in the Banach space $\mathbb{B}$, it is relatively compact. Thus, we can have finitely many open balls $\mathcal{C}_i,(i=1,2,...,m)$ in $\mathbb{B}$ with centers $x_{p_i}\in\mathcal{H}$ and radii ε (small enough) such that $\mathcal{H}\subset\bigcup\limits_{i=1}^{m}\mathcal{C}_i$. Set $\mathcal{I}_i=\{t\in\mathbb{T}:c^{-t/\omega}x_{op}(t)\in\mathcal{C}_i\}$, then $\mathbb{T}=\bigcup\limits_{i=1}^{m}\mathcal{I}_i$. Let $\mathcal{I}_1'=\mathcal{I}_1$, and $\mathcal{I}_i'=\mathcal{I}_i\backslash\bigcup\limits_{j=1}^{i-1}\mathcal{I}_j, 2\le m$. Note that when $i\neq j$, $\mathcal{I}_i'\cap\mathcal{I}_j'=\emptyset$. Now, for any $l>l_0$, we have

$$\begin{aligned}&\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|f_\varphi(t,x_{op}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\quad=\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,x_{op}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\quad\le\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,x_{op}(t))-f_\varphi(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\qquad+\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta\\
&\quad\le\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f(t,x_{op}(t))-f(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\qquad+\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_{op}(t,x_{op}(t))-f_{op}(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\qquad+\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\\
&\quad\le\lim_{l\to\infty}\frac{L_f}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|x_{op}(t)-c^{t/\omega}x_{p_i}(t)\|_{\omega,c}\nu(t)\Delta t\\
&\qquad+\lim_{l\to\infty}\frac{L_{f_{op}}}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|x_{op}(t)-c^{t/\omega}x_{p_i}(t)\|_{\omega,c}\nu(t)\Delta t\\
&\qquad+\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{l\to\infty}\frac{L_f}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|c^{-t/\omega}x_{op}(t)-x_{p_i}(t)\|_{\mathbb{B}}\nu(t)\Delta t\\
&\quad+\lim_{l\to\infty}\frac{L_{f_{op}}}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|c^{-t/\omega}x_{op}(t)-x_{p_i}(t)\|_{\mathbb{B}}\nu(t)\Delta t\\
&\quad+\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\sum_{i=1}^{m}\int_{\mathcal{I}_i'\cap[t_0-l,t_0+l]_{\mathbb{T}}}\|f_\varphi(t,c^{t/\omega}x_{p_i}(t))\|_{\omega,c}\nu(t)\Delta t\\
&<\lim_{l\to\infty}\frac{\mathfrak{m}(\nu,l,t_0)}{\mathfrak{m}(\varrho,l,t_0)}(L_f+L_{f_{op}})\varepsilon+m\varepsilon\\
&\leq (k(L_f+L_{f_{op}})+m)\varepsilon.
\end{aligned}
$$

Hence,

$$
\lim_{l\to\infty}\frac{1}{\mathfrak{m}(\varrho,l,t_0)}\int_{t_0-l}^{t_0+l}\|f_\varphi(t,x_{op}(t))\|_{\omega,c}\nu(t)\Delta t=0.
$$

This completes the proof. □

5.3 Model description

We consider a piecewise impulsive neutral-type evolution system, defined on an ω-periodic time scale $\mathbb{T}_\omega$, given as

$$
\begin{aligned}
(x(t)+\mathsf{b}(t)x(t-\gamma(t)))^{\Delta} &= -\mathsf{a}(t)x(t)+f(t,x(t-\tau(t)))+\mathsf{e}(t), \quad t\geq t_0,\ t\neq t_j \\
x(t_j{}^-)-x(t_j{}^+) &= I_j(x(t_j)), \quad t=t_j,\ j=1,2,\ldots,q,
\end{aligned}
\tag{5.3.1}
$$

where the coefficients are subject to the following condition:

H1: Assume that there exist $c\in\mathbb{F}_{\mathbb{B}}\setminus\{0\}$, $\varrho,\nu\in\mathfrak{U}_\infty^{inv}$ with $0<\lim\limits_{l\to\infty}\dfrac{\mathfrak{m}(\nu,l,t_0)}{\mathfrak{m}(\varrho,l,t_0)}=k<\infty$, where $l\in\Pi_{\mathbb{T}_\omega}$, so that

(a) $\mathsf{a}\in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega,\mathbb{R}^+)$, $\gamma,\tau\in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega,\Pi^{0+}_{\mathbb{T}_\omega})$, $\mathsf{b}\in \mathrm{WPP}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega,\mathbb{B})$ with $|\mathsf{b}(t)|\neq 1$ for any $t\in\mathbb{T}_\omega$, and $\mathsf{e}\in \mathrm{WPP}^{\varrho,\nu}_{\omega,\mathrm{c}}(\mathbb{T}_\omega,\mathbb{B})$.

(b) $f=f_{op}+f_\varphi$ such that $f_{op}\in \mathrm{P}_{\omega,\mathrm{c}}(\mathbb{T}_\omega\times\mathbb{B},\mathbb{B})$, and for every x in any bounded subset of $\mathbb{B}$, $f_\varphi(t,c^{t/\omega}x)\in \mathrm{WPP}^{\varrho,\nu}_{\omega,\mathrm{c}}(\mathbb{B})$ uniformly in $t\in\mathbb{T}_\omega$.

Let the initial condition of (5.3.1) is given as

$$
x(s)=\phi(s), \quad s\in[-\theta,t_0]_{\mathbb{T}_\omega},
$$

where $\phi(s)\in \mathrm{C}_{\mathrm{rd}}([-\theta,t_0]_{\mathbb{T}_\omega},\mathbb{B})$, $\theta=\max\{\bar\gamma,\bar\tau\}$ with $\bar\gamma=\sup\limits_{t\in\mathbb{T}_\omega}\gamma(t)$ and $\bar\tau=\sup\limits_{t\in\mathbb{T}_\omega}\tau(t)$.

Consider the sets

$$\mathrm{PC_{rd}}(\mathbb{T}_\omega, \mathbb{B}) = \{x(t) : x(t) \in \mathrm{C_{rd}}((\mathrm{t_j}, \mathrm{t_{j+1}})_{\mathbb{T}_\omega}, \mathbb{B}),\ \mathrm{j} = 1, 2, \ldots, \mathrm{q}\},$$

and

$$\mathbb{E} = \{x \in \mathrm{PC_{rd}}(\mathbb{T}_\omega, \mathbb{B}) : x \in \mathrm{WPP}^{\varrho,\nu}_{\omega,\mathrm{c}}(\mathbb{T}_\omega, \mathbb{B}),\ \text{where } \varrho, \nu \in \mathfrak{U}^{\mathrm{inv}}_\infty, 0 < \lim_{|\mathrm{l}| \to \infty} \frac{\mathfrak{m}(\nu, \mathrm{l}, \mathrm{t_0})}{\mathfrak{m}(\varrho, \mathrm{l}, \mathrm{t_0})} < \infty\}.$$

Clearly, the space $\mathbb{E}$ equipped the norm $\|x\|_{p\omega c}$ forms a Banach space. Define an operator on $\mathbb{E}$ as

$$(\mathcal{F}x)(t) = x(t) + \mathtt{b}(t)x(t - \gamma(t)). \tag{5.3.2}$$

Lemma 5.3.1. *Assume that $|\mathtt{b}(t)| \neq 1$. Then $\mathcal{F}$ has a bounded inverse $\mathcal{F}^{-1}$ on $\mathbb{E}$. In fact, for all $x \in \mathbb{E}$, we have*

(a).

$$\mathcal{F}^{-1}x(t) = \begin{cases} x(t) + \sum\limits_{j>0} \prod\limits_{0 \le i \le j-1} (-1)^j \mathtt{b}(D_i) x\left(t - \sum\limits_{i=0}^{j-1} \gamma(D_i)\right), & \textit{for } |\mathtt{b}(t)| < 1 \\ -\dfrac{x(t+\gamma)}{\mathtt{b}(t+\gamma)} - \sum\limits_{j>0} \dfrac{x(t + \gamma(t) + \sum\limits_{i=0}^{j-1} \gamma(D_i'))}{\mathtt{b}(t+\gamma(t)) \prod\limits_{i=0}^{j-1} \mathtt{b}(D_i')}, & \textit{for } |\mathtt{b}(t)| > 1, \end{cases}$$

where $D_j := t - \sum\limits_{i=0}^{j-1} \gamma(D_i)$ and $D_j' := t + \sum_{i=0}^{j-1} \gamma(D_i)$ with $D_0 = t = D_0'$.

(b).

$$\|\mathcal{F}^{-1}x\| \le \begin{cases} \dfrac{\|x\|}{1 - \overline{\mathtt{b}}}, & \textit{for } \overline{\mathtt{b}} < 1 \\ \dfrac{\|x\|}{\underline{\mathtt{b}} - 1}, & \textit{for } \underline{\mathtt{b}} > 1, \end{cases}$$

where $\overline{\mathtt{b}} = \sup\limits_{t \in \mathbb{T}_\omega} |\mathtt{b}(t)|$ and $\underline{\mathtt{b}} = \inf\limits_{t \in \mathbb{T}_\omega} |\mathtt{b}(t)|$.

Obviously, both the operators $\mathcal{F}^{-1}$ and $\mathcal{F}$ are linear. Let

$$(\mathcal{F}x)(t) = u(t).$$

Then, we can use $\mathcal{F}^{-1}$ to change (5.3.1) into the following form

$$\begin{aligned} u^\Delta(t) &= -\mathtt{a}(t)(\mathcal{F}^{-1}u)(t) + f(t, (\mathcal{F}^{-1}u)(t - \tau(t))) + \mathtt{e}(t), \quad t \ge t_0,\ t \ne t_j \\ (\mathcal{F}^{-1}u)(t_j{}^-) - (\mathcal{F}^{-1}u)(t_j{}^+) &= I_j((\mathcal{F}^{-1}u)(t_j)), \quad t = t_j,\ j = 1, 2, \ldots, q. \end{aligned} \tag{5.3.3}$$

Using $(\mathcal{F}^{-1}u)(t) = u(t) - \mathtt{b}(t)(\mathcal{F}^{-1}u)(t-\gamma)$, (5.3.3) can be changed into

$$\begin{aligned} u^{\Delta}(t) &= -\mathtt{a}(t)u(t) + \mathtt{a}(t)\mathtt{b}(t)(\mathcal{F}^{-1}u)(t-\gamma) + f(t, (\mathcal{F}^{-1}u)(t-\tau(t))) + \mathtt{e}(t), \quad t \geq t_0, \ t \neq t_j \\ u(t_j{}^{-}) - u(t_j{}^{+}) &= (\mathcal{F}I_j\mathcal{F}^{-1})u(t_j), \quad t = t_j, \ j = 1, 2, \ldots, q. \end{aligned} \tag{5.3.4}$$

Since system (5.3.1) is equivalent to system (5.3.4), we will investigate the weighted pseudo (ω, c)-periodic solutions for system (5.3.4).

5.4 Weighted pseudo (ω, c)-periodic solution

In this section, we examine the existence and uniqueness of weighted pseudo (ω, c)-periodic solution for a piecewise impulsive neutral-type dynamic equation on time scales, given by (5.3.4), and obtain conditions so the system becomes exponentially synchronized.

5.4.1 Existence

We first study the existence of a unique weighted pseudo (ω, c)-periodic solution for the system (5.3.1) when the coefficients are according to the description in **H1**. We employ the Banach fixed point theorem and some inequalities of time scale calculus to prove the result.

Before we state our main theorem for the existence of the solution of (5.3.4), we present some additional results that will be required to prove the main result.

Lemma 5.4.1. *Let $u \in \mathbb{E}$ be a solution of system (5.3.4), then it satisfies*

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} e_{-\mathtt{a}}(t, \sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s \\ &\quad - \sum_{j=1,\ t_j \in [t_0, t)}^{q} e_{-\mathtt{a}}(t, t_j)(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j). \end{aligned} \tag{5.4.1}$$

Proof. Let $u(t)$ be a solution of (5.3.4). Then, for any $t_0 \in \mathbb{T}_\omega$, there exists $j \in \mathbb{Z}$ such that t_j is the first impulsive point after t_0. Using Theorem 1.4.3, for $t \in [t_0, t_j]_{\mathbb{T}_\omega}$, we have

$$u(t) = e_{-\mathtt{a}}(t, t_0)u(t_0) + \int_{t_0}^{t} e_{-\mathtt{a}}(t, \sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s.$$

Thus,

$$u(t_j) = e_{-\mathtt{a}}(t_j, t_0)u(t_0) + \int_{t_0}^{t_j} e_{-\mathtt{a}}(t_j, \sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s.$$

Next, using Theorem 1.4.3 for $t \in (t_j, t_{j+1}]_{\mathbb{T}_\omega}$, we get

$$\begin{aligned} u(t) &= e_{-\mathtt{a}}(t,t_j)u(t_j^+) + \int_{t_j}^{t} e_{-\mathtt{a}}(t,\sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s,(\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s \\ &= e_{-\mathtt{a}}(t,t_0)u(t_0) - e_{-\mathtt{a}}(t,t_j)(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j) + \int_{t_0}^{t} e_{-\mathtt{a}}(t,\sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) \\ &\qquad + f(s,(\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s. \end{aligned}$$

Continuing the above process, for $t \in [t_0, \infty)_{\mathbb{T}_\omega}$ we obtain

$$\begin{aligned} u(t) = e_{-\mathtt{a}}(t,t_0)u(t_0) - \sum_{j=1,\ t_j\in[t_0,t)}^{q} e_{-\mathtt{a}}(t,t_j)(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j) + \int_{t_0}^{t} e_{-\mathtt{a}}(t,\sigma(s)) \\ \Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s,(\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s. \end{aligned}$$

Now, let us consider

$$\begin{aligned} v(t) &:= \int_{-\infty}^{t} e_{-\mathtt{a}}(t,\sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s,(\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\Big)\Delta s \\ &\quad - \sum_{j=1,t_j\in[t_0,t)}^{q} e_{-\mathtt{a}}(t,t_j)(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j), \quad \text{for} \ \ t > t_0 \end{aligned}$$

and $v(t_0) = u(t_0)$. Since, for every rd-continuous function f, we have

$$\int_{t_0}^{t} e_{-\mathtt{a}}(t,\sigma(s))f(s)\Delta s = \int_{-\infty}^{t} e_{-\mathtt{a}}(t,\sigma(s))f(s)\Delta s - \int_{-\infty}^{t_0} e_{-\mathtt{a}}(t,\sigma(s))f(s)\Delta s,$$

follows that $v(t) \equiv u(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}_\omega}$. This completes our proof. □

Lemma 5.4.2. *Let $\varrho, \nu \in \mathfrak{U}_\infty^{inv}$, $0 \le \lim\limits_{l\to\infty} \dfrac{\mathfrak{m}(\nu,l,t_0)}{\mathfrak{m}(\varrho,l,t_0)} < \infty$, and $\tau \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega, \Pi_{\mathbb{T}_\omega}^{+})$ be such that $0 < \tau^{\Delta}(t) < 1$ for all $t \in \mathbb{T}_\omega$. If $f \in \mathrm{WPP}_{\omega,c}^{\varrho,\nu}(\mathbb{T}_\omega, \mathbb{B})$, then $f(\cdot - \tau(\cdot)) \in \mathrm{WPP}_{\omega,c}^{\varrho,\nu}(\mathbb{T}_\omega, \mathbb{B})$.*

Proof. Let $f = f_{op} + f_\varphi$, where $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$ and $f_\varphi \in \mathcal{WE}_{\omega,c}^{\varrho,\nu}(\mathbb{T}_\omega, \mathbb{B})$. Then

$$f(t-\tau(t)) = f_{op}(t-\tau(t)) + f_\varphi(t-\tau(t)) := f_1(t) + f_2(t).$$

As $\tau \in \Pi_{\mathbb{T}_\omega}$, it follows that $t - \tau(t) \in \mathbb{T}_\omega$ for all $t \in \mathbb{T}_\omega$, and we have

$$f_1(t+\omega) = f_{op}(t+\omega-\tau(t+\omega)) = f_{op}(t-\tau(t)+\omega) = cf_{op}(t-\tau(t)) = cf_1(t),$$

that is, $f_1 \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$. Let $\mathfrak{B} = \left(\sup\limits_{t\in\mathbb{T}_\omega} |c|^{-\tau(t)/\omega}\right)\left(\sup\limits_{t\in\mathbb{T}} \dfrac{1}{(1-\tau^{\Delta}(t))}\right)\left(1 + 2\limsup\limits_{|t|\to\infty} \dfrac{\nu(t+\tau(t))}{\nu(t)}\right)$. By assumptions taken on ϱ, ν and τ, we have $\mathfrak{B} < \infty$ and

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \|f_2(t)\|_{\omega,c}\nu(t)\Delta t \\ &\quad = \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \|c^{-t/\omega}f(t-\tau(t))\|_{\mathbb{B}}\nu(t)\Delta t \end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} |c|^{-\tau(t)/\omega} \|c^{-(t-\tau(t))/\omega} f(t-\tau(t))\|_{\mathbb{B}} \nu(t)\Delta t \\
&\leq \mathfrak{B} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l-\tau(t_0-l)}^{t_0+l-\tau(t_0+l)} \|f(t)\|_{\omega,c} \nu(t)\Delta t \\
&\leq \mathfrak{B} \limsup_{l\to\infty} \frac{\mathfrak{m}(\varrho,l+\bar{\tau},t_0)}{\mathfrak{m}(\varrho,l,t_0)} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l+\bar{\tau},t_0)} \int_{t_0-l-\bar{\tau}}^{t_0+l+\bar{\tau}} \|f(t)\|_{\omega,c} \nu(t)\Delta t \\
&= 0,
\end{aligned}
$$

which implies that $f_2 \in \mathrm{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. Hence, we get $f(\cdot-\tau(\cdot)) \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. □

Theorem 5.4.3. *Consider $\mathbb{T}_\omega$ to be an ω-periodic time scale. Suppose $\alpha \in \mathrm{P}_{\omega,1}(\mathbb{T}_\omega,\mathbb{R}^+)$ is regressive and $f \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$, where $\varrho,\nu \in \mathfrak{U}^{inv}_\infty$ and $0 < \lim_{l\to\infty} \dfrac{\mathfrak{m}(\nu,l,t_0)}{\mathfrak{m}(\varrho,l,t_0)} < \infty$. Then $\mathcal{Z}(t) := \int_{-\infty}^t e_{-\alpha(s)}(t,\sigma(s))f(s) \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ provided $0 < \underline{\alpha} < \frac{1}{\bar{\mu}}$ and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{\alpha}\bar{\mu})}$.*

Proof. By definition, we can decompose f into $f = f_{op} + f_\varphi$, where $f_{op} \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ and $f_\varphi \in \mathcal{WE}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. Thus, we have

$$
\mathcal{Z}(t) = \int_{-\infty}^t e_{-\alpha}(t,\sigma(s))f(s)\Delta s := \mathcal{I}_1(t) + \mathcal{I}_2(t),
$$

where

$$
\mathcal{I}_1(t) = \int_{-\infty}^t e_{-\alpha}(t,\sigma(s))f_{op}(s)\Delta s, \quad \text{and}
$$

$$
\mathcal{I}_2(t) = \int_{-\infty}^t e_{-\alpha}(t,\sigma(s))f_\varphi(s)\Delta s.
$$

By Theorem 2.3.5, we can deduce that $\mathcal{I}_1 \in \mathrm{P}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$. It only remains to prove $\mathcal{I}_2 \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$.

Note that $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})} < |c| \leq e^{-\underline{\alpha}\omega}$ implies $\frac{\ln|c|}{\omega} + \underline{\alpha} < 0$ and $\frac{\ln|c|}{\omega} + \underline{\alpha} \in \mathcal{R}^+$. Thus, Theorem 1.4.29 can be applied to get the inequality

$$
e^{-\left(\frac{\ln|c|}{\omega}(t-s)\right)(t-s)} \leq e_{\ominus\left(\frac{\ln|c|}{\omega}\right)}(t,s) \quad \text{for all} \quad t,s \in \mathbb{T}_\omega, \quad t > s. \tag{5.4.2}
$$

Similarly for $e^{-\underline{\alpha}\omega} \leq |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{\alpha}\bar{\mu})}$, i.e., when $\frac{\ln|c|}{\omega} + \underline{\alpha} > 0$ and $-\left(\frac{\ln|c|}{\omega} + \underline{\alpha}\right) \in \mathcal{R}^+$, (5.4.2) follow. Consider a number $\tau > 0$ such that $t - s - \tau \in \mathbb{T}_\omega$ for all $s,t \in \mathbb{T}_\omega$. Then

$$
\begin{aligned}
&\lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \|\mathcal{I}_2(t)\|_{\omega,c} \nu(t)\Delta t \\
&\quad= \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \left\| \int_{-\infty}^t e_{-\alpha}(t,\sigma(s))f_\varphi(s)\Delta s \right\|_{\omega,c} \nu(t)\Delta t \\
&\quad\leq \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \left\| c^{-t/\omega} \int_{-\infty}^t e_{-\alpha}(t,\sigma(s))f_\varphi(s)\Delta s \right\|_{\mathbb{B}} \nu(t)\Delta t \\
&\quad\leq \frac{1}{1-\bar{\mu}\underline{\alpha}} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho,l,t_0)} \int_{t_0-l}^{t_0+l} \left(\int_{-\infty}^t e^{-\left(\frac{\ln|c|}{\omega}+\underline{\alpha}\right)(t-s)} \left\| f_\varphi(s) \right\|_{\omega,c} \Delta s \right) \nu(t)\Delta t
\end{aligned}
$$

$$\leq \frac{1}{1-\bar{\mu}\underline{\alpha}} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \left(\int_{\tau}^{\infty} e^{-(\frac{\ln|c|}{\omega}+\underline{\alpha})s} \Big\| f_{\varphi}(t-s-\tau) \Big\|_{\omega,c} \Delta s \right) \nu(t)\Delta t.$$

Consider

$$\mathcal{J}_l(s) = \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \| f_{\varphi}(t-s-\tau)\|_{\omega,c} \nu(t)\Delta t.$$

It is obvious that $\mathcal{J}_l(s) \in \mathrm{BC}_{\mathrm{rd}}(\mathbb{T}_\omega, \mathbb{B})$ is Δ-measurable, and by Theorem 5.2.8, we have

$$\lim_{l\to\infty} \mathcal{J}_l(s) = 0.$$

Thus, by Theorem 1.4.54 we get

$$\begin{aligned} \lim_{l\to\infty} \frac{1}{\mathfrak{m}(\varrho, l, t_0)} \int_{t_0-l}^{t_0+l} \|\mathcal{I}_2(t)\|_{\omega,c}\nu(t)\Delta t &\leq \frac{1}{1-\bar{\mu}\underline{\alpha}} \lim_{l\to\infty} \int_{\tau}^{\infty} e^{-(\frac{\ln|c|}{\omega}+\underline{\alpha})s} \mathcal{J}_l(s)\Delta s \\ &= \frac{1}{1-\bar{\mu}\underline{\alpha}} \int_{\tau}^{\infty} \lim_{l\to\infty} e^{-(\frac{\ln|c|}{\omega}+\underline{\alpha})s} \mathcal{J}_l(s)\Delta s \\ &= 0. \end{aligned}$$

The proof is complete. □

To prove our main result, which deals with the existence of a weighted pseudo (ω, c)-periodic solution of (5.3.1), we need to make some assumptions. These assumptions are as follows:

H2: $0 \leq \gamma^{\Delta}(t), \tau^{\Delta}(t) < 1$ for all $t \in \mathbb{T}_\omega$.

H3: For all $t \in \mathbb{T}_\omega^+$ and $x, y \in \mathbb{B}$, the nonlinear function f and its (w, c)-periodic part f_{op} satisfy the following inequalities:

$$\|f(t,x) - f(t,y)\|_{\mathbb{B}} \leq l_f \|x-y\|_{\mathbb{B}},$$

$$\|f_{op}(t,x) - f_{op}(t,y)\|_{\mathbb{B}} \leq l_{f_{op}} \|x-y\|_{\mathbb{B}},$$

where l_f and $l_{f_{op}}$ are positive constants.

H4: For all $j = 1, 2, \ldots, q$ and $x, y \in \mathbb{B}$, the impulsive terms I_j satisfy the inequality

$$\|I_j(x(t_j)) - I_j(y(t_j))\|_{\mathbb{B}} \leq l_j \|x-y\|_{\mathbb{B}} \text{ with } I_j(0) = 0.$$

H5: There exists $\underline{\mathsf{a}} > 0$ such that $-\underline{\mathsf{a}} \in \mathcal{R}^+$ and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\mathsf{a}}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{\mathsf{a}}\bar{\mu})}$. Furthermore, the regressive function a satisfies $0 < \underline{\mathsf{a}} \leq \mathsf{a}(t)$ for all $t \in \mathbb{T}_\omega$.

We now present our main theorem regarding the existence of a weighted pseudo (ω, c)-periodic solution to the system (5.3.4). The proof of this theorem relies on several preliminary results established earlier in this chapter.

Theorem 5.4.4. *Under assumptions **H1**–**H5**, the piecewise impulsive dynamic equation (5.3.4) has a unique weighted pseudo (ω, c)-periodic solution $u(\cdot) \in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega, \mathbb{B})$, provided that*

$$\textbf{H6: } \mathfrak{C} := \left\{ \frac{|c|^{\gamma^*/\omega} \dfrac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} + |c|^{\tau^*/\omega} \dfrac{l_f}{|1-\mathtt{b}^*|}}{(1-\bar{\mu}\underline{\mathtt{a}}) \left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)} + \sum_{j=1}^{q} \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} \right\} < 1,$$

where $\mathtt{b}^* = \bar{\mathtt{b}}$ *when* $\bar{\mathtt{b}} < 1$ *and* $\mathtt{b}^* = \underline{\mathtt{b}}$ *when* $\underline{\mathtt{b}} > 1$*, and for* $\alpha \in \{\tau, \delta\}$ *we have used*

$$\alpha^* = \begin{cases} \bar{\alpha} = \sup\limits_{\mathbb{T}_\omega^+} \alpha(t), & \textit{when } |c| \geq 1, \\ \underline{\alpha} = \inf\limits_{\mathbb{T}_\omega^+} \alpha(t), & \textit{when } |c| \leq 1. \end{cases}$$

Proof. Consider the following operator:

$$\begin{aligned} (\Pi u)(t) \quad = \quad & \int_{-\infty}^{t} e_{-\mathtt{a}}(t, \sigma(s)) \left(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s)))\right) \Delta s \\ & - \sum_{j=1}^{q} e_{-\mathtt{a}}(t, t_j)(\mathcal{F} I_j \mathcal{F}^{-1})u(t_j). \end{aligned} \tag{5.4.3}$$

Clearly, based on the assumptions made and the results proven so far in this article, we can easily show that the operator Π is well defined, i.e., $\Pi : \mathbb{E} \to \mathbb{E}$.

Next, we show that Π is a contraction mapping in $\mathbb{E}$. Indeed, if we have $u, v \in \mathbb{E}$, then

$$\begin{aligned} &\|(\Pi u)(t) - (\Pi v)(t)\|_{p\omega c} \\ &\quad = \sup_{t \in \mathbb{T}_\omega} \|c^{-t/\omega} \left((\Pi u) - (\Pi v)\right)(t)\|_{\mathbb{B}} \\ &\quad = \sup_{t \in \mathbb{T}_\omega} \Bigg\| \int_{-\infty}^{t} c^{-(t-s)/\omega} e_{-\mathtt{a}}(t, \sigma(s)) c^{-s/\omega} \Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}(u-v))(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s))) \\ &\qquad - f(s, (\mathcal{F}^{-1}v)(s-\tau(s)))\Big) \Delta s - c^{-t/\omega} \sum_{j=1, t_j \in [t_0, t)}^{q} e_{-\mathtt{a}}(t, t_j)(\mathcal{F} I_j \mathcal{F}^{-1} u - \mathcal{F} I_j \mathcal{F}^{-1} v)(t_j) \Bigg\|_{\mathbb{B}} \\ &\quad \leq \frac{1}{1-\bar{\mu}\underline{\mathtt{a}}} \sup_{t \in \mathbb{T}_\omega} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)}(t, s) |c|^{-s/\omega} \left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} \|(u-v)(s-\gamma)\|_{\mathbb{B}} + \frac{l_f}{|1-\mathtt{b}^*|} \|(u-v)(s-\tau(s))\|_{\mathbb{B}} \right) \Delta s \\ &\qquad + \sup_{t \in \mathbb{T}_\omega} \sum_{j=1, t_j \in [t_0, t)}^{q} e_{-\underline{\mathtt{a}}}(t, t_j) \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} |c|^{-t/\omega} \|(u-v)(t_j)\|_{\mathbb{B}} \\ &\quad \leq \Bigg\{ \frac{1}{1-\bar{\mu}\underline{\mathtt{a}}} \sup_{t \in \mathbb{T}_\omega} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)}(t, s) \left(|c|^{\gamma/\omega} \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} + |c|^{\tau/\omega} \frac{l_f}{|1-\mathtt{b}^*|} \right) \Delta s \\ &\qquad + \sup_{t \in \mathbb{T}_\omega} \sum_{j=1, t_j \in [t_0, t)}^{q} e_{-\underline{\mathtt{a}}}(t, t_j) \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} |c|^{-(t-t_j)/\omega} \Bigg\} \|(u-v)\|_{p\omega c} \\ &\quad \leq \Bigg\{ \frac{1}{1-\bar{\mu}\underline{\mathtt{a}}} \sup_{t \in \mathbb{T}_\omega} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)}(t, s) \left(|c|^{\gamma/\omega} \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} + |c|^{\tau/\omega} \frac{l_f}{|1-\mathtt{b}^*|} \right) \Delta s \\ &\qquad + \sup_{t \in \mathbb{T}_\omega} \sum_{j=1, t_j \in [t_0, t)}^{q} e_{\ominus\left(\underline{\mathtt{a}} \oplus \frac{\ln|c|}{\omega}\right)}(t, t_j) \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} \Bigg\} \|(u-v)\|_{p\omega c} \\ &\quad \leq \Bigg\{ \frac{1}{1-\bar{\mu}\underline{\mathtt{a}}} \sup_{t \in \mathbb{T}_\omega} \int_{-\infty}^{t} e_{\ominus\left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)}(t, s) \left(|c|^{\gamma/\omega} \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} + |c|^{\tau/\omega} \frac{l_f}{|1-\mathtt{b}^*|} \right) \Delta s + \sum_{j=1}^{q} \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} \Bigg\} \|(u-v)\|_{p\omega c} \end{aligned}$$

$$\leq \left\{ \frac{|c|^{\gamma^*/\omega} \dfrac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{|1-\mathtt{b}^*|} + |c|^{\tau^*/\omega} \dfrac{l_f}{|1-\mathtt{b}^*|}}{(1-\bar{\mu}\underline{\mathtt{a}})\left(\frac{\ln|c|}{\omega} + \underline{\mathtt{a}}\right)} + \sum_{j=1}^{q} \frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} \right\} \|(u-v)\|_{p\omega c},$$

implies that Π is a contraction mapping, along with condition **H6**. Therefore, by the Banach contraction principle, we conclude that there exists a unique $u \in \mathbb{E}$ such that $\Pi(u) = u$. Hence, the unique weighted pseudo (ω, c)-periodic solution of the piecewise impulsive dynamic equation (5.3.4) is

$$u(t) = \int_{-\infty}^{t} e_{-\mathtt{a}}(t, \sigma(s)) \big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}u)(s-\gamma) + f(s, (\mathcal{F}^{-1}u)(s-\tau(s))) + \mathtt{e}(s)\big) \Delta s \\ - \sum_{j=1}^{q} e_{-\mathtt{a}}(t, t_j)(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j),$$

and $\mathcal{F}^{-1}u$ is the unique weighted pseudo (ω, c)-periodic solution of (5.3.1). The theorem is proved. □

5.4.2 Weighted pseudo (ω, c)-periodic synchronization

By designing a controller, employing analytic techniques and constructing an appropriate Lyapunov function, we investigate the exponential synchronization of dynamic equations with weighted pseudo (ω, c)-periodic coefficients and piecewise impulsive effects on time scales, as described in (5.3.1), under the assumptions outlined in **H1**. For this, we consider the system (5.3.1) as a drive (master) system and design a response (slave) system as

$$\begin{aligned} (y(t) + \mathtt{b}(t)y(t-\gamma(t)))^{\Delta} &= -\mathtt{a}(t)y(t) + f(t, y(t-\tau(t))) + \mathtt{e}(t) + \mathcal{C}(t), \quad t > t_0,\ t \neq t_j \\ y(t_j{}^-) - y(t_j{}^+) &= I_j(y(t_j)), \quad t = t_j,\ j = 1, 2, \ldots, q, \end{aligned} \tag{5.4.4}$$

where $\mathcal{C}(t)$ is a controlled input. Letting $(\mathcal{F}y)(t) = v(t)$, we can change (5.4.4) in the form

$$\begin{aligned} v^{\Delta}(t) &= -\mathtt{a}(t)u(t) + \mathtt{a}(t)\mathtt{b}(t)(\mathcal{F}^{-1}v)(t-\gamma) + f(t, (\mathcal{F}^{-1}v)(t-\tau(t))) + \mathtt{e}(t) + \mathcal{C}(t),\ t > t_0,\ t \neq t_j \\ v(t_j{}^-) - v(t_j{}^+) &= (\mathcal{F}I_j\mathcal{F}^{-1})v(t_j), \quad t = t_j,\ j = 1, 2, \ldots, q. \end{aligned}$$

Let $w(t) = v(t) - u(t)$, so that we can get the error system as

$$\begin{aligned} w^{\Delta}(t) &= -\mathtt{a}(t)w(t) + \mathtt{a}(t)\mathtt{b}(t)(\mathcal{F}^{-1}w)(t-\gamma) + (f(t, (\mathcal{F}^{-1}v(t-\tau(t)))) \\ &\quad - f(t, (\mathcal{F}^{-1}u(t-\tau(t))))) + \mathcal{C}(t), \quad t > t_0,\ t \neq t_j \\ w(t_j{}^-) - w(t_j{}^+) &= (\mathcal{F}I_j\mathcal{F}^{-1})w(t_j), \quad t = t_j,\ j = 1, 2, \ldots, q. \end{aligned} \tag{5.4.5}$$

The main idea of synchronization is that the response system (5.4.5) uses a feasible controller to synchronize itself with the drive system (5.3.4). To witness the weighted pseudo (ω, c) periodic synchronization between (5.3.1) and (5.4.4), we design the following state-feedback controller:

$$\mathcal{C}(t) = -a(t)w(t) + a(t)b(t)(\mathcal{F}^{-1}w)(t-\tilde{\gamma}) + \tilde{f}(t, \mathcal{F}^{-1}w(t-\tilde{\tau}(t)))). \tag{5.4.6}$$

Consequently, the expression for the error dynamics can be given as

$$\begin{aligned} w^{\Delta}(t) &= -(\mathtt{a}(t)+a(t))w(t)+\mathtt{a}(t)\mathtt{b}(t)(\mathcal{F}^{-1}w)(t-\gamma)+a(t)b(t)(\mathcal{F}^{-1}w)(t-\tilde{\gamma})+\\ &\quad (f(t,(\mathcal{F}^{-1}v(t-\tau(t))))-f(t,(\mathcal{F}^{-1}u(t-\tau(t)))))+\tilde{f}(t,\mathcal{F}^{-1}w(t-\tilde{\tau}(t)))),\ t>t_0,\ t\neq t_j\\ w(t_j{}^-)-w(t_j{}^+) &= (\mathcal{F}I_j\mathcal{F}^{-1})w(t_j),\quad t=t_j,\ j=1,2,\ldots,q. \end{aligned} \tag{5.4.7}$$

Let system (5.4.4) is suplemented with the initial value

$$y(s)=\psi(s),\quad s\in[-\tilde{\theta},t_0]_{\mathbb{T}_\omega},$$

where $\psi(s)\in \mathrm{C_{rd}}(\mathbb{T}_\omega,\mathbb{B})$, $\tilde{\theta}=\max\{\bar{\tilde{\gamma}},\bar{\tilde{\tau}}\}$ with $\bar{\tilde{\gamma}}=\sup_{t\in\mathbb{T}_\omega}\tilde{\gamma}(t)$ and $\bar{\tilde{\tau}}=\sup_{t\in\mathbb{T}_\omega}\tilde{\tau}(t)$.

Definition 5.4.5. *The response system (5.4.4) and the drive system (5.3.1) are said to be globally exponentially (ω,c)-synchronized if there exist positive constants $\mathtt{M}\geq 1$, and $\lambda>0$ such that*

$$\|x-y\|_{\omega,c}\leq \mathtt{M}e_{\ominus\lambda}(t,0)\sup_{\mathbb{T}_\omega}\|\phi-\psi\|_{\omega,c}\quad \textit{for all}\quad t\geq t_0. \tag{5.4.8}$$

Theorem 5.4.6. *Let **H1**–**H5** hold. Suppose further that*

H7: *$a\in P_{\omega,1}(\mathbb{T}_\omega,\mathbb{R}^+)$ is regressive, $\tilde{\gamma},\tilde{\tau}\in P_{\omega,1}(\mathbb{T},\Pi^{0+}_{\mathbb{T}_\omega})$, $b\in \mathrm{WPP}^{\varrho,\nu}_{\omega,1}(\mathbb{T}_\omega,\mathbb{B})$, and $\tilde{f}=\tilde{f}_{op}+\tilde{f}_{\varphi}$ such that $\tilde{f}_{op}\in \mathrm{P}_{\omega,c}(\mathbb{T}\times\mathbb{B},\mathbb{B})$, $\tilde{f}_{\varphi}$ is Lipchitz continuous with respect to the norm $\|\cdot\|_{\mathbb{B}}$ and for every x in any bounded subset of $\mathbb{B}$, $\tilde{f}_{\varphi}(t,c^{t/\omega}x)\in \mathrm{WPP}^{\varrho,\nu}_{\omega,c}(\mathbb{T}_\omega,\mathbb{B})$ uniformly in $t\in\mathbb{T}_\omega$.*

H8: *$\tilde{f}(t,0)=0$ for all $t\in\mathbb{T}_\omega$. Also, there exists a positive constant $L_{\tilde{f}}$ such that*

$$\|\tilde{f}(t,x)\|_{\mathbb{B}}\leq L_{\tilde{f}}\|\tilde{f}(x)\|_{\mathbb{B}}.$$

H9: *There exists a positive constant $\lambda\in\mathcal{R}^+$ such that*

(a) *for $t\neq t_j,\ j=1,2,\ldots,q$,*

$$\begin{aligned} \mathfrak{C}^* := \Big\{\lambda-(1+\underline{\mu}\lambda)(\underline{a}+\underline{\mathtt{a}})+\frac{1}{|1-\mathtt{b}^*|}\Big(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}e_\lambda(t+\theta,t)+\frac{\bar{a}\bar{b}}{\tilde{\alpha}}e_\lambda(t+\tilde{\theta},t)\\ +\frac{l_f}{\beta}e_\lambda(t+\theta,t)+\frac{L_{\tilde{f}}}{\tilde{\beta}}e_\lambda(t+\tilde{\theta},t)\Big)\Big\}<0, \end{aligned}$$

where $0<(1+\bar{\mu}\lambda)\alpha=1-\sup_{t\in\mathbb{T}_\omega}\{\gamma_+^{\Delta}(t)\}$, $0<(1+\bar{\mu}\lambda)\tilde{\alpha}=1-\sup_{t\in\mathbb{T}_\omega}\{\tilde{\gamma}_+^{\Delta}(t)\}$, $0<(1+\bar{\mu}\lambda)\beta=1-\sup_{t\in\mathbb{T}_\omega}\{\tau_+^{\Delta}(t)\}$, and $(1+\bar{\mu}\lambda)\tilde{\beta}=1-\sup_{t\in\mathbb{T}_\omega}\{\tilde{\tau}_+^{\Delta}(t)\}$.

(b) *for each $j=1,2,\ldots,q$,*

$$\begin{aligned} l_j\leq \frac{(1+\bar{\mathtt{b}})}{|1-\mathtt{b}^*|}\Bigg[1+\frac{1}{1+\lambda\mu(t_j)}+\frac{\mu(t_j)}{|1-\mathtt{b}^*|(1+\lambda\mu(t_j))}\\ \Bigg(\Big(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}+\frac{l_f}{\beta}\Big)e_\lambda(t_j+\theta,\mathtt{t}_j)+\Big(\frac{\bar{a}\bar{b}}{\tilde{\alpha}}+\frac{L_{\tilde{f}}}{\tilde{\beta}}\Big)e_\lambda(t_j+\tilde{\theta},\mathtt{t}_j)\Bigg)\Bigg]. \end{aligned}$$

Then (5.3.1) *and* (5.4.4) *are globally exponentially synchronized.*

Proof. We have seen that any solution of (5.4.7) satisfies

$$w(t) = e_{-(a+\mathtt{a})}(t,t_0)w(t_0)+\int_{t_0}^{t} e_{-(a+\mathtt{a})}(t,\sigma(s))\Big(\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}w)(s-\gamma)+f(s,(\mathcal{F}^{-1}w)(s-\tau(s)))+\mathtt{e}(s)\Big)\Delta s \\ -\sum_{j=1}^{q} e_{-(a+\mathtt{a})}(t,t_j)(\mathcal{F}I_j\mathcal{F}^{-1})w(t_j). \quad (5.4.9)$$

Construct a Lyapunov function as

$$V(t) = \|w(t)\|_{\omega,c}e_\lambda(t,t_0) + \nabla(t),$$

with

$$\begin{aligned}\nabla(t) \quad = \quad & \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\int_{t-\gamma(t)}^{t}\|\mathcal{F}^{-1}w(s)\|_{\omega,c}e_\lambda(s+\theta,t_0)\Delta s + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\int_{t-\tilde{\gamma}(t)}^{t}\|\mathcal{F}^{-1}w(s)\|_{\omega,c}e_\lambda(s+\tilde{\theta},t_0)\Delta s \\ & +\frac{l_f}{\beta}\int_{t-\tau(t)}^{t}\|\mathcal{F}^{-1}w(s)\|_{\omega,c}e_\lambda(s+\theta,t_0)\Delta s + \frac{L_{\tilde{f}}}{\tilde{\beta}}\int_{t-\tilde{\tau}(t)}^{t}\|\mathcal{F}^{-1}w(s)\|_{\omega,c}e_\lambda(s+\tilde{\theta},t_0)\Delta s.\end{aligned}$$

Let $D^+_\Delta w(t)$ denote the upper right Dini-Δ-derivative of w at t. Then, by (5.4.9), for any $t > t_0$ with $t \neq t_j$, $j = 1,2,\dots,q$, we have

$$\begin{aligned} & D^+_\Delta\|w(t)\|_{\omega,c} \\ & \quad \le D^+_\Delta\Bigg(e_{-(a+\mathtt{a})}(t,t_0)\Bigg(\|w(t_0)\|_{\omega,c} + \int_{t_0}^{t} e_{-(a+\mathtt{a})}(t_0,\sigma(s))\Big(\|\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}w)(s-\gamma)\|_{\omega,c} \\ & \qquad + \|a(s)b(s)(\mathcal{F}^{-1}w)(s-\tilde{\gamma})\|_{\omega,c} + \|(f(s,(\mathcal{F}^{-1}v(s-\tau(s)))) - f(s,(\mathcal{F}^{-1}u(s-\tau(s)))))\|_{\omega,c} \\ & \qquad + \|\tilde{f}(s,(\mathcal{F}^{-1}w(s-\tilde{\tau}(s))))\|_{\omega,c}\Big)\Delta s + \sum_{j=1}^{q} e_{-(a+\mathtt{a})}(t_0,t_j)\|(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j)\|_{\omega,c}\Bigg)\Bigg) \\ & \quad \le -(a(t)+\mathtt{a}(t))e_{-(a+\mathtt{a})}(t,t_0)\Bigg(\|w(t_0)\|_{\omega,c} + \int_{t_0}^{t} e_{-(a+\mathtt{a})}(t_0,\sigma(s))\Big(\|\mathtt{a}(s)\mathtt{b}(s)(\mathcal{F}^{-1}w)(s-\gamma)\|_{\omega,c} \\ & \qquad + \|a(s)b(s)(\mathcal{F}^{-1}w)(s-\tilde{\gamma})\|_{\omega,c} + \|(f(s,(\mathcal{F}^{-1}v(s-\tau(s)))) - f(s,(\mathcal{F}^{-1}u(s-\tau(s)))))\|_{\omega,c} \\ & \qquad + \|\tilde{f}(s,(\mathcal{F}^{-1}w(s-\tilde{\tau}(s)))\|_{\omega,c}\Big)\Delta s + \sum_{j=1}^{q} e_{-(a+\mathtt{a})}(t_0,t_j)\|(\mathcal{F}I_j\mathcal{F}^{-1})u(t_j)\|_{\omega,c}\Bigg) \\ & \qquad + e_{-(a+\mathtt{a})}(\sigma(t),\sigma(t))\Big(\|\mathtt{a}(t)\mathtt{b}(t)(\mathcal{F}^{-1}w)(t-\gamma)\|_{\omega,c} + \|a(t)b(t)(\mathcal{F}^{-1}w)(t-\tilde{\gamma})\|_{\omega,c} \\ & \qquad + \|(f(t,(\mathcal{F}^{-1}v(t-\tau(t)))) - f(t,(\mathcal{F}^{-1}u(t-\tau(t)))))\|_{\omega,c} + \|\tilde{f}(t,(\mathcal{F}^{-1}w(t-\tilde{\tau}(t)))\|_{\omega,c}\Big) \\ & \quad \le -(\underline{a}+\underline{\mathtt{a}})\|w(t)\|_{\omega,c} + \Big(\bar{\mathtt{a}}\bar{\mathtt{b}}\|\mathcal{F}^{-1}w(t-\gamma)\|_{\omega,c} + \bar{a}\bar{b}\|\mathcal{F}^{-1}w(t-\tilde{\gamma})\|_{\omega,c} \\ & \qquad + l_f\|\mathcal{F}^{-1}w(t-\tau(t))\|_{\omega,c} + L_{\tilde{f}}\|\mathcal{F}^{-1}w(t-\tilde{\tau}(t))\|_{\omega,c}\Big). \end{aligned}$$

For the upper right derivative of $V(t)$ at $t \neq t_j$, we have

$$
\begin{aligned}
&D_\Delta^+ V(t)\\
&= \lambda \|w(t)\|_{\omega,c} e_\lambda(t,t_0) + D_\Delta^+\left(\|w(t)\|_{\omega,c}\right) e_\lambda(\sigma(t),t_0) + \Bigg(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t_0)\\
&\quad + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t_0) + \frac{l_f}{\beta}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t_0) + \frac{L_{\tilde{f}}}{\tilde{\beta}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t_0)\\
&\quad - \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}(1-\gamma_+^\Delta(t))\|\mathcal{F}^{-1}w(t-\gamma(t))\|_{\omega,c} e_\lambda(t-\gamma(t)+\theta,t_0) - \frac{\bar{a}\bar{b}}{\tilde{\alpha}}(1-\tilde{\gamma}_+^\Delta(t))\|\mathcal{F}^{-1}w(t-\tilde{\gamma}(t))\|_{\omega,c}\\
&\quad e_\lambda(t-\tilde{\gamma}(t)+\tilde{\theta},t_0) - \frac{l_f}{\beta}(1-\tau_+^\Delta(t))\|\mathcal{F}^{-1}w(t-\tau(t))\|_{\omega,c} e_\lambda(t-\tau(t)+\theta,t_0)\\
&\quad - \frac{L_{\tilde{f}}}{\tilde{\beta}}(1-\tilde{\tau}_+^\Delta(t))\|\mathcal{F}^{-1}w(t-\tilde{\tau}(t))\|_{\omega,c} e_\lambda(t-\tilde{\tau}(t)+\tilde{\theta},t_0)\Bigg)\\
&\le \Bigg\{(\lambda-(1+\underline{\mu}\lambda)(\underline{a}+\underline{\mathtt{a}}))\|w(t)\|_{\omega,c} + (1+\bar{\mu}\lambda)\Big(\bar{\mathtt{a}}\bar{\mathtt{b}}\|\mathcal{F}^{-1}w(t-\gamma)\|_{\omega,c}(1-e_\lambda(t+\theta-\gamma(t),t))\\
&\quad + \bar{a}\bar{b}\|\mathcal{F}^{-1}w(t-\tilde{\gamma})\|_{\omega,c}(1-e_\lambda(t+\tilde{\theta}-\tilde{\gamma}(t),t)) + l_f\|\mathcal{F}^{-1}w(t-\tau(t))\|_{\omega,c}(1-e_\lambda(t+\theta-\tau(t),t))\\
&\quad + L_{\tilde{f}}\|\mathcal{F}^{-1}w(t-\tilde{\tau}(t))\|_{\omega,c}(1-e_\lambda(t+\tilde{\theta}-\tilde{\tau}(t),t))\Big)\Bigg\} e_\lambda(t,t_0) + \Bigg(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t_0)\\
&\quad + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t_0) + \frac{l_f}{\beta}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t_0) + \frac{L_{\tilde{f}}}{\tilde{\beta}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t_0)\Bigg)\\
&\le \Bigg\{(\lambda-(1+\underline{\mu}\lambda)(\underline{a}+\underline{\mathtt{a}}))\|w(t)\|_{\omega,c} + \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t) + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t)\\
&\quad + \frac{l_f}{\beta}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\theta,t) + \frac{L_{\tilde{f}}}{\tilde{\beta}}\|\mathcal{F}^{-1}w(t)\|_{\omega,c} e_\lambda(t+\tilde{\theta},t)\Bigg\} e_\lambda(t,t_0)\\
&\le \Bigg\{\lambda-(1+\underline{\mu}\lambda)(\underline{a}+\underline{\mathtt{a}}) + \frac{1}{|1-\mathtt{b}^*|}\Bigg(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} e_\lambda(t+\theta,t) + \frac{\bar{a}\bar{b}}{\tilde{\alpha}} e_\lambda(t+\tilde{\theta},t) + \frac{l_f}{\beta} e_\lambda(t+\theta,t)\\
&\quad + \frac{L_{\tilde{f}}}{\tilde{\beta}} e_\lambda(t+\tilde{\theta},t)\Bigg)\Bigg\}\|w(t)\|_{\omega,c} e_\lambda(t,t_0).
\end{aligned}
$$

In view of condition **H9**, for all $t \neq t_j$, we have

$$D_\Delta^+ V(t) \le 0.$$

On the other hand, for $t = t_j,\ j = 1,2,\ldots,q$, we have

$$w(t_j^+) = (I - \mathcal{F}I_j\mathcal{F}^{-1})w(t_j^-) = (I - \mathcal{F}I_j\mathcal{F}^{-1})w(t_j), \quad \text{and}$$

$$
\begin{aligned}
D_\Delta^+ V(t_j) &= \lambda e_\lambda(t_j,t_0) + e_\lambda(\sigma(t_j),t_0)\left(\|w(t_j^+)\|_{\omega,c} - \|w(t_j)\|_{\omega,c}\right) + \nabla(t_j^+) - \nabla(t_j)\\
&\le \left[\lambda + (1+\lambda\mu(t_j))\left(\frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} - 1\right)\right]\|w(t_j)\|_{\omega,c} e_\lambda(t_j,t_0)\\
&\quad + \frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\int_{t_j}^{t_j^+}\|\mathcal{F}^{-1}w(s)\|_{\omega,c} e_\lambda(s+\theta,t_0)\Delta s + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\int_{t_j}^{t_j^+}\|\mathcal{F}^{-1}w(s)\|_{\omega,c} e_\lambda(s+\tilde{\theta},t_0)\Delta s\\
&\quad + \frac{l_f}{\beta}\int_{t_j}^{t_j^+}\|\mathcal{F}^{-1}w(s)\|_{\omega,c} e_\lambda(s+\theta,t_0)\Delta s + \frac{L_{\tilde{f}}}{\tilde{\beta}}\int_{t_j}^{t_j^+}\|\mathcal{F}^{-1}w(s)\|_{\omega,c} e_\lambda(s+\tilde{\theta},t_0)\Delta s
\end{aligned}
$$

$$\begin{aligned}
&\leq \left[\lambda + (1+\lambda\mu(t_j))\left(\frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} - 1\right)\right]\|w(t_j)\|_{\omega,c}e_\lambda(t_j,t_0) \\
&\quad + \left[\left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} + \frac{l_f}{\beta}\right)e_\lambda(t_j+\theta,t_0) + \left(\frac{\bar{a}\bar{b}}{\tilde{\alpha}} + \frac{L_{\tilde{f}}}{\tilde{\beta}}\right)e_\lambda(t_j+\tilde{\theta},t_0)\right]\mu(t_j)\|\mathcal{F}^{-1}w(t_j)\|_{\omega,c} \\
&\leq \left[(\lambda + (1+\lambda\mu(t_j))\left(\frac{(1+\bar{\mathtt{b}})l_j}{|1-\mathtt{b}^*|} - 1\right) + \frac{1}{|1-\mathtt{b}^*|}\left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} + \frac{l_f}{\beta}\right)\mu(t_j)e_\lambda(t_j+\theta,\mathtt{t}_j)\right. \\
&\quad \left. + \frac{1}{|1-\mathtt{b}^*|}\left(\frac{\bar{a}\bar{b}}{\tilde{\alpha}} + \frac{L_{\tilde{f}}}{\tilde{\beta}}\right)\mu(t_j)e_\lambda(t_j+\tilde{\theta},\mathtt{t}_j)\right]\|w(t_j)\|_{\omega,c}e_\lambda(t_j,t_0).
\end{aligned}$$

Employing condition **H9**, for all $t = t_j$, we obtain

$$D_\Delta^+ V(t) \leq 0.$$

Hence,

$$V(t) \leq V(t_0) \text{ for all } t \geq t_0.$$

Furthermore, we have

$$\begin{aligned}
V(t_0) &\leq \sup_{\mathbb{T}_\omega}\|\mathcal{F}\phi - \mathcal{F}\psi\|_{\omega,c}\Bigg\{1 + \frac{1}{|1-\mathtt{b}^*|}\Bigg(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha}\int_{t_0-\gamma(t_0)}^{t_0} e_\lambda(s+\theta,t_0)\Delta s \\
&\quad + \frac{\bar{a}\bar{b}}{\tilde{\alpha}}\int_{t_0-\tilde{\gamma}(t_0)}^{t_0} e_\lambda(s+\tilde{\theta},t_0)\Delta s + \frac{l_f}{\beta}\int_{t_0-\tau(t_0)}^{t_0} e_\lambda(s+\theta,t_0)\Delta s + \frac{L_{\tilde{f}}}{\tilde{\beta}}\int_{t_0-\tilde{\tau}(t_0)}^{t_0} e_\lambda(s+\tilde{\theta},t_0)\Delta s\Bigg)\Bigg\} \\
&\leq \sup_{\mathbb{T}_\omega}\|\mathcal{F}\phi - \mathcal{F}\psi\|_{\omega,c}\Bigg\{1 + \frac{1}{|1-\mathtt{b}^*|}\Bigg(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha\lambda}(e_\lambda(t_0+\theta,t_0) - e_\lambda(t_0+\theta-\gamma(t_0),t_0)) \\
&\quad + \frac{\bar{a}\bar{b}}{\lambda\tilde{\alpha}}(e_\lambda(t_0+\tilde{\theta},t_0) - e_\lambda(t_0+\tilde{\theta}-\tilde{\gamma}(t_0),t_0)) + \frac{l_f}{\beta}(e_\lambda(t_0+\theta,t_0) \\
&\quad - e_\lambda(t_0+\theta-\tau(t_0),t_0)) + \frac{L_{\tilde{f}}}{\tilde{\beta}}(e_\lambda(t_0+\tilde{\theta},t_0) - e_\lambda(t_0+\tilde{\theta}-\tilde{\tau}(t_0),t_0))\Bigg)\Bigg\} \\
&:= \sup_{\mathbb{T}_\omega}\|\mathcal{F}\phi - \mathcal{F}\psi\|_{\omega,c}\,\mathtt{M}.
\end{aligned}$$

It is obvious that

$$\|w(t)\|_{\omega,c}e_\lambda(t,t_0) \leq V(t) \;\text{ for all } t \geq t_0,$$

which implies

$$\|u(t) - v(t)\|_{\omega,c} \leq V(t_0)e_{\ominus\lambda}(t,t_0) \leq \mathtt{M}e_{\ominus\lambda}(t,t_0)\sup_{\mathbb{T}_\omega}\|\mathcal{F}\phi - \mathcal{F}\psi\|_{\omega,c} \;\text{ for all } t \geq t_0,$$

where $\mathtt{M} \geq 1$ is defined above. Therefore, using Lemma 5.3.1 we have

$$\|x(t) - y(t)\|_{\omega,c} = \|\mathcal{F}^{-1}u(t) - \mathcal{F}^{-1}v(t)\|_{\omega,c} \leq \frac{1+\bar{\mathtt{b}}}{|1-\mathtt{b}^*|}\mathtt{M}e_{\ominus\lambda}(t,t_0)\sup_{\mathbb{T}}\|\phi - \psi\|_{\omega,c} \;\text{ for all } t \geq t_0,$$

where $\bar{\mathtt{b}}$ and $\mathtt{b}^*$ are defined in Theorem 5.4.4. Hence, the systems (5.3.1) and (5.4.4) are globally exponentially synchronized. The proof is complete. □

5.5 Numerical examples

We now provide some numerical examples and their simulated solutions (graphically) to demonstrate the effectiveness of our theoretical results.

Example 5.5.1. *Consider the drive system (5.3.1) with the coefficients defined as*

$$\mathtt{a}(t) = 0.05e^{\cos(\pi t)}, \quad \gamma(t) = 0.0625 + \frac{|\sin(\pi t)|}{16}, \quad \tau(t) = 0.125 + \frac{|\cos(\pi t)|}{8}, \quad \mathtt{b}(t) = \frac{e^{\sin(\pi t)}}{5} + \frac{\sin(\cos(t))}{(1+|t|)^2}$$

$$\mathtt{e}(t) = c^{t/2}\cos(\pi t) + c^{t/2}\frac{e^{\cos(t^2)}}{(1+|t|)^2},$$

$$f(t,x) = \frac{0.004}{3}\left(c^{t/2}\cos(\pi t) + x + \frac{x}{(1+|t|)^2}\right), \quad I_j(x(t_j)) = 0.008\left(\frac{2}{3}\right)^j x(t_j)$$

and the impulses applied at time $t_j = 1.5 - \bar{\mu}, 3 - \bar{\mu}, 7 - \bar{\mu}, 10 - \bar{\mu}, 15 - \bar{\mu}$.

We can easily compute the bounds of coefficients as follows:

$$\omega = 2, \quad \underline{\mathtt{a}} = 0.018, \quad \bar{\mathtt{a}} = 0.136, \quad \underline{\mathtt{b}} = 0.0736, \quad \bar{\mathtt{b}} = 0.55 < 1, \quad \bar{\gamma} = \frac{1}{8}, \quad \bar{\tau} = \frac{1}{4},$$

$$l_f = .004, \quad \text{and} \quad \sum_{j=1}^{q} l_j \le \sum_{j=1}^{\infty} 0.008\left(\frac{2}{3}\right)^j = 0.016.$$

Note that with respect to the weights $\varrho = 1 + t^2$ and $\nu = (1+|t|)^2$, the conditions H1–H5 are fulfilled for the above coefficients. Taking $c = \frac{3}{2}$, we discuss the example for three different periodic time scales $\mathbb{T}$: (a) a continuous time scale, (b) a discrete time scale, and (c) a hybrid time scale that is a combination of continuous and discrete points.

(a). *Consider $\mathbb{T} = \mathbb{R}$. For this case, we have $\mu \equiv 0$, which consequently implies that $\mathfrak{C} \le 0.87 < 1$. Thus, in view of Theorem 5.4.4, together with the above coefficients and weights, (5.3.1) has a unique weighted pseudo $(2, \frac{3}{2})$-periodic solution on the time scale $\mathbb{R}$ (see Figure 5.1a).*

(b). *Consider $\mathbb{T} = \frac{1}{8}\mathbb{Z}$, $\gamma \equiv \frac{1}{8}$ and $\tau \equiv \frac{1}{4}$, and other coefficients of (5.3.1) are as defined above. In this case, we have $\mu \equiv \frac{1}{8}$, which implies that $\mathfrak{C} \le 0.872 < 1$. Thus, along with the above coefficients and weights, (5.3.1) has a unique weighted pseudo $(2, \frac{3}{2})$-periodic solution on the time scale $\frac{1}{8}\mathbb{Z}$ (see Figure 5.1c).*

(c). *Consider $\mathbb{T} = \bigcup_{n \in \frac{1}{16}\mathbb{Z}} \left[2n, 2n + \frac{1}{16}\right]$, $\gamma \equiv \frac{1}{8}$ and $\tau \equiv \frac{1}{4}$, and other coefficients of (5.3.1) are as defined above. In this case, we have $\bar{\mu} = \frac{1}{16}$, which implies that $\mathfrak{C} \le 0.871 < 1$. Thus, along with the above coefficients and weights, (5.3.1) has a unique weighted pseudo $(2, \frac{3}{2})$-periodic solution on the time scale $\bigcup_{n \in \frac{1}{16}\mathbb{Z}} \left[2n, 2n + \frac{1}{16}\right]$ (see Figure 5.1e).*

Now, consider the response system (5.4.4) *where the coefficients are defined as*

$$a(t) = 0.6 + 0.9|\cos(\pi t)|, \quad \tilde{\gamma}(t) = 0.1875 + 0.0625\cos(\pi t), \quad \tilde{\tau}(t) = \frac{1}{8}\left(1 + |\sin(\pi t)|\right),$$

$$b(t) = \frac{\sin(\pi t)}{20} + \frac{\cos(\sin(t))}{(1+|t|)^2}, \quad \tilde{f}(t,x) = 0.01\left(x + \frac{\sin(x)}{(1+|t|)^2}\right).$$

Computing the bounds, we have

$$\underline{a} = 0.6, \quad \bar{a} = 1.5, \quad \underline{b} = -0.05, \quad \bar{b} = 0.05, \quad \bar{\tilde{\gamma}} = \frac{1}{4}, \quad \bar{\tilde{\tau}} = \frac{1}{4}, \quad L_{\tilde{f}} = .03,$$

$$\alpha = \frac{0.80365}{(1+\bar{\mu}\lambda)}, \quad \beta = \frac{0.6073}{(1+\bar{\mu}\lambda)}, \quad \tilde{\alpha} = \frac{0.80365}{(1+\bar{\mu}\lambda)}, \quad \tilde{\beta} = \frac{0.6073}{(1+\bar{\mu}\lambda)}, \quad \theta = \frac{1}{4}, \quad \tilde{\theta} = \frac{1}{4}.$$

(a). *When $\mathbb{T} = \mathbb{R}$, for $\lambda = 0.07$, we have*

$$\mathfrak{C}^* := \lambda - (\underline{a} + \underline{\mathtt{a}}) + \frac{1}{|1-\bar{\mathtt{b}}|}\left(\left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} + \frac{l_f}{\beta}\right)e^{\lambda\theta} + \left(\frac{\bar{a}\bar{b}}{\tilde{\alpha}} + \frac{L_{\tilde{f}}}{\tilde{\beta}}\right)e^{\lambda\tilde{\theta}}\right) < -0.000584 < 0.$$

Thus, in view of Theorem 5.4.6, with given coefficients (5.3.1) *and* (5.4.4) *are globally exponentially synchronized in the time scale $\mathbb{R}$ (see Figure 5.1b), i.e.,*

$$\|x - y\|_{\omega,c} \le 4.37e^{-t/10}\sup_{\mathbb{T}}\|\phi - \psi\|_{\omega,c} \quad \textit{for all} \quad t \ge 0.$$

(b). *When $\mathbb{T} = \frac{1}{8}\mathbb{Z}$, consider $\tilde{\gamma} \equiv \frac{1}{8}$ and $\tilde{\tau} \equiv \frac{1}{4}$, then with $\alpha = \beta = \tilde{\alpha} = \tilde{\beta} = 1/(1+\lambda/8)$ we have*

$$\mathfrak{C}^* := \lambda - \left(1 + \frac{\lambda}{8}\right)(\underline{a} + \underline{\mathtt{a}}) + \frac{1}{|1-\bar{\mathtt{b}}|}\left(\left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} + \frac{l_f}{\beta}\right)\left(1 + \frac{\lambda}{8}\right)^{8\theta} + \left(\frac{\bar{a}\bar{b}}{\tilde{\alpha}} + \frac{L_{\tilde{f}}}{\tilde{\beta}}\right)\left(1 + \frac{\lambda}{8}\right)^{8\tilde{\theta}}\right) < -0.1341 < 0.$$

Thus, along with the given coefficients, (5.3.1) *and* (5.4.4) *are globally exponentially synchronized on the time scale $\frac{1}{8}\mathbb{Z}$ (see Figure 5.1d), i.e.,*

$$\|x - y\|_{\omega,c} \le 4.36975(0.9875)^{8t}\sup_{\mathbb{T}}\|\phi - \psi\|_{\omega,c} \quad \textit{for all} \quad t \in \frac{1}{8}\mathbb{Z}^+ \cup \{0\}.$$

(c). *Consider $\mathbb{T} = \bigcup_{n \in \frac{1}{16}\mathbb{Z}}\left[2n, 2n + \frac{1}{16}\right]$, with $\tilde{\gamma} \equiv \frac{1}{8}$, then we have*

$$\mathfrak{C}^* := \lambda - (\underline{a} + \underline{\mathtt{a}}) + \frac{1}{|1-\mathtt{b}^*|}\left(\left(\frac{\bar{\mathtt{a}}\bar{\mathtt{b}}}{\alpha} + \frac{l_f}{\beta}\right)e_\lambda(t+\theta, t) + \left(\frac{\bar{a}\bar{b}}{\tilde{\alpha}} + \frac{L_{\tilde{f}}}{\tilde{\beta}}\right)e_\lambda(t+\tilde{\theta}, t)\right) < -0.02 < 0.$$

Thus, with the given coefficients, (5.3.1) *and* (5.4.4) *are globally exponentially synchronized on the time scale $\bigcup_{n \in \frac{1}{16}\mathbb{Z}}\left[2n, 2n + \frac{1}{16}\right]$ (see Figure 5.1f).*

Example 5.5.2. *Consider the drive system* (5.3.1) *with the coefficients defined as*

$$\mathtt{a}(t) = 0.005e^{\cos(\pi t)}, \quad \mathtt{b}(t) = 1.6\left(\frac{e^{\sin(\pi t)}}{5} + \frac{\sin(\cos(t))}{(1+|t|)^2}\right), \quad \mathtt{e}(t) = c^{t/2}\cos(\pi t) + c^{t/2}\frac{e^{\cos(t^2)}}{(1+|t|)^2},$$

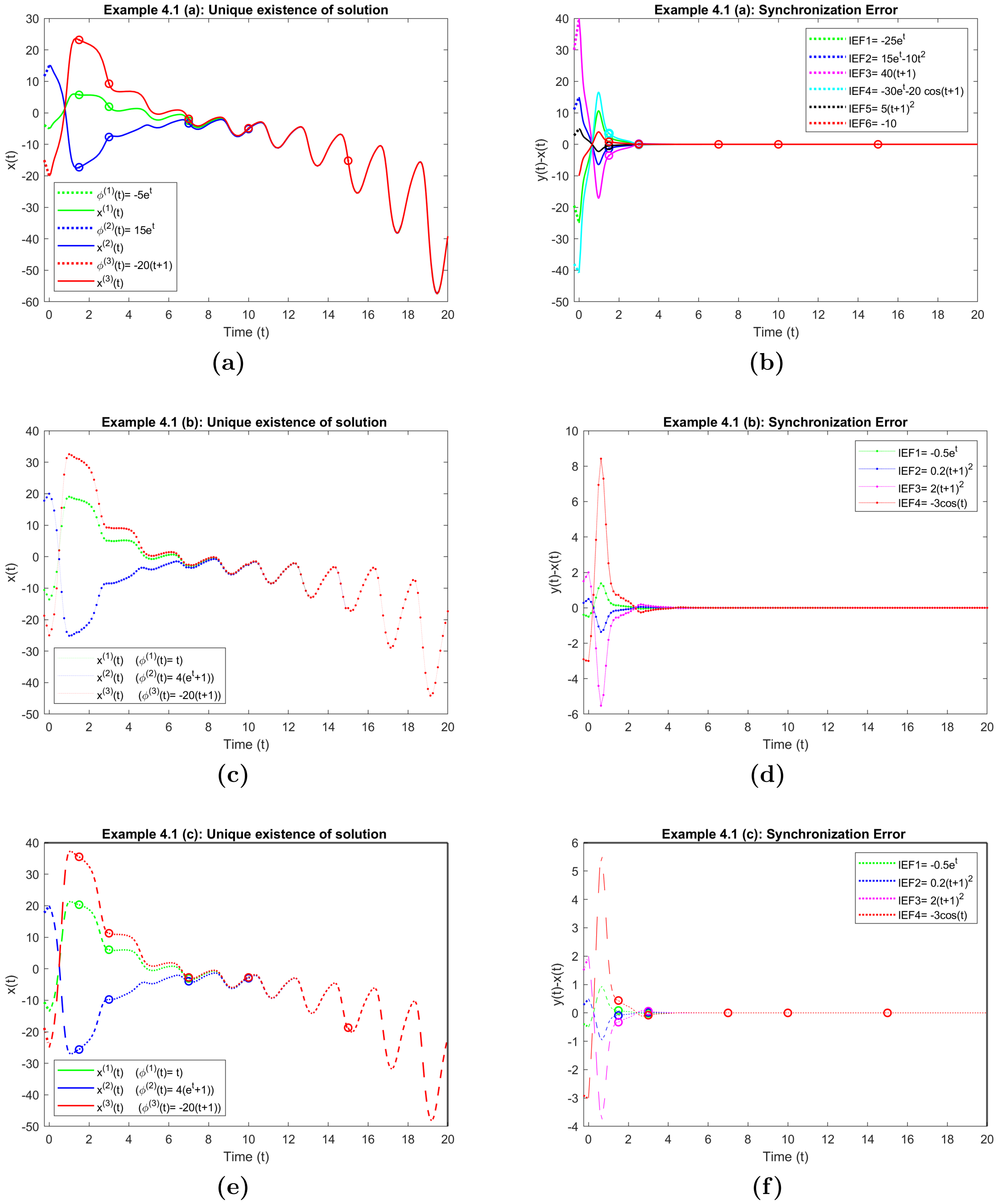


Figure 5.1: Trajectories of the solution of system (5.3.1) on different time scales with coefficients given in Example 5.5.1 are on the left. Here, for $k = 1, 2, ...,$ $x^{(k)}(t)$ denotes weighted $(2, 1.5)$- pseudoperiodic solution of the system corresponding to the initial function $\phi^{(k)}(t)$. On the right side, the synchronization error, which is difference of solutions of (5.4.4) and (5.3.1), is plotted for different initial error functions (IEF). In the figures, the circles mark the impulsive points where the solution experiences jump discontinuities.

$$\gamma(t) = 0.0625 + \frac{|\sin(\pi t)|}{16}, \quad \tau(t) = 0.125 + \frac{|\cos(\pi t)|}{8},$$

$$f(t,x) = \frac{0.004}{3}\left(c^{t/2}\cos(\pi t) + x + \frac{x}{(1+|t|)^2}\right), \quad I_j(x(t_j)) = 0.008\left(\frac{2}{3}\right)^j x(t_j)$$

and the impulses applied at time $t_j = 1.5, 3, 7, 10, 15$.

For $c = \frac{3}{2}$, *and* $\mathbb{T} = \mathbb{R}$, *we can easily compute the bounds of coefficients as*

$$\omega = 2, \quad \underline{\mathsf{a}} = 0.0018, \quad \bar{\mathsf{a}} = 0.0136, \quad \underline{\mathsf{b}} = 0.11776, \quad \bar{\mathsf{b}} = 0.88 < 1, \quad \bar{\gamma} = \frac{1}{8}, \quad \bar{\tau} = \frac{1}{4},$$

$$l_f = 0.004, \quad \text{and} \quad \sum_{j=1}^{q} l_j \leq \sum_{j=1}^{\infty} 0.008\left(\frac{2}{3}\right)^j = 0.016.$$

Similar to Example 5.5.1 , with the weights $\varrho = 1 + t^2$ *and* $\nu = (1+|t|)^2$, *conditions* ***H1***–***H5*** *are satisfied for the above coefficients. Furthermore, we obtain* $\mathfrak{C} \approx 0.92 < 1$. *Consequently, by Theorem 5.4.4, system* (5.3.1) *admits a unique weighted pseudo* $(2, \frac{3}{2})$*-periodic solution on the time scale* $\mathbb{R}$ *(see Figure 5.2).*

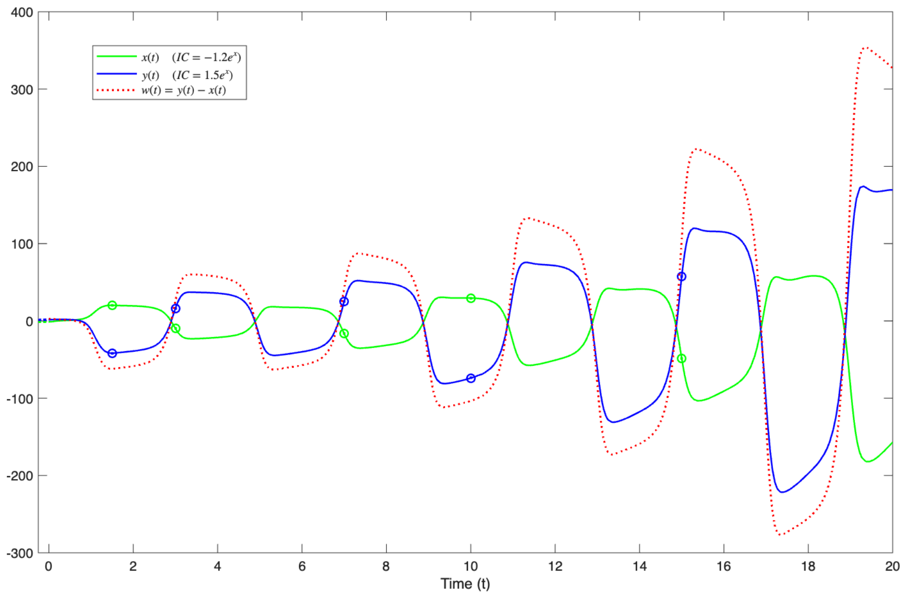


Figure 5.2: Solution of system (5.3.1), with coefficients in Example 5.5.2, corresponding to two different initial conditions and the associated synchronization error (without control). Here, $x(t)$ denotes the solution of system (5.3.1) with initial function $-1.2e^t$, $y(t)$ denotes the solution of same system with initial function $1.5e^t$, and $w(t)$ represents the synchronization error, that is, the solution of error system (5.4.7) (without control term, that is, $\mathcal{C}(t) \equiv 0$) with initial error function $2.7e^t$.

However, Figure 5.2 clearly shows that the system fails to synchronize, since the synchronizaion error corresponding to different initial conditions increases with time. This motivates the introduction of a control term satisfying conditions ***H7***–***H8***. *We consider the control function* (5.4.6) *with*

$$a(t) = 1.65 + 0.05|\cos(\pi t)|, \quad b(t) = \frac{\sin(\pi t)}{20} + \frac{\cos(\sin(t))}{(1+|t|)^2}, \quad \tilde{\gamma}(t) = 0.1875 + 0.0625\cos(\pi t),$$

$$\tilde{\tau}(t) = \frac{1}{8}\left(1 + |\sin(\pi t)|\right), \quad \tilde{f}(t,x) = 0.01\left(x + \frac{\sin(x)}{(1+|t|)^2}\right),$$

so that we have

$$\underline{a} = 1.65, \quad \bar{a} = 1.7, \quad \underline{b} = -0.05, \quad \bar{b} = 0.05, \quad \bar{\tilde{\gamma}} = \frac{1}{4}, \quad \bar{\tilde{\tau}} = \frac{1}{4}, \quad L_{\tilde{f}} = 0.03,$$

$$\alpha = 0.80365, \quad \beta = 0.6073, \quad \tilde{\alpha} = 0.80365, \quad \tilde{\beta} = 0.6073, \quad \theta = \frac{1}{4}, \quad \tilde{\theta} = \frac{1}{4}.$$

and for $\lambda = 0.05$, we obtain $\mathfrak{C}^* \approx -0.11 < 0$. Thus by Theorem 5.4.6, with the given coefficients in this example, (5.3.1) and (5.4.4) are globally exponentially synchronized on the time scale $\mathbb{R}$ (see Figure 5.3).

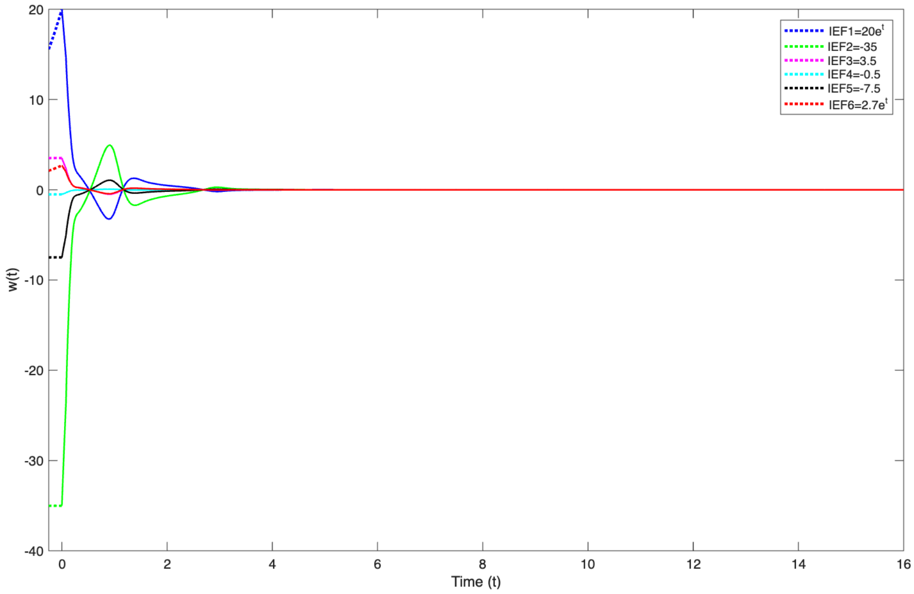


Figure 5.3: Synchronization errors for controlled systems. This figure illustrates the solutions of the error system (5.4.7), with coefficients in Example 5.5.2, for various initial error functions, confirming that synchronization is achieved under the proposed control scheme.

Example 5.5.3. *Consider the drive system (5.3.1) and the response system (5.4.4) on the time scale $\mathbb{T} = \mathbb{R}$, where the coefficients are given as*

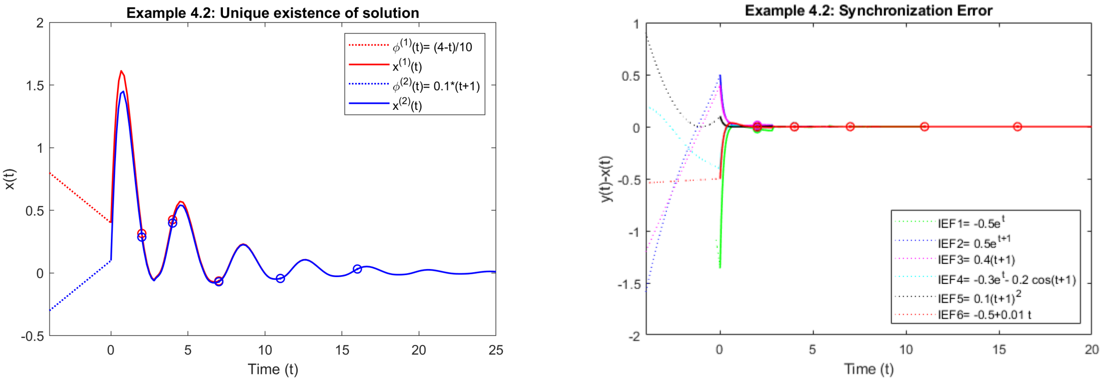
$$\mathtt{a}(t) = 0.6 + 0.4|\sin(\pi t/4)|, \quad \gamma(t) = 2 + |\sin(\pi t/4)|, \quad \tau(t) = 4 + |\cos(\pi t/4)|,$$

$$\mathtt{b}(t) = 0.15\left(\frac{\sin(\pi t/2)}{20} + \frac{\sin(\cos(t))}{e^{t^2}}\right), \quad \mathtt{e}(t) = c^{t/4}\cos(\pi t/2) + c^{t/4}\frac{e^{\cos^2(t)}}{1+t^2},$$

$$f(t,x) = 0.03\left(c^{t/4}\cos(\pi t) + x - \frac{x}{(1+|t|)^2}\right), \quad I_j(x(t_j)) = 2\,(0.042)^j\, x(t_j),$$

$$a(t) = 5 + \cos(\pi t/2), \quad \tilde{\gamma}(t) = 2 + |\cos(\pi t/4)|, \quad \tilde{\tau}(t) = 2 + |\sin(\pi t/4)|,$$

$$b(t) = \frac{1}{150}\left(\sin(\pi t/2) + e^{-|t|}\right), \quad \tilde{f}(t,x) = 0.025\left(x + \frac{x}{1+t^2}\right),$$



(a) **(b)**

Figure 5.4: This figure contains the solution trajectories $(x^{(k)}(t))$ of (5.3.1) with the coefficients on $\mathbb{R}$ as in Example 5.5.3 and initial functions $\phi^{(k)}(t)$ (left), and synchronization error for different initial error functions (right).

with impulses imposed on time $t_j = 2, 4, 7, 11, 16$. *We have*

$$\omega = 4, \quad \underline{\mathsf{a}} = 0.6, \quad \bar{\mathsf{a}} = 1, \quad \bar{\mathsf{b}} = 0.13, \quad \underline{\gamma} = 2, \quad \underline{\tau} = 4,$$

$$l_f = 0.06, \quad \sum_{j=1}^{q} l_j \le \sum_{j=1}^{\infty} = 0.09,$$

$$\underline{a} = 4, \quad \bar{a} = 6, \quad \bar{b} = 0.01, \quad L_{\tilde{f}} = 0.05,$$

$$\alpha = \beta = 0.2146, \quad \tilde{\alpha} = \tilde{\beta} = 0.2146, \quad \theta = 4, \quad \tilde{\theta} = 3.$$

It can be easily verified that with respect to the weights $\varrho = \frac{1}{2} + e^{-|t|}$ *and* $\nu = \frac{1}{2}$, *the conditions* ***H1***–***H5*** *are satisfied for the given coefficients. Furthermore, consider* $c = \frac{1}{2}$ *and* $\lambda = 0.14$, *then we have*

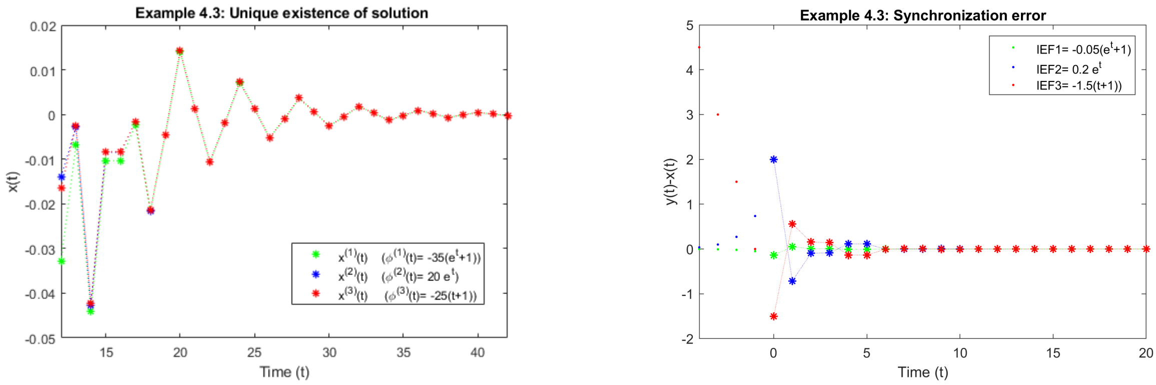
$$\mathfrak{C} < 0.4454 < 1 \quad \text{and} \quad \mathfrak{C}^* < -1.5474 < 0.$$

Thus, there exists a unique weighted pseudo $(4, \frac{1}{2})$*-periodic solution for* (5.3.1) *that is globally exponentially* (ω, c)*-synchronized with the solution of* (5.4.4) *(see Figure 5.4).*

Example 5.5.4. *For* $\mathbb{T} = \mathbb{Z}$, *the drive system* (5.3.1) *and the response system* (5.4.4), *where*

$$\gamma \equiv 2, \quad \tau \equiv 4, \quad \tilde{\tau} \equiv 2, \quad \tilde{\gamma} \equiv 2, \quad a(t) = 0.8 + 0.5\cos(\pi t/2),$$

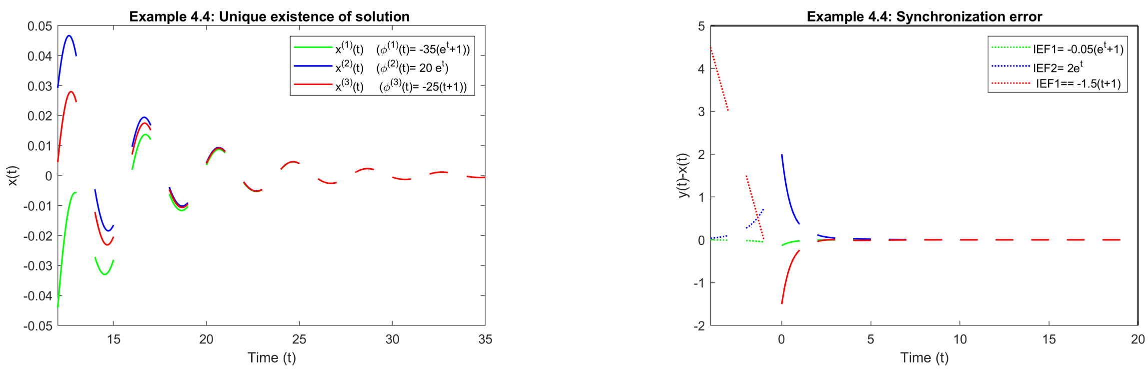
and all other coefficients are the same as given in Example 5.5.4. We can calculate $\mathfrak{C} = 0.9380 < 1$ *and* $\mathfrak{C}^* < -0.2957 < 0$. *Since all the requirements for Theorems 5.4.4 and 5.4.6 are satisfied, there exists a unique weighted pseudo* $(4, \frac{1}{2})$*-periodic solution for* (5.3.1) *that is globally exponentially synchronized with the solution of* (5.4.4) *on the considered time scale (see Figure 5.5).*



(a) **(b)**

Figure 5.5: This figure contains the solution trajectories $(x^{(k)}(t))$ of system (5.3.1) with the coefficients on $\mathbb{Z}$ as in Example 5.5.4 and initial functions $\phi^{(k)}(t)$ (left), and synchronization error for different initial error functions (right). For better visualization of the solution's behavior, the figures are shown for the existence result over the range $[15, 40]$. It can be observed that when time exceeds approximately 19, the solution begins to exhibit a behavior close to $(4, \frac{1}{2})$-periodic structure, although it was not so initially.

Example 5.5.5. *For* $\mathbb{T} = \bigcup_{n\in\mathbb{Z}} [2n, 2n+1]$, *with the coefficients given in Example 5.5.4, the weighted pseudo* $(4, \frac{1}{2})$*-periodic solutions of the drive system (5.3.1) and the response system (5.4.4) are exponentially synchronized (see Figure 5.6).*



(a) **(b)**

Figure 5.6: This figure contains the solution trajectories $(x^{(k)}(t))$ of system (5.3.1) with the coefficients on $\bigcup_{n\in\mathbb{Z}} [2n, 2n+1]$ as in Example 5.5.5 and initial functions $\phi^{(k)}(t)$ (left), and synchronization error for different initial error functions (right).

5.6 Conclusion

This chapter has investigated a class of neutral-type delayed evolution systems with piecewise impulsive effects on time scales, which arise naturally in the modeling of more general and realistic physical phenomena. Sufficient conditions for the existence of (doubly) weighted pseudo (ω, c)-periodic solutions have been established using the Banach fixed-point theorem. Since the notion of (doubly) weighted pseudo (ω, c)-periodicity has not been studied in the existing literature, this chapter has introduced the concept on general periodic time scales and has developed its fundamental analytical framework. Several essential properties, lemmas, and theorems have been derived to support the main existence results. Furthermore, by employing Lyapunov function methods, global exponential synchronization of weighted pseudo (ω, c)-periodic solutions has been obtained for neutral-type impulsive evolution systems. The theoretical results have been verified through several numerical examples and their graphical simulations on different time scales. The examples further verify that, even though the impulses are not weighted pseudo (ω, c)-periodic functions, the solution of the impulsive dynamical system can still exhibit weighted pseudo (ω, c)-periodic behavior. Moreover, the examples have illustrated how the introduction of a control term affects the system and leads to synchronization.

CHAPTER 6

S-Asymptotically (ω, c)-Periodic Solutions for Shunting Inhibitory Stochastic Delayed Cellular Neural Networks on Time Scales [1]

This chapter primarily investigates the stability of p-th mean S-asymptotically (ω, c)-periodic shunting inhibitory stochastic cellular neural networks with time-varying delays on an arbitrary periodic time scale. We introduce the concept of S-asymptotically (ω, c)-periodic processes on time scales and examine their fundamental properties. Considering that the system of dynamic equations concerned is driven by the Wiener process, we analyze the stability under the p-th mean exponential stability criterion in a linear expectation framework.

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6.1 Introduction

S-asymptotic periodicity is a generalization of periodicity that accommodates certain distortions in dynamical phenomena which cannot be captured by strict periodicity or some of its classical extensions. Consequently, the investigation of such generalized periodic behavior within broader dynamical frameworks becomes essential for accurately describing complex system dynamics. In practical applications, such systems are often influenced by random perturbations, and their dynamics tend to remain close to those of the corresponding deterministic models. These random disturbances can be naturally modeled by incorporating stochastic processes into the associated dynamic equations, leading to more realistic and accurate representations of the underlying phenomena. The inclusion of stochastic effects in dynamic equations on time scales naturally motivates the further extension of (ω, c)-periodicity, discussed in Chapter 2, to characterize stochastic processes on time scales, thereby providing a powerful tool for analyzing more realistic and complex dynamical behaviors.

Shunting inhibitory cellular neural networks (SICNNs), first introduced in 1993, have been widely applied in psychophysics, robotics, adaptive pattern recognition, optimization, vision, and image processing. Given their wide applicability, the study of their dynamical behaviors has attracted considerable research attention. Since the dynamical properties of neural networks are fundamental to their design, implementation, and application, understanding their oscillatory behavior is of great importance. Consequently, many studies have investigated the existence and stability of equilibrium points, as well as the occurrence of generalized periodic solutions of SICNNs [221, 222, 223]. Nevertheless, real nervous systems are inevitably influenced by stochastic perturbations and time delays arising from internal and external sources. These random fluctuations can significantly influence the dynamics of neural networks; sometimes stabilizing and, at other times, destabilizing the system; making the stability analysis of stochastic delayed neural networks essential for both theoretical understanding and practical applications [227]. Furthermore, (ω, c)-periodicity is an important dynamics of neural networks. Therefore, it is meaningful to study the (ω, c)-periodic and perturbed (ω, c)-periodic oscillations of SICNNs that incorporate time-delay effects and stochastic disturbances, as well as to analyze their stability properties.

In recent years, there has been growing interest in exploring generalized periodic behaviors in stochastic dynamical systems, with particular focus on the existence, uniqueness, and stability of different classes of solutions. Several studies have investigated almost automorphic and Stepanov-like weighted pseudo-almost automorphic solutions for stochastic cellular neural networks on time scales in the square-mean sense [92, 202]; the exponential stability of the unique almost periodic solution of Clifford-valued stochastic delayed neural networks [224]; square-mean robust exponential stability of discrete-time uncertain impulsive stochastic neural networks [225]; p-th-moment stability of stochastic Hopfield neural networks driven by G-Brownian motion [226]; and Stepanov almost periodic solutions of stochastic delayed SICNNs [227], among others. In addition, the existence of a unique periodic solution for stochastic differential equations driven by Levy noise has been investigated in [228]. Stepanov-like pseudo S-asymptotically (ω, c)-periodic solutions for a class of stochastic integro-differential equations have been analyzed in [229]. More recently, Stepanov-like weighted pseudo-S-asymptotically (ω, c)-periodic solutions for a class of fractional stochastic differential equations have been investigated in [230].

From the above literature survey, it is evident that research on generalized periodicity on time scales has focused primarily on the deterministic setting. Although some studies have investigated (ω, c)-periodicity in the continuous setting for nondeterministic systems, to the best of our knowledge, no results have been established in the stochastic framework regarding the existence, uniqueness, and stability of (ω, c)-periodic stochastic processes on time scales. Motivated by these observations, this chapter aims to develop an S-asymptotic (ω, c)-periodic analysis for a class of nonlinear systems governed by stochastic dynamic equations. In particular, we investigate stochastic SICNNs with time delays within the framework of time scales and derive conditions that ensure the existence of unique and stable S-asymptotically (ω, c)-periodic solutions in the p-th mean sense ($p \geq 2$), which provide a more comprehensive characterization of the stochastic dynamics by capturing variability and large deviations that the mean alone cannot reflect.

The organization of this chapter is as follows. Section 6.2 presents the notation, conventions, and preliminary results that will be used in the subsequent analysis. Section 6.3 briefly describes the stochastic SICNN model incorporating discrete and infinite-distributed delays on time scales. In Section 6.4, a comprehensive theory of S-asymptotically (ω, c)-periodic stochastic processes on time scales is developed in the p-th mean sense, providing the essential definitions and fundamental properties of such processes. Section 6.5 derives sufficient conditions ensuring the existence and uniqueness (in the pathwise sense) of p-th mean S-asymptotically (ω, c)-periodic solutions for stochastic SICNNs on time scales. The p-th mean exponential stability of this unique S-asymptotically (ω, c)-periodic solution has also been investigated in this Section. Section 6.6 presents numerical examples and simulations to validate the theoretical results. Finally, the chapter is concluded.

6.2 Notation, conventions and preliminary results

Throughout this chapter, $\mathbb{T}$ denotes a time scale, while $\mathbb{T}_\omega$ denotes an ω-periodic time scale. Let us assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}^{0+}}, \mathbb{P})$ denotes a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}^{0+}}$ that satisfies the usual conditions (it is increasing, rd-continuous, and $\mathcal{F}_{t_0}$ contains all $\mathbb{P}$-null sets, where $t_0 = \inf \mathbb{T}^{0+}$). Let $W(t, \zeta) = (W_1(t, \zeta), \dots, W_m(t, \zeta))^T$, $t \in \mathbb{T}, \zeta \in \Omega$, denote an m-dimensional Wiener process defined on this probability space. Furthermore, let $(\mathbb{H}_\mathbb{F}, \|\cdot\|)$ be a separable Hilbert space over the scalar field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. The expectation operator $\mathbb{E}[\cdot]$ denotes integration with respect to the probability measure $\mathbb{P}$ on Ω. For $p \in [1, \infty)$, $\mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$ denotes the space of all strongly measurable p-th integrable $\mathbb{H}_{\mathbb{F}}$-valued random variables X satisfying

$$\mathbb{E}\|X\|^p = \int_{\Omega} \|X(\zeta)\|^p d\mathbb{P}(\zeta) < \infty.$$

For $X \in \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$, we define $\|X\|_p := (\mathbb{E}\|X\|^p)^{1/p}$. It is importent to note that $\mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$, equipped with the norm $\|\cdot\|_p$, is a Banach space.

A stochastic process $\mathcal{X} : \mathbb{T}^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$, is said to be *stochastically continuous* (in the p-th mean) if, for every $s \in \mathbb{T}^{0+}$,

$$\lim_{t \to s} \mathbb{E}\big\|\mathcal{X}(t) - \mathcal{X}(s)\big\|^p = 0.$$

$\mathcal{X}$ is said to be *stochastically bounded* if there exists a constant $M > 0$ such that

$$\mathbb{E}\big\|\mathcal{X}(t)\big\| \leq M \quad \text{for all } t \in \mathbb{T}^{0+}.$$

We denote the space of all rd-continuous and bounded stochastic processes $\mathcal{X} : \mathbb{T}^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$ by $\mathrm{BC}_{\mathrm{rd}}(\mathbb{T}^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$. Equipped with the norm $\sup_{t \in \mathbb{T}_{\omega}^{0+}} (\mathbb{E}||\mathcal{X}(t)||^p)^{1/p}$, the space $\mathrm{BC}_{\mathrm{rd}}(\mathbb{T}^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is a Banach space.

Next we present some fundamental results that will be used in later sections. These results form a basis for stochastic analysis on time scales in Hilbert spaces and will be applied to our investigation of stochastic SICNNs.

Lemma 6.2.1. *Let $a_1, a_2, \ldots, a_n \in \mathbb{H}_{\mathbb{F}}$ and $p \geq 1$. Then*

$$\|a_1 + a_2 + \ldots + a_n\|^p \leq n^{p-1} \left(\|a_1\|^p + \|a_2\|^p + \ldots + \|a_n\|^p\right).$$

The inequality above is a direct consequence of the convexity of the function $x \mapsto \|x\|^p$ and will be frequently applied in subsequent sections.

Lemma 6.2.2. *Let $W = \{W(t) : t \in \mathbb{T}^{0+}\}$ be an $\mathbb{H}_{\mathbb{F}}$-valued Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}, \mathbb{P})$. Let $\mathcal{X} : [0, T]_{\mathbb{T}} \to \mathcal{L}^2(\Omega, \mathbb{H}_{\mathbb{F}})$ be an $\{\mathcal{F}_t\}_{t \in \mathbb{T}^{0+}}$-adapted, rd-continuous stochastic process satisfying*

$$\mathbb{E}\left[\int_0^T \|\mathcal{X}(s)\|^2 \, \Delta s\right] < \infty.$$

Then the following assertions hold:

(i) The stochastic integral

$$\int_0^t \mathcal{X}(s) \, \Delta W(s), \quad t \in [0, T]_{\mathbb{T}},$$

is an $\mathbb{H}_{\mathbb{F}}$-valued, rd-continuous, square-integrable martingale, and the Itô isometry on time scales *holds:*

$$\mathbb{E}\Big\|\int_0^t \mathcal{X}(s) \, \Delta W(s)\Big\|^2 = \mathbb{E}\left[\int_0^t \|\mathcal{X}(s)\|^2 \, \Delta s\right], \quad t \in [0, T]_{\mathbb{T}}.$$

(ii) If $p > 2$, there exists a constant $C_p > 0$ such that the Burkholder–Davis–Gundy inequality *holds:*

$$\mathbb{E}\left(\sup_{0\le t\le T}\Big\|\int_0^t \mathcal{X}(s)\,\Delta W(s)\Big\|^p\right) \le C_p\,\mathbb{E}\left(\int_0^T \|\mathcal{X}(s)\|^2\,\Delta s\right)^{p/2}.$$

Proof. Let $\mathcal{S}$ be the class of simple predictable processes

$$\mathcal{X}^{(n)}(t,\zeta) = \sum_{k=1}^{n} \xi_{k-1}(\zeta)\,\mathbf{1}_{(t_{k-1},t_k]_{\mathbb{T}}}(t),$$

on $[0,T]_{\mathbb{T}}$ with partition $0 = t_0 < t_1 < \cdots < t_n = T$, where each $t_k \in \mathbb{T}$ and $\xi_{k-1} \in \mathbb{H}_{\mathbb{F}}$ is $\mathcal{F}_{t_{k-1}}$-measurable. For $t \in [0,T]_{\mathbb{T}}$, we define

$$I_n(t) := \int_0^t \mathcal{X}^{(n)}(s)\,\Delta W(s) := \sum_{k=1}^{n} \xi_{k-1}\big(W(\min(t_k,t)) - W(\min(t_{k-1},t))\big).$$

Using independence and zero mean of disjoint Wiener increments, we can easily compute

$$\begin{aligned}
\mathbb{E}\|I_n(t)\|^2 &= \sum_{k=1}^{n} \mathbb{E}\Big[\|\xi_{k-1}\|^2\,\mathbb{E}\big[(W(\min(t_k,t)) - W(\min(t_{k-1},t)))^2\big]\Big] \\
&= \sum_{k=1}^{n} \mathbb{E}\Big[\|\xi_{k-1}\|^2\big((\min(t_k,t)) - (\min(t_{k-1},t))\big)\Big] \\
&= \mathbb{E}\Big[\int_0^t \|\mathcal{X}^{(n)}(s)\|^2\,\Delta s\Big].
\end{aligned}$$

This implies that the Itô isometry holds for simple processes. Next, consider

$$\mathcal{H} := \left\{\mathcal{Y} : \ \mathcal{Y} \text{ is progressively measurable and } \mathbb{E}\left[\int_0^T \|\mathcal{Y}(t)\|^2 \Delta t\right] < \infty\right\}$$

with inner product $\langle \mathcal{Y}_1, \mathcal{Y}_2\rangle_{\mathcal{H}} := \mathbb{E}\left[\int_0^T \langle \mathcal{Y}_1(t), \mathcal{Y}_2(t)\rangle \Delta t\right]$. Then, by arguments similar to those in the classical theory, it can be easily shown that $\mathcal{S}$ is dense in $\mathcal{H}$ with respect to the norm $\|\cdot\|_{\mathcal{H}}$ induced by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

Let $\{\mathcal{X}^{(n)}\} \subset \mathcal{S}$ and $\|\mathcal{X}^{(n)} - \mathcal{X}\|_{\mathcal{H}} \to 0$. By the isometry for simple processes, for each fixed $t \in [0,T]_{\mathbb{T}}$,

$$\mathbb{E}\|I_n(t) - I_m(t)\|^2 = \mathbb{E}\Big[\int_0^t \|\mathcal{X}^{(n)}(s) - \mathcal{X}^{(m)}(s)\|^2\,\Delta s\Big] \to 0,$$

imply that $\{I_n(t)\}$ is Cauchy in $\mathcal{L}^2(\Omega;\mathbb{H}_{\mathbb{F}})$ and converges to a limit $M(t)$. For $t \in [0,T]_{\mathbb{T}}$, we define

$$M(t) := \int_0^t \mathcal{X}(s)\,\Delta W(s)$$

as $\mathcal{L}^2$-limit of $I_n(t)$. Taking the limit in the isometry relation of I_n gives the Itô isometry as

$$\mathbb{E}\|M(t)\|^2 = \mathbb{E}\Big[\int_0^t \|\mathcal{X}(s)\|^2\,\Delta s\Big], \qquad \forall t \in [0,T]_{\mathbb{T}}.$$

Note that each $I_n(t)$ is a square-integrable $\{\mathcal{F}_t\}$-martingale because the increments after time u are independent of $\mathcal{F}_u$ and have mean zero, which forces the conditional expectation of future increments to vanish and implies $\mathbb{E}[I_n(t)|_{\mathcal{F}_u}] = I_n(u)$. Now, we decompose $[0,T]_{\mathbb{T}}$ into the left-dense part $\bigcup_j (a_j, b_j]$ and the set S of left-scattered points. In each subinterval $(a_j, b_j]$, the integral coincides with the classical Itô integral and is thus almost surely continuous there. At a left-scattered point $0 < s \in S$, the left-endpoint construction of the simple integrals yields a jump

$$\Delta M(s) = M(s) - M(\rho(s)) = \mathcal{X}(\rho(s))\big(W(s) - W(\rho(s))\big),$$

and the limiting process retains this structural form. Hence M is rd-continuous with explicit Gaussian jumps at left-scattered points. Since M has the predictable quadratic variation given as

$$[M]_t \;=\; \int_0^t \|\mathcal{X}(s)\|^2 \, \Delta s,$$

the *Burkholder–Davis–Gundy inequality* yields

$$\mathbb{E}\Big(\sup_{0\le t\le T} \|M(t)\|^p\Big) \le C_p \, \mathbb{E}\Big(\int_0^T \|\mathcal{X}(s)\|^2 \, \Delta s\Big)^{p/2}.$$

□

6.3 Model statement

In SICNNs, the processing units, referred to as cells, arranged in a two-dimensional array, where each cell interacts only with its neighboring units. This biologically inspired structure enables SICNNs to capture both local connectivity and nonlinear inhibitory effects, making them suitable for modeling complex dynamical behaviors. To mathematically describe such dynamics in an arbitrary time domain, which incorporates leakage, shunting inhibition, external input, time-varying delays, and stochastic perturbations, consider a two-dimensional grid of processing cells. Let $C_{ij}, i = 1, \dots, m,\ j = 1 \dots n$, denote the cell at the $(i,j)^{\text{th}}$-position of the lattice and $N_{r_1}(i,j)$ represent the r_1–neighborhood of C_{ij} such that

$$N_{r_1}(i,j) = \Big\{ C_{kl} : \max\{|k-i|, |l-j|\} \le r_1, 1 \le k \le m, 1 \le l \le n \Big\},$$

and $N_{r_2}(i,j)$ and $N_{r_3}(i,j)$ are specified similarly. In SICNNs, interactions between cells arise through recurrent shunting inhibition, where the inhibitory conductance (synaptic weight) of each cell is modulated by the membrane potentials (voltages) of its neighboring cells. Consequently, the dynamics of each cell C_{ij} of stochastic SICNNs, driven by the Wiener process, with discrete and infinitely distributed time-varying delays on a time scale, is governed by the following nonlinear dynamic equation:

$$\Delta x_{ij}(t) \quad = \quad \Big[- a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t - \eta_{kl}(t)))x_{ij}(t) \qquad (6.3.1)$$

$$
\begin{aligned}
&- \sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t) \int_0^\infty \mathcal{K}_{ij}(u) g(x_{kl}(t-u)) \Delta u x_{ij}(t) + L_{ij}(t) \Bigg] \Delta t \\
&+ \sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(t) \delta_{ij}(t, x_{ij}(t)) \Delta w_{ij}(t), \quad t \geq 0,\ t \in \mathbb{T}_\omega.
\end{aligned}
$$

Here $\mathbb{T}_\omega$ is an ω-periodic time scale and x_{ij} denotes the activity of the cell C_{ij}. The term $L_{ij}(t)$ represents the external input to C_{ij}; while $a_{ij}(t) > 0$ is the passive decay rate of cell activity. The coefficients $C_{ij}^{kl}, B_{ij}^{kl}, D_{ij}^{kl} \geq 0$ represent the coupling (connection) strengths of postsynaptic activity transmitted from the cell C_{kl} to the cell C_{ij}. The activation functions $f(x_{kl})$ and $g(x_{kl})$ are continuous positive functions that describe the output or firing rate of the cell C_{kl}. The term η_{ij} corresponds to the transmission delay and $\mathcal{K}_{ij}$ is the kernel function that governs the distributed delay in cell (i,j). Stochastic perturbations are represented by the $m \times n$-dimensional Wiener process

$$
w(t) = (w_{11}(t), w_{12}(t)), \ldots, w_{mn}(t))^T
$$

defined in a complete probability space, and the diffusion coefficients δ_{ij} are Δ-measurable functions on the time scale $\mathbb{T}_\omega$.

The initial condition associated with (6.3.1) is defined as follows:

$$
x_{ij}(s) = \psi_{ij}(s), \quad s \in (-\infty, 0]_{\mathbb{T}_\omega},
$$

where $i = 1, \ldots, m$, $j = 1 \ldots n$, and ψ_{ij} are Δ-differentiable.

Since $\mathbb{T}_\omega$ is periodic, the graininess operator μ is bounded. We use the notation $\bar{\mu}$ and $\underline{\mu}$ to denote the supremum and infimum of μ, respectively. Throughout this paper, we impose the following assumptions on the model parameters, including all coefficients, kernel functions, activation functions, delays, and diffusion terms:

(A1) Let $p \geq 2$. There exist $c \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$ such that for each i, j

- **(i)** $a_{ij} \in \mathrm{C_{rd}}(\mathbb{T}_\omega, \mathbb{R})$ are ω-periodic, positively regressive functions, bounded below by positive constants $\underline{a}_{ij}$ satisfying $\underline{a}_{ij} \in \mathcal{R}^+$ and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{a}_{ij}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{a}_{ij}\bar{\mu})}$.
- **(ii)** $\eta_{ij} \in \mathrm{C_{rd}}(\mathbb{T}_\omega, \Pi^{0+}_{\mathbb{T}_\omega})$ are ω-periodic functions.
- **(iii)** $C_{ij}^{kl}, B_{ij}^{kl}, D_{ij}^{kl} \in \mathrm{C_{rd}}(\mathbb{T}_\omega, \mathbb{F}^+)$ are bounded ω-periodic functions with upper bounds $\bar{C}_{ij}^{kl}, \bar{B}_{ij}^{kl}$ and $\bar{D}_{ij}^{kl}$, respectively.
- **(iv)** $L_{ij} : \mathbb{T}_\omega \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ are p-th mean S-asymptotically (ω, c)-periodic stochastic processes with $\sup\limits_{t\in\mathbb{T}_\omega} \mathbb{E}\|c^{-t/\omega} L_{ij}(t)\|^p = \hat{L}_{ij}^p < \infty$ for some constants $\hat{L}_{ij}$.

(A2) The activation functions $f, g : \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}) \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ are bounded almost surely and Lipschitz continuous with respect to the norm in the sense that we can always find positive constants $\hat{f}$, $\mathtt{L}_f$,

$\hat{g}$, and $\mathtt{L}_g$ such that for all $\mathcal{X}, \mathcal{Y}$ satisfying $c^{-t/\omega}\mathcal{X}(t), c^{-t/\omega}\mathcal{Y}(t) \in (\mathbb{T}_\omega, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ we have

$$\sup_{\mathbb{T}_\omega} \|f\|_\infty \le \hat{f}, \quad \left(\mathbb{E}\|f(\mathcal{X}(t)) - f(\mathcal{Y}(t))\|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \le \mathtt{L}_f \mathbb{E}\left(\|c^{-t/\omega}(\mathcal{X}(t) - \mathcal{Y}(t))\|^p\right)^{1/p},$$

and

$$\sup_{\mathbb{T}_\omega} \|g\|_\infty \le \hat{g}, \quad \left(\mathbb{E}\|g(\mathcal{X}(t)) - g(\mathcal{Y}(t))\|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \le \mathtt{L}_g \mathbb{E}\left(\|c^{-t/\omega}(\mathcal{X}(t) - \mathcal{Y}(t))\|^p\right)^{1/p},$$

with $f(0) = g(0) = 0$. Here $\|\cdot\|_\infty := \text{ess sup}_\Omega \|\cdot\|$.

(A3) For each $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $\delta_{ij}(t) : \mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}) \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ are continuous, (ω, c)-stochastically bounded uniformly for all $\mathcal{X} \in \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$, p-th mean S-asymptotically (ω, c)-periodic and Lipschitz with respect to the second variable in the sense that

$$\sup_{t\in\mathbb{T}_\omega} (\mathbb{E}||c^{-t/\omega}\delta_{ij}(t, \mathcal{X}(t))||^p)^{1/p} \le \hat{\delta}_{ij}, \quad (\mathbb{E}||c^{-t/\omega}\left(\delta_{ij}(t, \mathcal{X}(t)) - \delta_{ij}(t, \mathcal{Y}(t))\right)||^p)^{1/p} \le \ell_{ij}(\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t) - \mathcal{Y}(t))\|^p)^{1/p}$$

for some $\hat{\delta}_{ij}, \ell_{ij} > 0$. In addition to this, we assume that all the x-dependent terms of $\delta_{ij}(t, x)$ vanish at $x = 0$.

(A4) $\mathcal{K}_{ij} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F}^+)$ satisfies $\int_0^\infty \mathcal{K}_{ij}(s)\Delta s \le \bar{\mathcal{K}}_{ij}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

The concept of p-th mean S-asymptotically (ω, c)-periodicity on time scales are introduced and discussed in the next section.

Remark 6.3.1. *The activation functions f, g and the diffusion coefficients δ_{ij} are taken to satisfy Lipschitz and linear growth conditions. Along with the assumptions taken on other coefficients, these conditions ensure that the solution is rd-continuous, $\mathcal{F}_t$-adapted and square-integrable. These assumptions guaranty well-defined stochastic integrals and allow the application of Itô isometry and Burkholder–Davis–Gundy inequalities.*

Remark 6.3.2. *The boundedness condition given in Assumption* (**A4**)*, ensures that the cumulative influence of the distributed delays remains finite, so the integral term in the system* (6.3.1) *remains bounded. This prevents the infinite distributed delays from accumulating unbounded effects that could destabilize the system. If the kernel decays too slowly or is non-integrable (e.g., $\mathcal{K}_{ij}(s) \approx s^{-\alpha}$ with $\alpha \le 1$ for large s), the integral could diverge, allowing delayed contributions to grow without bound and potentially destroy the stability of the system.*

6.4 p-th mean S-asymptotically (ω, c)-periodic stochastic processes on time scales

Extending the notion of (ω, c)-periodic functions, in this section we introduce the concepts of p-th mean S-asymptotically (ω, c)-periodic stochastic processes on time scales and establish some fundamental results.

Definition 6.4.1. *Consider the collection of stochastic processes $\mathcal{X} := \{\mathcal{X}(t) : t \in \mathbb{T}_\omega^{0+}\}$ such that for $c \in \mathbb{F}^*$ and $\omega \in \Pi_\mathbb{T}$, the mapping $c^{-t/\omega}\mathcal{X} : \mathbb{T}_\omega^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is well defined. We call such a stochastic process stochastically (ω, c)-bounded if there exists a positive number M such that*

$$\mathbb{E}\|c^{-t/\omega}\mathcal{X}(t)\|^p < M \quad \text{for all } t \in \mathbb{T}_\omega^{0+}.$$

We denote the space of all rd-continuous and (ω, c)-bounded stochastic processes $\mathcal{X}$, where $c^{-t/\omega}\mathcal{X} : \mathbb{T}_\omega^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is well defined, by $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$.

Proposition 6.4.2. *The space $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ equipped with the norm*

$$\|\mathcal{X}\|_\infty := \sup_{t \in \mathbb{T}_\omega^{0+}} (\mathbb{E}\|c^{-t/\omega}\mathcal{X}(t)\|^p)^{1/p}, \tag{6.4.1}$$

forms a Banach space for all $p \geq 1$.

Proof. It is easy to verify that $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ forms a vector space and $\|\cdot\|_\infty$ constitutes a norm on $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$. Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ with respect to the norm $\|\cdot\|_\infty$ and $\mathcal{Y}$ be the point-wise limit of $c^{-t/\omega}\mathcal{X}_n(t)$ with respect to $\|\cdot\|_p$, i.e.,

$$\lim_{n\to\infty} \|c^{-t/\omega}\mathcal{X}_n(t) - \mathcal{Y}(t)\|_p = 0 \quad \text{for all} \quad t \in \mathbb{T}_\omega^{0+}. \tag{6.4.2}$$

This limit $\mathcal{Y}$ always exists by the completeness of $\mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ with respect to $\|\cdot\|_p$. Since $(\mathcal{X}_n)$ is Cauchy with respect to $\|\cdot\|_\infty$, the convergence in (6.4.2) is uniform for $t \in \mathbb{T}_\omega^{0+}$.

Define $\mathcal{X}(t) = c^{t/\omega}\mathcal{Y}(t)$. Clearly, the mapping $c^{-t/\omega}\mathcal{X}(t) : \mathbb{T}_\omega^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is well defined. We now verify that $c^{-t/\omega}\mathcal{X}$ is stochastically rd-continuous. In fact, for $t, s \in \mathbb{T}_\omega^{0+}$ we have

$$\begin{aligned}\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t) - \mathcal{X}(s))\|^p \quad &\leq \quad 3^{p-1}\mathbb{E}\|c^{-t/\omega}(\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}(t)\|^p + 3^{p-1}\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}_n(s)\|^p \\ &\quad +3^{p-1}\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(s) - c^{-t/\omega}\mathcal{X}(s)\|^p.\end{aligned}$$

Thus, by the uniform convergence of $c^{-t/\omega}\mathcal{X}_n$ to $c^{-t/\omega}\mathcal{X}$ with respect to $\|\cdot\|_p$ and the stochastic rd-continuity of $c^{-t/\omega}\mathcal{X}_n$, the stochastic rd-continuity of $c^{-t/\omega}\mathcal{X}$ follows.

We further have

$$\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t)\|^p < M \quad \text{for some } M > 0.$$

The uniform convergence of $c^{-t/\omega}\mathcal{X}_n$ to $c^{-t/\omega}\mathcal{X}$ then implies that $\mathcal{X}$ is also stochastically (ω, c)-bounded in $\mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$ and

$$\lim_{n\to\infty}\sup_{t\in\mathbb{T}}\Big(\mathbb{E}(\|c^{-t/\omega}\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}(t)\|^p\Big)^{1/p} = 0.$$

From this it follows that $(\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is a Banach space with respect to the norm $\|\cdot\|_\infty$. □

Assuming $p \geq 1$, we next define the S-asymptotically (ω, c)-periodic processes on time scales in the p-th mean sense.

Definition 6.4.3. *A stochastic process $\mathcal{X} \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is said to be p-th mean S- asymptotically (ω, c)-periodic if*

$$\lim_{t\in\mathbb{T}_\omega^{0+},\ t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p = 0. \tag{6.4.3}$$

We denote the set of all p-th mean S-asymptotically (ω, c)-periodic stochastic processes in the space $(\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ by $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$.

Example 6.4.4. *Consider a stochastic process $\mathcal{X}(t,\zeta) = \frac{1}{2^{t/(2\pi)}}\big[\cos t + \sin(\sqrt{t})X(\zeta)\big]$, $t \in \mathbb{R}$, where $X \in N(0,1)$ is a real-valued random variable with a standard normal distribution. Then,*

$$\Big|\frac{1}{2^{-t/(2\pi)}}\Big(\mathcal{X}(t+2\pi) - \frac{1}{2}\mathcal{X}(t)\Big)\Big| = \Big|\frac{1}{2}(\sin(\sqrt{t+2\pi}) - \sin(\sqrt{t}))X\Big|.$$

Now, we have

$$\begin{aligned}\Big|\frac{1}{2}(\sin(\sqrt{t+2\pi}) - \sin(\sqrt{t}))\Big| &\leq \Big|\cos\Big(\frac{\sqrt{t+2\pi}+\sqrt{t}}{2}\Big)\sin\Big(\frac{\sqrt{t+2\pi}-\sqrt{t}}{2}\Big)\Big| \\ &\leq \frac{\sqrt{t+2\pi}-\sqrt{t}}{2} \leq \frac{\pi}{2\sqrt{t}} + O(1/\sqrt{t}) \ \to 0 \text{ as } t\to\infty.\end{aligned}$$

Also, $X \in N(0,1)$ implies that $\mathbb{E}|X|^p = 2^{p/2}\frac{\Gamma(p+1/2)}{\sqrt{\pi}} < \infty$ for any $p > -1$. Hence,

$$\lim_{t\to\infty}\mathbb{E}\Big|\frac{1}{2^{-t/2\pi}}(\mathcal{X}(t+2\pi) - \frac{1}{2}\mathcal{X}(t))\Big|^p = 0,$$

i.e., $\mathcal{X} \in S\mathrm{AP}_{2\pi,\frac{1}{2}}(\mathbb{R}_0^+, \mathcal{L}^p(\Omega, \mathbb{R}))$ for all $p \geq 1$.

Example 6.4.5. *Consider $\Omega = \bigcup_{n=0}^{\infty} I_n$, $I_n = [n, n+\frac{1}{2}]$, so that each interval I_n is equipped with Borel σ-algebra $\mathcal{B}$ and the normalized Lebesgue measure $\mathscr{M}_n(A) = \frac{\mathscr{M}(B)}{\mathscr{M}(I_n)} = 2\mathscr{M}(A)$, $A \subset I_n$, where $\mathscr{M}$ is the Lebesgue measure. Let us define the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B})$ by $\mathbb{P}(B) = \sum_{n=0}^{\infty}\frac{\mathscr{M}_n(B\cap I_n)}{2^{n-1}}$, $B \in \mathcal{B}$. For each $\zeta \in \Omega$, let $n(\zeta)$ denote the unique index with $\zeta \in I_{n(\zeta)}$. Now, Define a stochastic process $X : \mathbb{R}\times\Omega \to \mathbb{R}$ by*

$$X(t,\zeta) = 3^{t/2\pi}\exp(\sin t) + 3^{t/2\pi}h\big(t - n(\zeta)\big), \quad \textit{where} \ \ h(t) = \frac{1}{1+t} \ \textit{for} \ (t \geq 0).$$

We then have

$$\mathbb{E}\Big|3^{-t/(2\pi)}\left(\mathcal{X}(t+2\pi) - 3\mathcal{X}(t)\right)\Big|^2 = 9\int_\Omega \Big|\big(h(t+2\pi - n(\zeta)\big) - h\big(t - n(\zeta)\big)\big)\Big|^2 d\mathbb{P}(\zeta).$$

Now, using the normalization of μ_n, for a fixed $t \in \mathbb{R}$ we can have

$$\int_\Omega \Big|\big(h(t+2\pi - n(\zeta)\big) - h\big(t - n(\zeta)\big)\big)\Big|^2 d\mathbb{P}(\zeta) \quad = \quad \sum_{n=0}^{\infty} \frac{1}{2^{n-1}}\Big|\big(h(t+2\pi - n(\zeta)\big) - h\big(t - n(\zeta)\big)\big)\Big|^2.$$

Therefore, by dominated convergence theorem, we have

$$\lim_{t\to\infty} \mathbb{E}\Big|3^{-t/(2\pi)}\left(\mathcal{X}(t+2\pi) - 3\mathcal{X}(t)\right)\Big|^2 = 0,$$

which implies that $\mathcal{X} \in S\mathrm{AP}_{2\pi,3}(\mathbb{R}_0^+, \mathcal{L}^2(\Omega, \mathbb{R}))$.

Remark 6.4.6. *By the monotonicity of the $\mathcal{L}^p$-norm in a probability space, if $\mathcal{X} \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, then, in fact, $\mathcal{X} \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^q(\Omega, \mathbb{H}_\mathbb{F}))$ for any $1 \le q \le p$. However, this implication does not always hold for $q > p$. A counterexample is given below.*

Example 6.4.7. *Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \mathscr{M})$, where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$. For a sequence of pairwise disjoint measurable sets $\{S_n\}_{n\ge1} \subset \mathcal{B}([0,1])$ such that $\mathbb{P}(S_n) = \frac{1}{n^3}$, define a stochastic process $\mathcal{X} : \mathbb{Z}_0^+ \times \Omega \to \mathbb{R}$ by*

$$\mathcal{X}(0,\zeta) = 2^{n/4}\cos(\pi n/2), \quad \mathcal{X}(n,\zeta) = 2^{n/4}\cos(\pi n/2) + 2^{n/4}\sum_{k=1}^{n} k\boldsymbol{1}_{S_k}(\zeta), \quad n \ge 1.$$

Then for all $n \ge 0$, we have

$$2^{-n/4}\left(\mathcal{X}(n+4,\zeta) - 2\mathcal{X}(n,\zeta)\right) = 2(n+4)\boldsymbol{1}_{S_{n+4}}(\zeta) = \begin{cases} 2(n+4) & \text{if } \zeta \in S_{n+4}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for $p \ge 1$ we obtain

$$\int_0^1 |2^{-n/4}(\mathcal{X}(n+4) - 2\mathcal{X}(n))|^p d\mathbb{P}(\zeta) = 2^p\sum_{k=1}^{4}(n+k)^p\mathbb{P}(S_{n+k}) = 2^p\sum_{k=1}^{4}(n+k)^{p-3} \to \begin{cases} 0 & \text{when } p < 3, \\ 8 & \text{when } p = 3, \\ \infty & \text{when } p > 3, \end{cases}$$

i.e., $\mathcal{X} \in SAP_{4,2}(\mathbb{Z}_0^+, \mathcal{L}^p(\mathcal{B}[0,1], \mathbb{R}))$ for all $1 \le p < 3$ and not for any $p \ge 3$.

Note that in the deterministic setting, Definition 6.4.3 reduces to the definition of S-asymptotically (ω, c)-periodic functions, and we obtain the following notion.

Definition 6.4.8. *A function $f \in \mathrm{C_{rd}}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F})$ is said to be (ω, c)-bounded if there exists a positive number M such that*

$$\|c^{-t/\omega} f(t)\| < M.$$

Let $(\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F})$ denote the collection of such functions. A function $f \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F})$ is said to be S-asymptotically (ω, c)-periodic if

$$\lim_{t\to\infty,\ t\in\mathbb{T}_\omega^{0+}} \|c^{-t/\omega}(f(t+\omega) - cf(t))\| = 0.$$

We denote the set of all S-asymptotically (ω, c)-periodic functions in $(\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F})$ by $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathbb{H}_\mathbb{F})$.

Remark 6.4.9. *The concept of S-asymptotically (ω, c)-periodic functions generalizes the notion of asymptotically (ω, c)-periodic functions defined in Chapter 4. This is because, after subtracting the (ω, c)-periodic part $f_{op}(t)$ from the asymptotically (ω, c)-periodic functions $f(t)$, the remaining part $f_{oa}(t)$ satisfies $\lim_{t\to\infty} c^{-t/\omega} f_{oa}(t) = 0$. However, the converse does not necessarily hold, as shown by the following examples.*

Example 6.4.10. *Consider a function $f(t) = \left(\frac{2}{3}\right)^{t/2\pi}[\sin t + \sin(\ln(1+t))]$, $t \in \mathbb{R}$. Then, we have*

$$\begin{aligned}
|(2/3)^{-t/2\pi}(f(t+2\pi) - (2/3)f(t))| &= (2/3)|\sin(\ln(1+t+2\pi)) - \sin(\ln(1+t))| \\
&\leq |\ln(1+t+2\pi) - \ln(1+t)| \\
&= \left|\ln\left(1 + \frac{2\pi}{1+t}\right)\right| \to 0 \text{ as } t \to \infty,
\end{aligned}$$

whereas, $\lim_{t\to\infty} \sin(\ln(1+t)) \not\to 0$. Thus, f is S-asymptotically $(2\pi, 2/3)$-periodic function but is not asymptotically $(2\pi, 2/3)$-periodic.

Example 6.4.11. *Consider a function $f(t) = \left(\frac{3}{2}\right)^{t/2}[\sin \pi t + \cos(1 + \ln t)]$, $t \in \bigcup_{n\in\mathbb{Z}}[4n, 4n+2]$. Since, $\lim_{t\to\infty} \cos(1+\ln t) \not\to 0$, f is not asymptotically $(2, 3/2)$-periodic function. However,*

$$\begin{aligned}
|(3/2)^{-t/2}(f(t+2) - (3/2)f(t))| &= (3/2)|\cos(1+\ln(t+2)) - \cos(1+\ln t)| \\
&= (3/2)|\cos(1+\ln t + \ln(1+2/t)) - \cos(1+\ln t)| \\
&\to 0 \text{ as } t \to \infty,
\end{aligned}$$

i.e., f is S-asymptotically $(2, 3/2)$-periodic function in $\bigcup_{n\in\mathbb{Z}}[4n, 4n+2]$.

The next lemma describes some algebraic properties of the set $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$.

Lemma 6.4.12. *If $\mathcal{X}$, $\mathcal{X}_1, \mathcal{X}_2 \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ are all p-th mean S-asymptotically (ω, c) periodic stochastic processes, then*

(i) $\mathcal{X}_1 + \mathcal{X}_2$ is p-th mean S-asymptotically (ω, c) periodic.

(ii) $\lambda\mathcal{X}$ *is p-th mean S-asymptotically* (ω, c) *periodic for every scalar* λ.

(iii) $\mathcal{X}_\tau(t) := \mathcal{X}(\tau + t)$ *is p-th mean S-asymptotically* (ω, c) *periodic for each* $t \in \mathbb{T}_\omega^{0+}$ *and* $\tau \in \Pi_{\mathbb{T}}$.

Establishing the completeness of the space $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$, equipped with a suitable norm, is essential for the fixed-point methods used in the next section. The following theorem investigates the Banach-space structure of $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$.

Theorem 6.4.13. $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ *is a Banach space when it is equipped with the norm* $\|\cdot\|_\infty$ *defined in* (6.4.1).

Proof. By Lemma 6.4.12, $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is a vector space. It is also clear that $\|\cdot\|_\infty$ is a norm in $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$. We only need to show that this normed linear space is complete. Since $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})) \subset (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ and $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is a Banach space with respect to the norm $\|\cdot\|_\infty$, it is enough to show that $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is closed. To this end, assume that $(\mathcal{X}_n)_{n\in\mathbb{N}} \subset S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ is an arbitrary sequence such that for stochastic process $\mathcal{Y} \in \mathbb{T}_\omega^{0+} \to \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$

$$\lim_{n\to\infty} \mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t) - \mathcal{Y}(t)\|^p = 0 \quad \text{for each } t \in \mathbb{T}_\omega^{0+}. \tag{6.4.4}$$

Consider $\mathcal{X}(t) = c^{t/\omega}\mathcal{Y}(t)$. By the completeness of the space $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$, $\mathcal{X}$ is stochastically rd-continuous and (ω, c)-bounded. We need to show that $\mathcal{X}$ satisfies (6.4.3). Indeed, by

$$\begin{aligned} c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t)) \quad &\le \quad c^{-t/\omega}\mathcal{X}(t+\omega) - c^{-t/\omega}\mathcal{X}_n(t+\omega) + c^{-t/\omega}(\mathcal{X}_n(t+\omega) - c\mathcal{X}_n(t)) \\ &\quad +c(c^{-t/\omega}\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}(t))), \end{aligned}$$

for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p \quad &\le \quad 3^{p-1}\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t+\omega) - c^{-t/\omega}\mathcal{X}(t+\omega)\|^p + 3^{p-1}\mathbb{E}\|c^{-t/\omega}(\mathcal{X}_n(t+\omega) \\ &\quad -c\mathcal{X}_n(t))\|^p + 3^{p-1}|c|^p\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}(t))\|^p. \end{aligned} \tag{6.4.5}$$

Also, we have $\mathcal{X}_n \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$, i.e.,

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}_n(t+\omega) - c\mathcal{X}_n(t))\|^p \quad = \quad 0. \tag{6.4.6}$$

Using (6.4.6), the inequality (6.4.5) yields

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p \quad &\le \quad 3^{p-1}\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t+\omega) - c^{-t/\omega}\mathcal{X}(t+\omega)\|^p \\ &\quad +3^{p-1}|c|^p\mathbb{E}\|c^{-t/\omega}\mathcal{X}_n(t) - c^{-t/\omega}\mathcal{X}(t))\|^p. \end{aligned}$$

Finally, employing (6.4.4) in the above, we finally have

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p = 0.$$

The proof is complete. □

We next extend the notion of p-th mean S-asymptotic (ω, c)-periodicity for processes in $\mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ and then establish two composition theorems for stochastic processes possessing p-th mean S-asymptotic (ω, c)-periodicity.

Definition 6.4.14. *Consider that the mapping $c^{-t/\omega}\mathcal{F}(t, \mathcal{X}) : \mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}) \to \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is well defined for $t \in \mathbb{T}_\omega^{0+}$ and $\mathcal{X} \in \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$. The stochastic process $\mathcal{F}$ is said to be stochastically (ω, c)-bounded if there exists a constant $M > 0$ such that*

$$\mathbb{E}\|c^{-t/\omega}\mathcal{F}(t, \mathcal{X})\|^p < M$$

for all $t \in \mathbb{T}_\omega^{0+}$ and all $\mathcal{X} \in K$, where $c^{-t/\omega}K := \{c^{-t/\omega}\mathcal{X}(t) : \mathcal{X} \in K\} \subset \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is bounded in the sense that $\sup\limits_{\mathcal{X} \in K} \|c^{-t/\omega}\mathcal{X}(t)\|_p < \infty$.

In this chapter, $(\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}), \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ denotes the space of all stochastic processes $\mathcal{X}$ that are rd-continuous in t and stochastically (ω, c)-bounded, for which the mapping $(t, \mathcal{X}) \mapsto c^{-t/\omega}\mathcal{X}(t)$ is well-defined and takes values in $\mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$.

Definition 6.4.15. *A stochastic process $\mathcal{F} \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}), \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ is said to be p-th mean S-asymptotically (ω, c)-periodic, uniformly in $\mathcal{X}$, if*

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t + \omega, \mathcal{X}) - c\mathcal{F}(t, c^{-1}\mathcal{X}))\|^p = 0$$

for all $\mathcal{X} \in K$, where $c^{-t/\omega}K \subset \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is bounded.

Theorem 6.4.16 (Composition Theorem 1)**.** *Let $\mathcal{F} \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}), \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ be a p-th mean S-asymptotically (ω, c)-periodic stochastic process in $t \in \mathbb{T}_\omega^{0+}$. Further, assume that $\mathcal{F}$ satisfies the Lipschitz condition in the sense*

$$\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t, \mathcal{X}) - \mathcal{F}(t, \mathcal{Y}))\|^p \leq \mathtt{L}\mathbb{E}\|c^{-t/\omega}(\mathcal{X} - \mathcal{Y})\|^p \quad \text{for all } \ \mathcal{X}, \mathcal{Y} \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})),$$

where $\mathtt{L} > 0$ is independent of t. Then, for any p-th mean S-asymptotically (ω, c)-periodic stochastic process $\mathcal{X} \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, the stochastic process $\mathcal{F}(t, \mathcal{X}(t))$ is p-th mean S-asymptotically (ω, c)-periodic.

Proof. By p-th mean S-asymptotically (ω, c)-periodicity of $\mathcal{F}$ and $\mathcal{X}$, we have

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t + \omega, \mathcal{X}) - c\mathcal{F}(t, c^{-1}\mathcal{X}))\|^p = 0 \tag{6.4.7}$$

for all $\mathcal{X}$ in any set K such that $c^{-t/\omega}K \subset \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})$ is bounded, and

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t + \omega) - c\mathcal{X}(t))\|^p = 0 \tag{6.4.8}$$

for all $\mathcal{X}$ where $c^{-t/\omega}\mathcal{X}(t) \in \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$.

Now, using the assumptions of the theorem, we can obtain

$$\begin{aligned}
\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega)) - c\mathcal{F}(t, \mathcal{X}(t)))\|^p &\leq 2^{p-1}\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega) - c\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega)))\|^p \\
&\quad +2^{p-1}\mathbb{E}\|c^{-t/\omega}(c\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega)) - c\mathcal{F}(t, \mathcal{X}(t)))\|^p \\
&\leq 2^{p-1}\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega) - c\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega))\|^p \\
&\quad +2^{p-1}\mathrm{L}\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p.
\end{aligned}$$

Thus, we can deduce from (6.4.7) and (6.4.8) that

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega)) - c\mathcal{F}(t, \mathcal{X}(t)))\|^p = 0,$$

i.e., $\mathcal{F}(\cdot, \mathcal{X}(\cdot)) \in SAP_{\omega,c}(\mathbb{T}^+ \times \mathbb{T}^+, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$. □

Definition 6.4.17. *A stochastically continuous process $\mathcal{F}$, with a well defined mapping $c^{-t/\omega}\mathcal{F} : \mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}) \to \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$, is said to be p-th mean S-asymptotically (ω, c)-uniformly continuous on bounded sets if for every $\varepsilon > 0$ and every bounded set B, there exist $T = T(\varepsilon, B) \in \mathbb{T}_\omega^{0+}$ and $\delta = \delta(\varepsilon, B) > 0$ such that for all $t \geq T$ and $\mathcal{X}, \mathcal{Y} \in \{c^{t/\omega}B : t \in \mathbb{T}_\omega^{0+}\}$,*

$$\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t, \mathcal{X}) - \mathcal{F}(t, \mathcal{Y})\|^p < \varepsilon \quad \textit{whenever} \quad \mathbb{E}\|c^{-t/\omega}(\mathcal{X} - \mathcal{Y})\|^p < \delta.$$

Theorem 6.4.18 (Composition Theorem 2)**.** *Consider a mapping $c^{-t/\omega}\mathcal{F} : \mathbb{T}_\omega^{0+} \times \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}) \to \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$, so that the stochastic process $\mathcal{F}$ is uniformly p-th mean S-asymptotically (ω, c)-periodic on bounded sets and p-th mean S-asymptotically (ω, c)-uniformly continuous on bounded sets. Let $\mathcal{X} \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ be an S-asymptotically (ω, c)-periodic process in the p-th mean sense. Then $x(\cdot) := \mathcal{F}(\cdot, \mathcal{X}(\cdot))$ is a p-th mean S-asymptotically (ω, c)-periodic process.*

Proof. Since the stochastic process $\mathcal{X}$ is p-th mean S-asymptotically (ω, c)-periodic, for each $\delta > 0$ there exists an $T_1 = T_1(\delta) \in \mathbb{T}_\omega^{0+}$ such that for all $\mathcal{X}$ where $c^{-t/\omega}\mathcal{X}(t) \in \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})$,

$$\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p < |c|\delta \quad \text{for all} \quad t \geq T_1. \tag{6.4.9}$$

Thus, by p-th mean S-asymptotic-(ω, c)-uniform continuity of $\mathcal{F}$ in the bounded set $B := \{c^{-t/\omega}\mathcal{X}(t) : \mathcal{X} \in SAP_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}})), t \in \mathbb{T}_\omega^{0+}\}$, for any $\varepsilon > 0$ there exists a $T_2 = T_2(\varepsilon, B) \in \mathbb{T}_\omega^{0+}$ such that

$$\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega)) - \mathcal{F}(t, \mathcal{X}(t)))\|^p < \frac{\varepsilon}{2^p|c|^p} \quad \text{for all} \quad t \geq T_2. \tag{6.4.10}$$

Furthermore, the p-th mean S-asymptotic (ω, c)-periodicity of $\mathcal{F}$ implies that there exist $T_3 = T_3(\varepsilon, B) \in \mathbb{T}_\omega^{0+}$ such that

$$\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t) - c\mathcal{F}(t, c^{-1}\mathcal{X}(t)))\|^p < \frac{\varepsilon}{2^p} \quad \text{for all} \quad t \geq T_3. \tag{6.4.11}$$

As a consequence of (6.4.10) and (6.4.11), for $t \geq T = \max\{T_1, T_2, T_3\}$ we have

$$\begin{aligned}\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega)) - c\mathcal{F}(t, \mathcal{X}(t)))\|^p &\leq 2^{p-1}\mathbb{E}\|c^{-t/\omega}(\mathcal{F}(t+\omega, \mathcal{X}(t+\omega) - c\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega)))\|^p \\ &\quad +2^{p-1}\mathbb{E}\|c^{-t/\omega}(c\mathcal{F}(t, c^{-1}\mathcal{X}(t+\omega)) - c\mathcal{F}(t, \mathcal{X}(t)))\|^p \\ &< \varepsilon.\end{aligned}$$

This completes the proof. $\square$

Next, we identify the classes of functions whose pointwise product belongs to the space $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$.

Lemma 6.4.19. *If $P \in SAP_{\omega,1}(\mathbb{T}_\omega^{0+}, \mathbb{F})$ and $\mathcal{X} \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, then the pointwise product $P\mathcal{X} \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$.*

Proof. From the definitions, it follows that $|P|_\infty := \sup_{t\in\mathbb{T}_\omega^{0+}} |P(t)| < \infty$ and $\|\mathcal{X}\|_\infty < \infty$. This implies that for all $t \in \mathbb{T}_\omega^{0+}$, $\mathbb{E}\|c^{-t/\omega}P(t)\mathcal{X}(t)\|^p \leq |P|_\infty^p\mathbb{E}\|c^{-t/\omega}\mathcal{X}(t)\|^p$ is bounded by a fixed constant. Also, for each $\varepsilon > 0$ we can always find a $T > 0$ so that for all $t \geq T$ we can have

$$\begin{aligned}\mathbb{E}\Big\|c^{-t/\omega}\left(P(t+\omega)\mathcal{X}(t+\omega) - cP(t)\mathcal{X}(t)\right)\Big\|^p &\leq \mathbb{E}\Big(\|\left(P(t+\omega) - P(t)\right)c^{-t/\omega}\mathcal{X}(t+\omega)\| \\ &\quad +\|P(t)c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|\Big)^p \\ &\leq 2^{p-1}|c|^p|P(t+\omega) - P(t)|^p\mathbb{E}\|c^{-(t+\omega)/\omega}\mathcal{X}(t+\omega)\|^p \\ &\quad +2^{p-1}|P|_\infty^p\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p \\ &\leq 2^{p-1}|c|^p\|\mathcal{X}\|_\infty^p|P(t+\omega) - P(t)|^p \\ &\quad +2^{p-1}|P|_\infty^p\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p \\ &< \varepsilon.\end{aligned}$$

Thus, $\lim_{t\to\infty}\mathbb{E}\Big\|c^{-t/\omega}\left(P(t+\omega)\mathcal{X}(t+\omega) - cP(t)\mathcal{X}(t)\right)\Big\|^p = 0$, which completes the proof. $\square$

Lemma 6.4.20. *Let $\mathcal{P} \in \mathrm{C}(\mathbb{T}_\omega^{0+}, \mathcal{L}^\infty(\Omega, \mathbb{H}_\mathbb{F}))$ and $\mathcal{X} \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, then we have $\mathcal{P}(t)\mathcal{X}(t) \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ provided that $\mathcal{P}(t+\omega) = \mathcal{P}(t)$ for all $t \in \mathbb{T}_\omega^{0+}$.*

Proof. Since $\mathcal{P}(t) \in \mathcal{L}^\infty(\Omega, \mathbb{H}_\mathbb{F})$, it follows by definition that $\|\mathcal{P}\|_\infty < \infty$. Consequently, we have

$$\begin{aligned}\mathbb{E}\Big\|c^{-t/\omega}\left(\mathcal{P}(t+\omega)\mathcal{X}(t+\omega) - c\mathcal{P}(t)\mathcal{X}(t)\right)\Big\|^p &\leq \mathbb{E}\Big(\|\mathcal{P}(t)c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|\Big)^p \\ &\leq \|\mathcal{P}\|_\infty^p\mathbb{E}\|c^{-t/\omega}(\mathcal{X}(t+\omega) - c\mathcal{X}(t))\|^p \\ &\to 0 \quad (\text{as } t \to \infty.)\end{aligned}$$

$\square$

In the next two lemmas, we show that a Volterra type integral operator associated with f, with kernel $e_{-\alpha(s)}(t, \sigma(s))$, leaves the space $SAP_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$ invariant under both the standard integral and the Wiener integral on time scales. These operators form an integral part of the system (6.3.1).

Lemma 6.4.21. *Suppose $f \in SAP_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, $p > 1$. Then the function defined by*

$$\mathfrak{F} := \int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta s$$

belongs to space $SAP_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, provided that there exists $\underline{\alpha}$, a positive lower bound of $\alpha(t)$, such that $-\underline{\alpha} \in \mathcal{R}^+$ and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{\alpha}\bar{\mu})}$.

Proof. As $f \in SAP_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, we have

$$\mathbb{E}\|c^{-t/\omega} f(t)\|^p < M < \infty \quad \text{for all} \quad t \in \mathbb{T}_\omega^{0+}, \quad \& \quad \lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(f(t+\omega) - cf(t))\|^p = 0.$$

From the continuity of f, it follows that $\mathfrak{F}$ is stochastically continuous. Moreover,

$$\begin{aligned}
\mathbb{E}\left\|c^{-t/\omega} \int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta s\right\|^p &\leq \mathbb{E}\left(\int_0^t c^{-(t-s)/\omega} e_{-\alpha(s)}(t, \sigma(s)) \|c^{-s/\omega} f(s)\| \Delta s\right)^p \\
&\leq \frac{1}{(1-\bar{\mu}\underline{\alpha})^p} \mathbb{E}\left(\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) \|c^{-s/\omega} f(s)\| \Delta s\right)^p \\
&\leq \frac{\left(\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s)\Delta s\right)^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p} \int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) \mathbb{E}\|c^{-s/\omega} f(s)\|^p \Delta s \\
&\leq \frac{M}{(1-\bar{\mu}\underline{\alpha})^p} \left(\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) \Delta s\right)^p \\
&\leq \frac{M}{(1-\bar{\mu}\underline{\alpha})^p \left(\underline{\alpha} + \frac{ln|c|}{\omega}\right)^p} < \infty,
\end{aligned}$$

suggest that f is stochastically (ω, c)-bounded. Next, we can have

$$\begin{aligned}
\mathbb{E}\|c^{-t/\omega}(\mathfrak{F}(t+\omega) - c\mathfrak{F}(t))\|^p &= \mathbb{E}\left\|c^{-t/\omega}\left(\int_0^{t+\omega} e_{-\alpha(s)}(t+\omega, \sigma(s)) f(s) \Delta s - c\int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta s\right)\right\|^p \\
&= \mathbb{E}\left\|c^{-t/\omega}\left(\int_0^{\omega} e_{-\alpha(s)}(t+\omega, \sigma(s)) f(s) \Delta s\right.\right. \\
&\quad \left.\left. + \int_0^t e_{-\alpha(s+\omega)}(t+\omega, \sigma(s+\omega)) f(s+\omega) \Delta s - c\int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta s\right)\right\|^p \\
&\leq 2^{p-1}\mathbb{E}\left\|c\int_0^{\omega} c^{-(t+\omega-s)/\omega} e_{-\alpha(s)}(t+\omega, \sigma(s)) c^{-s/\omega} f(s) \Delta s\right\|^p \\
&\quad + 2^{p-1}\mathbb{E}\left\|\int_0^t c^{-(t-s)/\omega} e_{-\underline{\alpha}}(t, \sigma(s)) c^{-s/\omega}(f(s+\omega) - cf(s)) \Delta s\right\|^p \\
&\leq \frac{|c|^p 2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p} \mathbb{E}\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega, 0) \int_0^{\omega} e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(s, 0) \|c^{-s/\omega} f(s)\| \Delta s\right)^p
\end{aligned}$$

$$+\frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\mathbb{E}\Bigg(\int_0^t \left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\right)^{\frac{p-1}{p}}\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\right)^{\frac{1}{p}} \|c^{-s/\omega}(f(s+\omega)-cf(s))\|\Delta s\Bigg)^p$$

As p and $\frac{p}{p-1}$ are conjugate exponents, using Hölder's inequality, we can obtain

$$\begin{aligned}\mathbb{E}\|c^{-t/\omega}(\mathfrak{F}(t+\omega)-c\mathfrak{F}(t))\|^p \leq\ & \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\Bigg[\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega,0)\right)^p\left(\int_0^\omega e_{\frac{p}{p-1}\odot(\underline{\alpha}+\frac{ln|c|}{\omega})}(s,0)\Delta s\right)^{p-1} \\ & \int_0^\omega \mathbb{E}\|c^{-s/\omega}f(s)\|^p\Delta s+\left(\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\Delta s\right)^{p-1} \\ & \int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\mathbb{E}\|c^{-s/\omega}(f(s+\omega)-cf(s))\|^p\Delta s\Bigg].\end{aligned}$$

We next use the fact that for any $\varepsilon>0$ there exists $T_\varepsilon>0$ such that $\mathbb{E}\|c^{-t/\omega}(f(t+\omega)-cf(t))\|^p \leq \frac{\varepsilon\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)^p(1-\bar{\mu}\underline{\alpha})^p}{2^{p-1}}$ whenever $t>T_\varepsilon$. Thus, for $t>T_\varepsilon$ we have

$$\begin{aligned}&\mathbb{E}\|c^{-t/\omega}(\mathfrak{F}(t+\omega)-c\mathfrak{F}(t))\|^p \\ &\leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\Bigg[|c|^pM\omega\left(\frac{\bar{\mu}\left(\left(e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(\omega,0)\right)^{\frac{p}{p-1}}-1\right)}{\left(1+\underline{\mu}\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)\right)-1}\right)^{p-1}\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega,0)\right)^p \\ &\quad+\left(\int_0^t\frac{\ominus\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,\sigma(s))}{-\underline{\alpha}-\frac{ln|c|}{\omega}}\Delta s\right)^{p-1}\Bigg(\int_0^{T_\varepsilon}e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s) \\ &\quad\mathbb{E}\|c^{-s/\omega}(f(s+\omega)-cf(s))\|^p\Delta s+\int_{T_\varepsilon}^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\mathbb{E}\|c^{-s/\omega}(f(s+\omega)-cf(s))\|^p\Delta s\Bigg)\Bigg] \\ &\leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)^{p-1}}\Bigg(\int_0^{T_\varepsilon}e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\mathbb{E}\|c^{-s/\omega}(f(s+\omega)-cf(s))\|^p\Delta s \\ &\quad+\frac{\varepsilon(1-\bar{\mu}\underline{\alpha})^p\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)^p}{2^{p-1}}\int_{T_\varepsilon}^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\Delta s\Bigg)+C\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega,0)\right)^p \\ &\leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)^{p-1}}\Bigg(2^{p-1}(1+|c|)Me_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,0)\int_0^{T_\varepsilon}e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(s,0)\Delta s \\ &\quad+\frac{\varepsilon(1-\bar{\mu}\underline{\alpha})^p\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)^{p-1}}{2^{p-1}}\Bigg)+C\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega,0)\right)^p,\end{aligned}$$

where $C:=\frac{M\omega|c|^p2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\left(\bar{\mu}\left(\left(e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(\omega,0)\right)^{\frac{p}{p-1}}-1\right)\Big/\left(\left(1+\underline{\mu}\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)\right)-1\right)\right)^{p-1}$.

Since $\underline{\alpha}+\frac{ln|c|}{\omega}>0$, $\lim_{t\to\infty}e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,0)=0$, we consequently have

$$\lim_{n\to\infty}\mathbb{E}\|c^{-t/\omega}(\mathfrak{F}(t+\omega)-c\mathfrak{F}(t))\|^p=0.$$

This completes the proof. $\square$

Lemma 6.4.22. *Let $f \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$, $p \geq 2$. Assume that there exists a positive lower bound $\underline{\alpha}$ of a regressive function $\alpha(t)$, such that $-\underline{\alpha} \in \mathcal{R}^+$ and $e^{-\frac{\omega}{\bar{\mu}}(1+\underline{\alpha}\bar{\mu})} < |c| < e^{\frac{\omega}{\bar{\mu}}(1-\underline{\alpha}\bar{\mu})}$. Then the function defined by*

$$\mathfrak{G}(t) := \int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta W(s)$$

belongs to the space $S\mathrm{AP}_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F}))$.

Proof. It is easy to see that $\mathfrak{G}$ is a continuous stochastic process defined in $\mathbb{T}_\omega$. Furthermore, using a similar approach to that in the proof below, one can easily verify that $\mathfrak{G}$ is stochastically (ω, c)-bounded. Now, we have

$$\begin{aligned}
&\mathbb{E}\|c^{-t/\omega}(\mathfrak{G}(t+\omega) - c\mathfrak{G}(t))\|^p \\
&\quad = \mathbb{E}\Big\|c^{-t/\omega}\Big(\int_0^{t+\omega} e_{-\alpha(s)}(t+\omega, \sigma(s)) f(s)\Delta W(s) - c\int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s) \Delta W(s)\Big)\Big\|^p \\
&\quad = \mathbb{E}\Big\|c^{-t/\omega}\Big(\int_0^{\omega} e_{-\alpha(s)}(t+\omega, \sigma(s)) f(s)\Delta W(s) \\
&\qquad + \int_0^t e_{-\alpha(s+\omega)}(t+\omega, \sigma(s+\omega)) f(s+\omega)\Delta W(s+\omega) - c\int_0^t e_{-\alpha(s)}(t, \sigma(s)) f(s)\Delta W(s)\Big)\Big\|^p \\
&\quad \leq 2^{p-1}\mathbb{E}\Big\|c^{-t/\omega}\int_0^\omega e_{-\alpha(s)}(t+\omega, \sigma(s)) f(s)\Delta W(s)\Big\|^p \\
&\qquad + 2^{p-1}\mathbb{E}\Big\|c^{-t/\omega}\int_0^t e_{-\alpha(s)}(t, \sigma(s))\left(f(s+\omega) - cf(s)\right)\Delta W(s)\Big\|^p \\
&\quad \leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega, 0)\right)^p \mathbb{E}\Big\|\int_0^\omega e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(s, 0)\left(c^{-s/\omega}f(s)\right)\Delta W(s)\Big\|^p \\
&\qquad + \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\mathbb{E}\Big\|\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) c^{-s/\omega}\left(f(s+\omega) - cf(s)\right)\Delta W(s)\Big\|^p \\
&\quad := \mathbb{E}\mathfrak{K}(t) + \mathbb{E}\mathfrak{I}(t),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\mathfrak{K}(t) &:= \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t+\omega, 0)\right)^p \mathbb{E}\Big\|\int_0^\omega e_{(\underline{\alpha}+\frac{ln|c|}{\omega})}(s, 0)\left(c^{-s/\omega}f(s)\right)d\mathbb{P}(s)\Big\|^p, \text{ and} \\
\mathbb{E}\mathfrak{I}(t) &:= \frac{2^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\mathbb{E}\Big\|\int_0^t e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) c^{-s/\omega}\left(f(s+\omega) - cf(s)\right)d\mathbb{P}(s)\Big\|^p.
\end{aligned}$$

It is clear that $\lim_{t\to\infty}\mathbb{E}\mathfrak{K}(t) = 0$. Let $\varepsilon > 0$. Since $\lim_{t\to\infty}\mathbb{E}\|c^{-t/\omega}(f(t+\omega) - cf(t))\|^p = 0$, there exists $T_\varepsilon > 0$ such that $\mathbb{E}\|c^{-t/\omega}(f(t+\omega) - cf(t))\|^p \leq \frac{\varepsilon\left(\left(1+\underline{\mu}(\underline{\alpha}+\frac{ln|c|}{\omega})\right)^2 - 1\right)^{p/2}}{C_p(\bar{\mu})^{p/2}}$ whenever $t > T_\varepsilon$, where the constant C_p will be precise later in this proof. Now, we can write

$$\mathbb{E}\mathfrak{I}(t) \leq \frac{4^{p-1}}{(1-\bar{\mu}\underline{\alpha})^p}\Bigg(\mathbb{E}\Big\|\int_0^{T_\varepsilon} e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t, s) c^{-s/\omega}\left(f(s+\omega) - cf(s)\right)d\mathbb{P}(s)\Big\|^p$$

$$\begin{aligned}&+\mathbb{E}\Big\| \int_{T_\varepsilon}^{t} e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)d\mathbb{P}(s)\Big\|^{p}\Bigg)\\ =\;& \frac{4^{p-1}}{(1-\bar{\mu}\underline{\alpha})^{p}}\left(\mathbb{E}\mathfrak{J}_1(t)+\mathbb{E}\mathfrak{J}_2(t)\right),\end{aligned}$$

with

$$\begin{aligned}\mathbb{E}\mathfrak{J}_1(t) &:= \mathbb{E}\Big\| \int_{0}^{T_\varepsilon} e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)d\mathbb{P}(s)\Big\|^{p}, \text{ and}\\ \mathbb{E}\mathfrak{J}_2(t) &:= \mathbb{E}\Big\| \int_{T_\varepsilon}^{t} e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)d\mathbb{P}(s)\Big\|^{p}.\end{aligned}$$

Note that for all $t\geq 0$,

$$c^{-t/\omega}(f(t+\omega)-cf(t))\in\mathcal{L}^{p}(\Omega,\mathbb{H}_{\mathbb{F}})\subseteq\mathcal{L}^{2}(\mathbb{P},\mathbb{H}_{\mathbb{F}})$$

and

$$\begin{aligned}\mathbb{E}\int_{0}^{t}\|e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)c^{-s/\omega}(f(s+\omega)-cf(s))\|^{2}ds &\leq 2(1+|c|)\sup_{t\geq 0}\mathbb{E}\|c^{-t/\omega}f(t)\|^{2}\int_{0}^{t}e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,s)\Delta s\\ &\leq \frac{2(1+|c|)}{\underline{\alpha}+\frac{ln|c|}{\omega}}\sup_{t\geq 0}\mathbb{E}\|c^{-t/\omega}f(t)\|^{2}\\ &< \infty.\end{aligned}$$

Let us first consider $p>2$. In this case using Hölder's inequality, with conjugate exponents $\frac{p}{p-2}$ and $\frac{p}{2}$, together with the *Burkholder–Davis–Gundy inequality* on time scales, we can have a constant C_p such that

$$\begin{aligned}\mathbb{E}\mathfrak{J}_1(t) &\leq C_p\mathbb{E}\left[\int_{0}^{T_\varepsilon} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^{2}\Delta s\right]^{p/2}\\ &\leq C_p\left(\int_{0}^{T_\varepsilon} e_{\frac{2p}{p-2}\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\Delta s\right)^{\frac{p-2}{2}}\int_{0}^{T_\varepsilon}\mathbb{E}\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^{p}\Delta s\\ &\leq C_p\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,0)\right)^{p}\left(\int_{0}^{T_\varepsilon} e_{\frac{2p}{p-2}\odot(\underline{\alpha}+\frac{ln|c|}{\omega})}(s,0)\Delta s\right)^{\frac{p-2}{2}} T_\varepsilon 2^{p-1}(1+|c|)\sup_{t\in\mathbb{T}_{\omega}^{0+}}\mathbb{E}\|c^{-t/\omega}f(t)\|^{p}.\end{aligned}$$

Therefore $\lim_{t\to\infty}\mathbb{E}\mathfrak{J}_1(t)=0$. Similarly,

$$\begin{aligned}\mathbb{E}\mathfrak{J}_2(t) &\leq C_p\mathbb{E}\left[\int_{T_\varepsilon}^{t} e_{\frac{2(p-2)}{p}\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)e_{\frac{4}{p}\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^{2}\Delta s\right]^{p/2}\\ &\leq C_p\left(\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\Delta s\right)^{\frac{p-2}{2}}\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\mathbb{E}\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^{p}\Delta s\\ &\leq C_p\left(\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\Delta s\right)^{\frac{p}{2}}\frac{\varepsilon\left(\left(1+\underline{\mu}\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)\right)^{2}-1\right)^{p/2}}{C_p\left(\bar{\mu}\right)^{p/2}}\\ &\leq \varepsilon,\end{aligned}$$

impose that $\lim_{t\to\infty} \mathbb{E}\mathfrak{J}_2(t) = 0$.

Now, for the case $p = 2$, using Itô isometry we can have

$$\begin{aligned}
\mathbb{E}\mathfrak{J}_1(t) &= \mathbb{E}\left[\int_0^{T_\varepsilon} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^2\Delta s\right] \\
&\le \left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,0)\right)^2 \int_0^{T_\varepsilon} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(s,0)\mathbb{E}\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^2\Delta s \\
&\le 2(1+|c|)\left(e_{\ominus(\underline{\alpha}+\frac{ln|c|}{\omega})}(t,0)\right)^2 \sup_{t\in\mathbb{T}_\omega^{0+}} \mathbb{E}\|c^{-t/\omega}f(t)\|^2 \int_0^{T_\varepsilon} e_{2\odot(\underline{\alpha}+\frac{ln|c|}{\omega})}(s,0)\Delta s,
\end{aligned}$$

which implies that $\lim_{t\to\infty} \mathbb{E}\mathfrak{J}_1(t) = 0$. Similarly for $p = 2$, using Itô isometry and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\mathfrak{J}_2(t) \\
&= \mathbb{E}\left[\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^2\Delta s\right] \\
&\le \left(\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\Delta s\right)^{1/2} \left(\int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\left(\mathbb{E}\|c^{-s/\omega}\left(f(s+\omega)-cf(s)\right)\|^2\Delta s\right)^2\right)^{1/2}.
\end{aligned}$$

Since, for $T_\varepsilon > 0$, $\mathbb{E}\|c^{-t/\omega}(f(t+\omega)-cf(t))\|^2 \le \frac{\varepsilon\left(\left(1+\underline{\mu}\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)\right)^2-1\right)}{\bar{\mu}}$, we can write

$$\mathbb{E}\mathfrak{J}_2(t) \le \frac{\varepsilon\left(\left(1+\underline{\mu}\left(\underline{\alpha}+\frac{ln|c|}{\omega}\right)\right)^2-1\right)}{\bar{\mu}} \int_{T_\varepsilon}^{t} e_{2\odot(\ominus(\underline{\alpha}+\frac{ln|c|}{\omega}))}(t,s)\Delta s \le \varepsilon.$$

Thus, $\lim_{t\to\infty} \mathbb{E}\mathfrak{J}_2(t) = 0$.

Hence, we can conclude that

$$\lim_{t\to\infty} \mathbb{E}\|c^{-t/\omega}(\mathfrak{G}(t+\omega)-c\mathfrak{G}(t))\|^p = 0.$$

This completes the proof. □

6.5 p-th mean S-asymptotically (ω, c)-periodic solution

For $p \ge 2$, we have seen in Section 6.4 that the class of p-th mean S-asymptotically (ω, c)-periodic processes contains the class of mean S-asymptotically (ω, c)-periodic processes. In this section, we investigate the existence and global exponential stability of the p-th mean S-asymptotically (ω, c)-periodic solution to stochastic SICNNs (6.3.1) on time scales.

Note that the system (6.3.1) is driven by a Wiener process. Consequently, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C([0,\infty),\mathbb{R})$ is the set of all continuous paths with $\zeta(0) = 0$ for all $\zeta \in \Omega$, $\mathcal{F}$

is the σ-algebra generated by the cylinder sets, and $\mathbb{P} = W$ is the Wiener measure. Although the Wiener process is inherently continuous, we restrict attention to the given time scale $\mathbb{T}_\omega$ through

$$W|_{\mathbb{T}_\omega} : \mathbb{T}_\omega \to \mathbb{R},\ W|_{\mathbb{T}_\omega}(t) = W(t) \text{ for } t \in \mathbb{T}_\omega.$$

This restriction does not affect the underlying Wiener process, which remains continuous on $[0, \infty]$. Moreover, the expectation of the stochastic process $\mathcal{X}(t)$, driven by the Wiener process and defined in $\mathbb{T}_\omega$, is taken as the classical expectation over the full sample space Ω, as in continuous-time theory.

Definition 6.5.1. *An $\mathcal{F}_t$-progressively measurable process $x(t) = (x_{11}(t), x_{12}(t), \ldots, x_{mn}(t))^T$ with $t \in \mathbb{T}$, is said to be a solution of system (6.3.1), if, for each $1 \le i \le m$ and $1 \le j \le n$, $x_{ij}(t)$ is the solution of the following integral equation:*

$$\begin{aligned} x_{ij}(t) \;\; = \;\; & e_{-a_{ij}}(t,0)x_{ij}(0) + \int_0^t e_{-a_{ij}}(t,\sigma(s)) \Bigg(- \sum_{C_{kl} \in \mathcal{N}_{r_1}(i,j)} C_{ij}^{kl}(s) f(x_{kl}(s - \eta_{kl}(s))) x_{ij}(s) \\ & - \sum_{C_{kl} \in \mathcal{N}_{r_2}(i,j)} B_{ij}^{kl}(t) \int_0^\infty \mathcal{K}_{ij}(u) g(x_{kl}(s-u)) \Delta u x_{ij}(s) + L_{ij}(s) \Bigg) \Delta s \\ & - \int_0^t e_{-a_{ij}}(t,\sigma(s)) \sum_{C_{kl} \in \mathcal{N}_{r_3}(i,j)} D_{ij}^{kl}(s) \delta_{ij}(s, x_{ij}(s)) \Delta w_{ij}(s). \end{aligned} \tag{6.5.1}$$

Lemma 6.5.2. *Suppose that the system (6.3.1) is supplied with an initial history function $x_{ij}(s) = \phi_{ij}(s)$ for $s \le 0$, then (6.5.1) can equivalently be written as*

$$\begin{aligned} x_{ij}(t) \;\; = \;\; & \int_{-\infty}^t e_{-a_{ij}}(t,\sigma(s)) \Bigg(- \sum_{C_{kl} \in \mathcal{N}_{r_1}(i,j)} C_{ij}^{kl}(s) f(x_{kl}(s - \eta_{kl}(s))) x_{ij}(s) \\ & - \sum_{C_{kl} \in \mathcal{N}_{r_2}(i,j)} B_{ij}^{kl}(t) \int_0^\infty \mathcal{K}_{ij}(u) g(x_{kl}(s-u)) \Delta u x_{ij}(s) + L_{ij}(s) \Bigg) \Delta s \\ & - \int_0^t e_{-a_{ij}}(t,\sigma(s)) \sum_{C_{kl} \in \mathcal{N}_{r_3}(i,j)} D_{ij}^{kl}(s) \delta_{ij}(s, x_{ij}(s)) \Delta w_{ij}(s). \end{aligned} \tag{6.5.2}$$

Proof. Let $F_{ij}(s)$ denote the deterministic integrand in (6.5.1). Since the system admits the history $x_{ij}(s) = \phi_{ij}(s)$ for $s \le 0$, the initial term, $e_{-a_{ij}}(t,0)x_{ij}(0)$, represents the accumulated effect of the past history, which can be written as an integral over $(-\infty, 0]_{\mathbb{T}}$ using the semigroup property of the time-scale exponential, i.e.,

$$e_{-a_{ij}}(t,0)x_{ij}(0) = \int_{-\infty}^0 e_{-a_{ij}}(t,\sigma(s)) F_{ij}(s) \Delta s.$$

Thus, using $\int_{-\infty}^t (\cdot)\Delta s = \int_{-\infty}^0 (\cdot)\Delta s + \int_0^t (\cdot)\Delta s$, and substituting into (6.5.1), we can obtain the representation given in (6.5.2). □

Remark 6.5.3. *Let* $\mathbb{X} := \{x = (x_{11}, x_{12}, \ldots, x_{mn}) \ : \ x_{ij} \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})), \ 1 \le i \le n, 1 \le j \le m\}$*. Then* $(\mathbb{X}, \|\cdot\|_\mathbb{X})$ *is a Banach space with*

$$\|x\|_\mathbb{X} := \max_{i,j} \|x_{ij}\|_\infty \tag{6.5.3}$$

and $\|\cdot\|_\infty$ *is defined in* (6.4.1)*.*

The next theorem establishes the existence of a unique (ω, c)-bounded solution of (6.3.1). The proof is based on defining an appropriate contraction mapping on a suitable Banach space and applying the fixed point theorem.

Theorem 6.5.4. *Consider that assumptions* **(A1)**–**(A4)** *hold. Furthermore, assume that there exists a positive constant* κ *such that for each* $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, *we have*

(A5) $\mathcal{N}_p \ge \frac{1}{\kappa^p}$, *where*

$$\mathcal{N}_p := \left[\frac{(1-\bar{\mu}\min\limits_{ij}\underline{a}_{ij})^p}{4^{p-1}} - mn\max_{i,j}\left(\left(\frac{\hat{f}}{\underline{a}_{ij}+\frac{ln|c|}{\omega}}\right)^p \mathfrak{S}_C + \left(\frac{\hat{g}\bar{\mathcal{K}}_{ij}}{\underline{a}_{ij}+\frac{ln|c|}{\omega}}\right)^p \mathfrak{S}_B \right.\right.$$
$$\left.\left. + C_p\left(\frac{\hat{\delta}_{ij}^2}{\underline{a}_{ij}+\frac{ln|c|}{\omega}}\right)^{p/2} \mathfrak{S}_D\right)\right] \Big/ \left[\max_{ij}\left(\frac{\hat{L}_{ij}}{\underline{a}_{ij}+\frac{ln|c|}{\omega}}\right)^p\right],$$

with $C_p = \begin{cases} \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{p/2} & \text{when } p > 2, \\ 1 & \text{when } p = 2, \end{cases}$ $\quad \mathfrak{S}_C := \left(\sum\limits_{C_{kl}\in N_{r_1}(i,j)} (\bar{C}_{ij}^{kl})^{\frac{p}{p-1}}\right)^{p-1}$,

$\mathfrak{S}_B := \left(\sum\limits_{C_{kl}\in N_{r_2}(i,j)} (\bar{B}_{ij}^{kl})^{\frac{p}{p-1}}\right)^{p-1}$ *and* $\mathfrak{S}_D := \left(\sum\limits_{C_{kl}\in N_{r_3}(i,j)} (\bar{D}_{ij}^{kl})^{\frac{p}{p-1}}\right)^{p-1}$.

(A6) $\mathcal{M}_p := \max_{i,j} \frac{4^{p-1}mn}{(1-\bar{\mu}\underline{a}_{ij})^p}\left[\frac{2^{p-1}(\hat{f}^p+\mathrm{L}_f^p\kappa^p)\mathfrak{S}_C}{(\underline{a}_{ij}+\frac{ln|c|}{\omega})^p} + \frac{2^{p-1}\bar{\mathcal{K}}_{ij}^p(\hat{g}^p+\mathrm{L}_g^p\kappa^p)\mathfrak{S}_B}{(\underline{a}_{ij}+\frac{ln|c|}{\omega})^p} + \frac{C_p l_{ij}^p \mathfrak{S}_D}{(\underline{a}_{ij}+\frac{ln|c|}{\omega})^{p/2}}\right] < 1$, *with* C_p, $\mathfrak{S}_C$, $\mathfrak{S}_B$, *and* $\mathfrak{S}_D$ *are defined in* **(A5)**.

Then, the system (6.3.1) *has a unique (in the pathwise sense) stochastically* (ω, c)*-bounded and* $\mathcal{L}^p$*-continuous solution in*

$$\mathbb{X}^* = \{x = (x_{11}, x_{12}, \ldots, x_{mn})^T : \ \text{each } x_{ij} \in (\mathrm{BC_{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_\mathbb{F})) \text{ and } \|x\|_\mathbb{X} \le \kappa\}.$$

Proof. Let us define a nonlinear operator Γ in $\mathbb{X}^*$ as

$$(\Gamma x)(t) = ((\Gamma_{11}x)(t), (\Gamma_{12}x)(t), \ldots, (\Gamma_{mn}x)(t))^T,$$

where

$$(\Gamma_{ij}x)(t) = \int_{-\infty}^t e_{-a_{ij}}(t, \sigma(s)) {S_x}^{ij}(s)\Delta s + \int_{-\infty}^t e_{-a_{ij}}(t, \sigma(s)) {T_x}^{ij}(s)\Delta w_{ij}(s) \tag{6.5.4}$$

with $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$

$$S_x^{ij}(t) := -\sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t-\eta_{kl}(t)))x_{ij}(t) - \sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t)\int_0^\infty \mathcal{K}_{ij}(u) g(x_{kl}(t-u))\Delta u x_{ij}(t) + L_{ij}(t),$$

and

$$T_x^{ij}(t) := -\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(t)\delta_{ij}(t, x_{ij}(t)).$$

First, we show that $\max_{i,j}\{\sup_{t\in\mathbb{T}} \mathbb{E}\|c^{-t/\omega}(\Gamma_{ij}x)(t)\|^p\} \le \kappa^p$ whenever $\max_{i,j}\{\sup_{t\in\mathbb{T}_\omega} \mathbb{E}\|c^{-t/\omega}x_{ij}(t)\|^p\} \le \kappa^p$. We can easily obtain the following:

$$\mathbb{E}\|c^{-t/\omega}(\Gamma_{ij}(x))(t)\|^p \le 4^{p-1}\left(\mathbb{E}[\mathscr{I}_{ij}{}^{(1)}] + \mathbb{E}[\mathscr{I}_{ij}{}^{(2)}] + \mathbb{E}[\mathscr{I}_{ij}{}^{(3)}] + \mathbb{E}[\mathscr{I}_{ij}{}^{(4)}]\right), \tag{6.5.5}$$

where for $1 \le k \le 4$, $\mathscr{I}_{ij}{}^{(k)}$ is defined and simplified below. Applying the identities and inequalities of time scales, respectively given in Theorem 1.4.25 and Lemma 1.4.29, and then using Hölder's inequality and Tonelli's theorem in probability theory, we can have

$$\begin{aligned}
\mathbb{E}[\mathscr{I}_{ij}{}^{(1)}] &:= \mathbb{E}\left\|c^{-t/\omega}\int_{-\infty}^t e_{-a_{ij}}(t,\sigma(s)) \sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(s) f(x_{kl}(s-\eta_{kl}(s)))x_{ij}(s)\Delta s\right\|^p \\
&\le \mathbb{E}\Bigg(\frac{1}{(1-\bar{\mu}\underline{a}_{ij})}\int_{-\infty}^t \left(e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\right)^{\frac{p-1}{p}} \left(e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\right)^{\frac{1}{p}} \\
&\qquad \sum_{C_{kl}\in N_{r_1}(i,j)} \bar{C}_{ij}^{kl}\|f(x_{kl}(s-\eta_{kl}(s)))c^{-s/\omega}x_{ij}(s)\|\Delta s\Bigg)^p \\
&\le \frac{\left(\int_{-\infty}^t e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\Delta s\right)^{p-1}}{(1-\bar{\mu}\underline{a}_{ij})^p} \\
&\qquad \int_{-\infty}^t e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\mathbb{E}\left[\sum_{C_{kl}\in N_{r_1}(i,j)} \bar{C}_{ij}^{kl}\|f(x_{kl}(s-\eta_{kl}(s)))c^{-s/\omega}x_{ij}(s)\|\right]^p \Delta s.
\end{aligned}$$

Applying again Hölder's inequality followed by Tonelli's theorem again in a similar manner, the above inequality yields

$$\begin{aligned}
\mathbb{E}[\mathscr{I}_{ij}{}^{(1)}] \le{}& \frac{\left(\int_{-\infty}^t e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\Delta s\right)^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p} \\
&\int_{-\infty}^t e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\sum_{C_{kl}\in N_{r_1}(i,j)} \mathbb{E}\left(\|f(x_{kl}(s-\eta_{kl}(s)))\|_\infty \|c^{-s/\omega}x_{ij}(s)\|\right)^p \Delta s
\end{aligned}$$

$$\leq mn\left(\frac{\hat{f}}{1-\bar{\mu}\underline{a}_{ij}}\right)^p \mathfrak{S}_C\left(\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^p \mathbb{E}\|x\|_{\mathbb{X}}^p. \quad \text{(by Assumption } (\mathbf{A2}))$$

Similarly,

$$\mathbb{E}[\mathscr{I}_{ij}{}^{(2)}] := \mathbb{E}\Big\|c^{-t/\omega}\int_{-\infty}^t e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(s)\int_0^\infty \mathcal{K}_{ij}(u)g(x_{kl}(s-u))\Delta u x_{ij}(s)\Delta s\Big\|^p$$

$$\leq \mathbb{E}\Bigg[\frac{1}{(1-\bar{\mu}\underline{a}_{ij})}\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)$$

$$\sum_{C_{kl}\in N_{r_2}(i,j)}\bar{B}_{ij}^{kl}\left(\int_0^\infty \mathcal{K}_{ij}(u)\|g(x_{kl}(s-u))\|_{\infty}\Delta u\right)\|c^{-s/\omega}x_{ij}(s)\|\Delta s\Bigg]^p$$

$$\leq mn\left(\frac{\hat{g}\bar{\mathcal{K}}_{ij}}{(1-\bar{\mu}\underline{a}_{ij})}\right)^p \mathfrak{S}_B\left(\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^p \mathbb{E}\|x\|_{\mathbb{X}}^p. \quad \text{(by Assumption } (\mathbf{A2}))$$

Moreover,

$$\begin{aligned}
\mathbb{E}[\mathscr{I}_{ij}{}^{(3)}] &:= \mathbb{E}\Big\|c^{-t/\omega}\int_{-\infty}^t e_{-a_{ij}}(t,\sigma(s))L_{ij}(s)\Delta s\Big\|^p \\
&\leq \mathbb{E}\left[\frac{1}{(1-\bar{\mu}\underline{a}_{ij})}\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\|c^{-s/\omega}L_{ij}(s)\|\Delta s\right]^p \\
&\leq \frac{1}{(1-\bar{\mu}\underline{a}_{ij})^p}\left(\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^p \sup_{t\in\mathbb{T}_\omega}\mathbb{E}\|c^{-t/\omega}L_{ij}(t)\|^p,
\end{aligned}$$

and

$$\mathbb{E}[\mathscr{I}_{ij}{}^{(4)}] := \mathbb{E}\Big\|c^{-t/\omega}\int_{-\infty}^t e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)\delta_{ij}(s,x_{ij}(s))\Delta w_{ij}(s)\Big\|^p$$

$$\leq \mathbb{E}\Big\|\frac{1}{(1-\bar{\mu}\underline{a}_{ij})}\int_{-\infty}^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)c^{-s/\omega}\delta_{ij}(s,x_{ij}(s))\Delta w_{ij}(s)\Big\|^p.$$

By Burkholder–Davis–Gundy inequality on time scales followed by Hölder's inequality, for $p>2$ we have

$$\mathbb{E}[\mathscr{I}_{ij}{}^{(4)}] \leq \frac{C_p}{(1-\bar{\mu}\underline{a}_{ij})^p}\mathbb{E}\left[\int_{-\infty}^t\left(e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\right)^2\left(\sum_{C_{kl}\in N_{r_3}(i,j)}\bar{D}_{ij}^{kl}\|c^{-s/\omega}\delta_{ij}(s,x_{ij}(s))\|\right)^2\Delta s\right]^{p/2}$$

$$\leq \frac{C_p}{(1-\bar{\mu}\underline{a}_{ij})^p}\mathbb{E}\Bigg[\int_{-\infty}^t e_{\frac{2(p-2)}{p}\odot\left(\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)\right)}(t,s)e_{\frac{4}{p}\odot\left(\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)\right)}(t,s)$$

$$\left(\sum_{C_{kl}\in N_{r_3}(i,j)}\bar{D}_{ij}^{kl}\|c^{-s/\omega}\delta_{ij}(s,x_{ij}(s))\|\right)^2\Delta s\Bigg]^{p/2}$$

$$\leq \frac{C_p}{(1-\bar{\mu}\underline{a}_{ij})^p}\left(\int_{-\infty}^t e_{2\odot\left(\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)\right)}(t,s)\Delta s\right)^{\frac{p-2}{2}}$$

$$\int_{-\infty}^t e_{2\odot\left(\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)\right)}(t,s)\mathbb{E}\left[\sum_{C_{kl}\in N_{r_3}(i,j)}\bar{D}_{ij}^{kl}\|c^{-s/\omega}\delta_{ij}(s,x_{ij}(s))\|\right]^p\Delta s$$

(by Hölder's inequality and Tonelli's theorem)

$$\leq \frac{C_p \mathfrak{S}_D}{(1-\bar{\mu}\underline{a}_{ij})^p} \left(\int_{-\infty}^{t} e_{2\odot(\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega}))}(t,s)\Delta s \right)^{\frac{p}{2}} \sum_{i,j} \sup_{t\in\mathbb{T}} \mathbb{E}\|c^{-t/\omega}\delta_{ij}(t, x_{ij}(t))\|^p$$

$$\leq \frac{mnC_p\hat{\delta}_{ij}}{(1-\bar{\mu}\underline{a}_{ij})^p} \mathfrak{S}_D \left(\int_{-\infty}^{t} e_{2\odot(\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega}))}(t,s)\Delta s \right)^{\frac{p}{2}}, \quad \text{(by Assumption } (\mathbf{A3}))$$

while, by Itô isometry on time scales, for $p = 2$, we obtain

$$\mathbb{E}[\mathscr{I}_{ij}{}^{(4)}] \leq \frac{1}{(1-\bar{\mu}\underline{a}_{ij})^2} \mathbb{E}\left[\int_{-\infty}^{t} e_{2\odot(\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega}))}(t,s) \left(\sum_{C_{kl}\in N_{r_3}(i,j)} \bar{D}_{ij}^{kl} \|c^{-s/\omega}\delta_{ij}(s, x_{ij}(s))\| \right)^2 \Delta s \right]$$

$$\leq \left(\frac{mn \sum_{C_{kl}\in N_{r_3}(i,j)} (\bar{D}_{ij}^{kl})^2}{(1-\bar{\mu}\underline{a}_{ij})^2} \right) \left(\int_{-\infty}^{t} e_{2\odot(\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega}))}(t,s)\Delta s \right) \hat{\delta}_{ij}.$$

Note that for $\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right) > 0$ we have

$$\int_{-\infty}^{t} e_{2\odot(\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega}))}(t,s)\Delta s \leq \int_{-\infty}^{t} e_{\ominus(\underline{a}_{ij}+\frac{ln|c|}{\omega})}(t,s)\Delta s \leq \frac{1}{\underline{a}_{ij} + \frac{ln|c|}{\omega}}.$$

Thus, from (6.5.5) we can obtain the following:

$$\mathbb{E}\|c^{-t/\omega}(\Gamma_{ij}(x))(t)\|^p \leq \frac{4^{p-1}mn}{(1-\bar{\mu}\underline{a}_{ij})^p} \left[\frac{\hat{f}^p \mathfrak{S}_C \kappa^p + \hat{g}^p \bar{\mathcal{K}}_{ij}^p \mathfrak{S}_B \kappa^p + \frac{\hat{L}_{ij}^p}{mn}}{\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)^p} + \frac{C_p \kappa^p \hat{\delta}_{ij}^p \mathfrak{S}_D}{\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)^{p/2}} \right] \text{(when } p > 2),$$

and

$$\mathbb{E}\|c^{-t/\omega}(\Gamma_{ij}(x))(t)\|^2 \leq \frac{4mn}{(1-\bar{\mu}\underline{a}_{ij})^2} \Bigg[\Bigg(\frac{\hat{f}^2}{\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)^2} \sum_{C_{kl}\in N_{r_1}(i,j)} (\bar{C}_{ij}^{kl})^2 \kappa^2$$
$$+ \hat{g}^2 \bar{\mathcal{K}}_{ij}^2 \sum_{C_{kl}\in N_{r_2}(i,j)} (\bar{B}_{ij}^{kl})^2 \kappa^2 + \frac{\hat{L}_{ij}^2}{mn} \Bigg) + \frac{\hat{\delta}_{ij}^2 \kappa^2}{\underline{a}_{ij} + \frac{ln|c|}{\omega}} \sum_{C_{kl}\in N_{r_3}(i,j)} (\bar{D}_{ij}^{kl})^2 \Bigg].$$

In view of $(\mathbf{A5})$, this finally yields

$$\|\Gamma_{ij}(x)\|_{\mathbb{X}}^p \leq \kappa^p \quad \text{for all} \quad p \geq 2.$$

In addition to this, by a similar approach, Γx is stochastically $\mathcal{L}^p$-rd-continuous. Therefore, we have $\Gamma(\mathbb{X}^*) \subset \mathbb{X}^*$.

Next, we show that Γ is a contraction mapping. In fact, in view of $(\mathbf{A1})$–$(\mathbf{A5})$, for any $x_{ij}, y_{ij} \in B^*$, we easily obtain that

$\mathbb{E}[\mathscr{H}_{ij}{}^{(1)}]$

$$:= \mathbb{E}\Big\|c^{-t/\omega}\int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s)) \sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(s)\left(f(x_{kl}(s-\eta_{kl}(s)))x_{ij}(s) - f(y_{kl}(s-\eta_{kl}(s)))y_{ij}(s)\right)\Delta s\Big\|^{p}$$

$$\leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{a}_{ij})^{p}}\Bigg[\mathbb{E}\Big[\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s) \sum_{C_{kl}\in N_{r_1}(i,j)} \bar{C}_{ij}^{kl}\|f(x_{kl}(s-\eta_{kl}(s)))c^{-s/\omega}\left(x_{ij}(s)-y_{ij}(s)\right)\|\Delta s\Big]^{p}$$

$$+\mathbb{E}\Big[\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s) \sum_{C_{kl}\in N_{r_1}(i,j)} \bar{C}_{ij}^{kl}\|\left(f(x_{kl}(s-\eta_{kl}(s)) - f(y_{kl}(s-\eta_{kl}(s)))\right)c^{-s/\omega}y_{ij}(s)\|\Delta s\Big]^{p}\Bigg]$$

$$\leq \frac{2^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^{p}}\Bigg[\hat{f}^{p}\left(\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^{p}\sum_{i,j}\sup_{t\in\mathbb{T}}\mathbb{E}\|c^{-t/\omega}(x_{ij}(t)-y_{ij}(t))\|^{p}$$

$$+\left(\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^{p-1}\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\sum_{C_{kl}\in N_{r_1}(i,j)}\Big(\mathbb{E}\|f(x_{kl}(s-\eta_{kl}(s))$$

$$-f(y_{kl}(s-\eta_{kl}(s)))\|^{\frac{p}{p-1}}\Big)^{p-1}\mathbb{E}\|c^{-s/\omega}y_{ij}(s)\|^{p}\Delta s\Bigg] \quad \text{(by Hölder's inequality)}$$

$$\leq \frac{2^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^{p}}\left(\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^{p}\Bigg[mn\hat{f}^{p}\|x-y\|_{\mathbb{X}}^{p} + \mathtt{L}_f^{p}\sup_{t\in\mathbb{T}}\mathbb{E}\|c^{-t/\omega}y_{ij}(t)\|^{p}$$

$$\sum_{C_{kl}\in N_{r_1}(i,j)}\mathbb{E}\|c^{-(t-\eta_{kl}(t))/\omega}\left(x_{kl}(t-\eta_{kl}(t)) - y_{kl}(t-\eta_{kl}(t))\right)\|^{p}\Bigg] \quad \text{(by Assumption } (\mathbf{A2}))$$

$$\leq \frac{mn2^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^{p}\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p}}\left(\hat{f}^{p}+\mathtt{L}_f^{p}\kappa^{p}\right)\|x-y\|_{\mathbb{X}}^{p}.$$

Similarly, we can have

$$\mathbb{E}[\mathscr{H}_{ij}{}^{(2)}] := \mathbb{E}\Big\|c^{-t/\omega}\int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(s)\left(\int_0^{\infty}\mathcal{K}_{ij}(u)g(x_{kl}(s-u))\Delta u x_{ij}(s)\right.$$

$$\left.-\int_0^{\infty}\mathcal{K}_{ij}(u)g(y_{kl}(s-u))\Delta u y_{ij}(s)\right)\Delta s\Big\|^{p}$$

$$\leq \frac{2^{p-1}}{(1-\bar{\mu}\underline{a}_{ij})^{p}}\mathbb{E}\Bigg[\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\sum_{C_{kl}\in N_{r_2}(i,j)}\bar{B}_{ij}^{kl}\bigg(\int_0^{\infty}\mathcal{K}_{ij}(u)\|\big(g(x_{kl}(s-u))$$

$$-g(y_{kl}(s-u))\big)c^{-s/\omega}x_{ij}(s)\|\Delta u\bigg)\Delta s\Bigg]^{p} + \frac{2^{p-1}}{(1-\bar{\mu}\underline{a}_{ij})^{p}}\mathbb{E}\Bigg[\int_0^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)$$

$$\sum_{C_{kl}\in N_{r_2}(i,j)}\bar{B}_{ij}^{kl}\left(\int_0^{\infty}\mathcal{K}_{ij}(u)\|g(x_{kl}(s-u))\|_{\infty}\Delta u\right)\|c^{-s/\omega}\left(x_{ij}(s)-y_{ij}(s)\right)\|\Delta s\Bigg]^{p}$$

$$\leq \frac{2^{p-1}\mathfrak{S}_B}{(1-\bar{\mu}\underline{a}_{ij})^{p}\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p-1}}\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Bigg[\left(\int_0^{\infty}\mathcal{K}_{ij}(u)\Delta u\right)^{p-1}$$

$$\sum_{C_{kl}\in N_{r_2}(i,j)}\left(\int_0^{\infty}\mathcal{K}_{ij}(u)\mathtt{L}_g^{p}\mathbb{E}\|c^{-(s-u)/\omega}\big(x_{kl}(s-u)-y_{kl}(s-u)\big)\|^{p}\mathbb{E}\|c^{-s/\omega}x_{ij}(s)\|^{p}\right)\Delta u\Bigg]\Delta s$$

$$+\frac{mn2^{p-1}\bar{\mathcal{K}}_{ij}^{p}\hat{g}^{p}\mathfrak{S}_B}{(1-\bar{\mu}\underline{a}_{ij})^{p}\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p}}\|x-y\|_{\mathbb{X}}^{p}$$

$$\leq \frac{mn2^{p-1}\bar{\mathcal{K}}_{ij}^{p}\mathfrak{S}_B}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^p}\left(\hat{g}^p+\mathrm{L}_g^p\kappa^p\right)\|x-y\|_{\mathbb{X}}^p, \quad \text{(by Assumptions (}\mathbf{A2}\text{) and (}\mathbf{A4}\text{))}$$

and

$$\begin{aligned}
&\mathbb{E}[\mathscr{H}_{ij}{}^{(3)}]\\
&:= \mathbb{E}\Bigg\|c^{-t/\omega}\int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)\left(\delta_{ij}(s,x_{ij}(s)-\delta_{ij}(s,y_{ij}(s))\right)\Delta w_{ij}(s)\Bigg\|^p\\
&\leq \mathbb{E}\Bigg\|\frac{1}{(1-\bar{\mu}\underline{a}_{ij})}\int_{-\infty}^{t} e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)c^{-s/\omega}\left(\delta_{ij}(s,x_{ij}(s)-\delta_{ij}(s,y_{ij}(s))\right)\Delta w_{ij}(s)\Bigg\|^p.
\end{aligned}$$

For $p>2$, by the Burkholder-Davis-Gundy inequality on time scales and Assumption ($\mathbf{A3}$), the above yields

$$\begin{aligned}
\mathbb{E}[\mathscr{H}_{ij}{}^{(3)}] &\leq \frac{C_p mn l_{ij}^p\mathfrak{S}_D}{(1-\bar{\mu}\underline{a}_{ij})^p}\left(\int_{-\infty}^{t} e_{2\odot\left(\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)\right)}(t,s)\Delta s\right)^{\frac{p}{2}}\sup_{t\in\mathbb{T}}\mathbb{E}\|c^{-t/\omega}(x_{ij}(t)-y_{ij}(t))\|^p\\
&\leq \frac{C_p mn l_{ij}^p\mathfrak{S}_D}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p/2}}\|x-y\|_{\mathbb{X}}^p.
\end{aligned}$$

and for $p=2$, by Itô isometry on time scales together with Assumption ($\mathbf{A3}$), we obtain

$$\mathbb{E}[\mathscr{H}_{ij}{}^{(3)}] \leq \frac{mnl_{ij}^2\sum_{C_{kl}\in N_{r_3}(i,j)}(\bar{D}_{ij}^{kl})^2}{(1-\bar{\mu}\underline{a}_{ij})^2\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}\|x-y\|_{\mathbb{X}}^2.$$

Notice that

$$\mathbb{E}\|c^{-t/\omega}(\Gamma x_{ij}(t)-\Gamma y_{ij}(t))\|^p \leq 3^{p-1}\left(\mathbb{E}[\mathscr{H}_{ij}{}^{(1)}]+\mathbb{E}[\mathscr{H}_{ij}{}^{(2)}]+\mathbb{E}[\mathscr{H}_{ij}{}^{(3)}]\right).$$

Thus, for $p>2$ the above inequalities yield

$$\begin{aligned}
\mathbb{E}\|c^{-t/\omega}(\Gamma x_{ij}(t)-\Gamma y_{ij}(t))\|^p \leq \frac{3^{p-1}mn}{(1-\bar{\mu}\underline{a}_{ij})^p}&\left[\frac{2^{p-1}\left(\hat{f}^p+\mathrm{L}_f^p\kappa^p\right)\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^p}\right.\\
&\left.+\frac{2^{p-1}\bar{\mathcal{K}}_{ij}^p\left(\hat{g}^p+\mathrm{L}_g^p\kappa^p\right)\mathfrak{S}_B}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^p}+\frac{C_p l_{ij}^p\mathfrak{S}_D}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p/2}}\right]\|x-y\|_{\mathbb{X}}^p,
\end{aligned}$$

and, for $p=2$ we get

$$\begin{aligned}
\mathbb{E}\|c^{-t/\omega}(\Gamma x_{ij}(t)-\Gamma y_{ij}(t))\|^2 \leq \frac{3mn}{(1-\bar{\mu}\underline{a}_{ij})^2}&\left[\frac{2\left(\hat{f}^2+\mathrm{L}_f^2\kappa^2\right)}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^2}\sum_{C_{kl}\in N_{r_1}(i,j)}(\bar{C}_{ij}^{kl})^2+\frac{2\bar{\mathcal{K}}_{ij}^2\left(\hat{g}^2+\mathrm{L}_g^2\kappa^2\right)}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^2}\right.\\
&\left.\sum_{C_{kl}\in N_{r_2}(i,j)}(\bar{B}_{ij}^{kl})^2+\frac{l_{ij}^2}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}\sum_{C_{kl}\in N_{r_3}(i,j)}(\bar{D}_{ij}^{kl})^2\right]\|x-y\|_{\mathbb{X}}^2.
\end{aligned}$$

Hence, in view of (**A6**), we have

$$\|\Gamma x_{ij} - \Gamma y_{ij}\|_{\mathbb{X}}^p < \|x - y\|_{\mathbb{X}}^p \text{ for all } p \geq 2,$$

which implies that Γ is a contraction mapping. Therefore, Γ has a unique fixed point x in $\mathbb{X}^*$, that is, (6.3.1) has a unique solution x in $\mathbb{X}^*$. $\square$

Remark 6.5.5. *The constant $\mathcal{M}_p$ introduced in Assumption (**A6**) represents the contraction coefficient of the operator Γ corresponding to the stochastic SICNN model* (6.3.1)*. It reflects the balance between the Lipschitz effects of the nonlinear terms f, g, and δ_{ij} and the stabilizing decay rate a_{ij} of the network. The requirement $\mathcal{M}_p < 1$ ensures that the intrinsic decay dominates the nonlinear and stochastic interactions, thereby guaranteeing that Γ is a contraction. If the combined Lipschitz effects of f, g, and δ_{ij} become too large relative to the decay rate a_{ij}, then $\mathcal{M}_p$ may exceed one and the operator may fail to be contractive. In this case, the Banach fixed-point theorem used to establish existence and uniqueness is no longer applicable, and alternative analytical methods would be required to investigate the existence and uniqueness of solutions.*

Remark 6.5.6. *It is important to emphasize that the uniqueness we obtain in Theorem 6.5.4 is a strong uniqueness, that is, a pathwise uniqueness. This means that the solution is unique as a function of time ($t \in \mathbb{T}_\omega^{0+}$) once the underlying randomness is fixed. Different sample paths of the driving noise may lead to different realizations of the solution, but for a fixed* Wiener path *and fixed random variables (if any), the solution is uniquely determined. Formally, if $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are two solutions of* (6.3.1) *corresponding to the same* Wiener path *and the same fixed random inputs, then*

$$\mathbb{P}\Big(\mathcal{X}(t) - \mathcal{Y}(t) \textit{ for all } t \in \mathbb{T}_\omega^{0+}\Big) = 1.$$

Next, we show that under the same conditions as in Theorem 6.5.4, system (6.3.1) admits a unique S-asymptotically (ω, c)-periodic solution in the p-th mean sense.

Theorem 6.5.7. *Consider that assumptions (**A1**)–(**A6**) hold. Then, the system* (6.3.1) *has a unique(in the pathwise sense) p-th mean S-asymptotically (ω, c)-periodic solution in $\mathbb{X}^* = \{x = (x_{11}, x_{12}, \dots, x_{mn})^T : x_{ij} \in (\mathrm{BC}_{\mathrm{rd}})_{\omega,c}(\mathbb{T}_\omega^{0+}, \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$ and $\|x\|_{\mathbb{X}} \leq \kappa\}$.*

Proof. Theorem 6.5.4 ensures that system (6.3.1) admits a unique solution x in $\mathbb{X}^*$, given as

$$x(t) = \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) {S_x}^{ij}(s)\Delta s + \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) {T_x}^{ij}(s)\Delta w_{ij}(s).$$

We only need to show that x is S-asymptotically (ω, c)-periodic. In fact, we can rewrite the operator defined in (6.5.4) as

$$(\Gamma_{ij}x)(t) = e_{-a_{ij}}(t, 0)\int_{-\infty}^{0} e_{-a_{ij}}(0, \sigma(s)) {S_x}^{ij}(s)\Delta s + \int_{0}^{t} e_{-a_{ij}}(t, \sigma(s)) {S_x}^{ij}(s)\Delta s$$

$$+ e_{-a_{ij}}(t,0)\int_{-\infty}^{0} e_{-a_{ij}}(0,\sigma(s)){T_x}^{ij}(s)\Delta w_{ij}(s) + \int_0^t e_{-a_{ij}}(t,\sigma(s)){T_x}^{ij}(s)\Delta w_{ij}(s)$$
$$= e_{-a_{ij}}(t,0)(\Gamma_{ij}x)(0) + \int_0^t e_{-a_{ij}}(t,\sigma(s)){S_x}^{ij}(s)\Delta s + \int_0^t e_{-a_{ij}}(t,\sigma(s)){T_x}^{ij}(s)\Delta w_{ij}(s).$$

Now, since $a_{ij}(t+\omega) = a_{ij}(t)$ holds for all $t \in \mathbb{T}_\omega$ and for each $i = 1,2,\ldots,n, j = 1,2,\ldots,m$, it follows that

$$\begin{aligned}
&\|c^{-t/\omega}(e_{-a_{ij}}(t+\omega,0)(\Gamma_{ij}x)(0) - ce_{-a_{ij}}(t,0)(\Gamma_{ij}x)(0))\|^p \\
&\quad\le 2^{p-1}|(\Gamma_{ij}x)(0)|^p\left(\|c^{-t/\omega}e_{-a_{ij}}(t+\omega,\omega)e_{-a_{ij}}(\omega,0)\|^p + |c|^p\|c^{-t/\omega}e_{-a_{ij}}(t,0)\|^p\right) \\
&\quad\le 2^{p-1}|(\Gamma_{ij}x)(0)|^p\left(e_{p\odot(-\underline{a}_{ij})}(\omega,0) + |c|^p\right)\left(e_{\ominus\left(\underline{a}_{ij}+\frac{\ln|c|}{\omega}\right)}(t,0)\right)^p.
\end{aligned}$$

Thus,

$$\lim_{t\to\infty}\mathbb{E}\|c^{-t/\omega}(e_{\ominus a_{ij}}(t+\omega,0)x(0) - ce_{\ominus a_{ij}}(t,0)x(0)))\|^p = 0.$$

Again, using some mathematical computation along with assumption (**A2**), we can easily obtain

$$\begin{aligned}
\mathbb{E}[{\mathscr{K}_{ij}}^{(1)}] &:= \mathbb{E}\Bigg\|c^{-t/\omega}\Bigg(\sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t+\omega)f(x_{kl}(t+\omega-\eta_{kl}(t+\omega)))x_{ij}(t+\omega) \\
&\qquad - c\sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t-\eta_{kl}(t)))x_{ij}(t)\Bigg)\Bigg\|^p \\
&\le 2^{p-1}\Bigg[\mathbb{E}\Bigg\|\sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t+\omega-\eta_{kl}(t)))\Big(c^{-t/\omega}(x_{ij}(t+\omega) - cx_{ij}(t))\Big)\Bigg\|^p \\
&\quad + \mathbb{E}\Bigg\|\sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(t)c\Big(f(x_{kl}(t+\omega-\eta_{kl}(t))) - f(x_{kl}(t-\eta_{kl}(t)))\Big)c^{-t/\omega}x_{ij}(t)\Bigg\|^p\Bigg] \\
&\le 2^{p-1}mn\mathfrak{S}_C\Bigg[\hat{f}_{ij}^p\mathbb{E}\|c^{-t/\omega}(x_{ij}(t+\omega) - cx_{ij}(t))\|^p \\
&\qquad + |c|^{p-1}\kappa^p\mathbb{E}\|c^{-(t-\eta_{kl}(t)+\omega)/\omega}(x_{kl}(t-\eta_{kl}(t)) - cx_{kl}(t-\eta_{kl}(t)))\|^p\Bigg],
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[{\mathscr{K}_{ij}}^{(2)}] &:= \mathbb{E}\Bigg\|c^{-t/\omega}\Bigg(\sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t+\omega)\int_0^\infty \mathcal{K}_{ij}(u)g(x_{kl}(t+\omega-u))\Delta u x_{ij}(t+\omega) \\
&\qquad - c\sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t)\int_0^\infty \mathcal{K}_{ij}(u)g(x_{kl}(t-u))\Delta u x_{ij}(t)\Bigg)\Bigg\|^p \\
&\le 2^{p-1}mn\mathfrak{S}_B\Bigg(\hat{g}_{ij}^p\left(\int_0^\infty \mathcal{K}_{ij}(u)\Delta u\right)^p\mathbb{E}\|c^{-t/\omega}\left(x_{ij}(t+\omega) - cx_{ij}(t)\right)\|^p + \\
&\quad |c|^p\mathbb{E}\left[\int_0^\infty \mathcal{K}_{ij}(u)\|g(x_{ij}(t+\omega-u)) - g(x_{ij}(t-u))\|\Delta u\right]^p\sup_t\|c^{-t/\omega}x_{ij}(t)\|^p\Bigg)
\end{aligned}$$

$$\leq 2^{p-1} mn\bar{\mathcal{K}}_{ij}^{p} \mathfrak{S}_B \Bigg[\hat{g}_{ij}^{p} \mathbb{E} \| c^{-t/\omega} \left(x_{ij}(t+\omega) - c x_{ij}(t) \right) \|^p + |c|^{p-1} \kappa^p \mathbb{E} \Bigg[\sup_{u \in \Pi_{\mathbb{T}} \cap \mathbb{T}_0^+} \| c^{-(t-u)/\omega} (x_{ij}(t-u+\omega) - c x_{ij}(t-u)) \|^p \Bigg] \Bigg],$$

and

$$\mathbb{E}[\mathscr{K}_{ij}{}^{(3)}] := \mathbb{E} \| c^{-t/\omega} (L_{ij}(t+\omega) - c L_{ij}(t)) \|^p.$$

Consequently, under assumption (**A1**), if $x \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_{\omega}^{0+} \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$, then the above inequalities imply

$$\lim_{t\to\infty} \| c^{-t/\omega} (S_x{}^{ij}(t+\omega) - c S_x{}^{ij}(t)) \|^p \leq 3^{p-1} \left(\lim_{t\to\infty} \mathbb{E}[\mathscr{K}_{ij}{}^{(1)}] + \lim_{t\to\infty} \mathbb{E}[\mathscr{K}_{ij}{}^{(2)}] + \lim_{t\to\infty} \mathbb{E}[\mathscr{K}_{ij}{}^{(3)}] \right) = 0,$$

that is, S_x^{ij} is p-th mean S-asymptotically (ω, c)-periodic. Following Lemma 6.4.21 we conclude that $\int_0^t e_{-a_{ij}}(t, \sigma(s)) S_x{}^{ij}(s) \Delta s$ is p-th mean S-asymptotically (ω, c)-periodic. Again, applying assumption (**A3**) together with Theorem 6.4.16 and Lemma 6.4.19, and subsequently using Lemma 6.4.22, it follows that $\int_0^t e_{-a_{ij}}(t, \sigma(s)) T_x{}^{ij}(s) \Delta w_{ij}(s)$ is p-th mean S-asymptotically (ω, c)-periodic whenever $x \in S\mathrm{AP}_{\omega,c}(\mathbb{T}_{\omega}^{0+} \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$. Thus, we have $\Gamma(S\mathrm{AP}_{\omega,c}(\mathbb{T}_{\omega}^{0+} \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))) \subset S\mathrm{AP}_{\omega,c}(\mathbb{T}_{\omega}^{0+} \mathcal{L}^p(\Omega, \mathbb{H}_{\mathbb{F}}))$. □

Next, we establish the global stability conditions for the unique p-th mean S-asymptotically (ω, c)-periodic solution of system (6.3.1) in terms of its coefficients. The following theorem ensures the exponential attractivity of the unique S-asymptotically (ω, c)-periodic solution of (6.3.1) in the p-th mean sense.

Theorem 6.5.8. *Under the assumptions of Theorem 6.5.7, let $x \in \mathbb{X}^*$ be the unique p-th mean S-asymptotically (ω, c)-periodic solution of system (6.3.1) with initial value ϕ. Let y be an arbitrary solution of the perturbed system corresponding to (6.3.1), with an arbitrary initial value ψ. Then, there exist positive constants λ_p and M such that*

$$\max_{i,j} \mathbb{E} \| c^{-t/\omega} (x_{i,j}(t) - y_{i,j}(t)) \|^p \leq M e_{\ominus \lambda_p}(t, 0) \max_{i,j} \left(\sup_{s \in (-\theta, 0]_{\mathbb{T}_\omega}} \mathbb{E} \| c^{-s/\omega} (\phi_{ij}(s) - \psi_{ij}(s)) \|^p \right) \text{ for all } t > 0, \tag{6.5.6}$$

i.e., the solution x of the system (6.3.1) is globally exponentially stable in the p-th mean sense.

Proof. Let $z = y - x$. Then, from (6.3.1), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}\Delta z_{ij}(t) = \Bigg[&- a_{ij}(t) z_{ij}(t) - \sum_{C_{kl} \in N_{r_1}(i,j)} C_{ij}^{kl}(t) \left(f(x_{kl}(t-\eta_{kl}(t))) x_{ij}(t) - f(y_{kl}(t-\eta_{kl}(t))) y_{ij}(t) \right) - \\ &\sum_{C_{kl} \in N_{r_2}(i,j)} B_{ij}^{kl}(t) \left(\int_0^\infty \mathcal{K}_{ij}(u) g(x_{kl}(t-u)) \Delta u x_{ij}(t) - \int_0^\infty \mathcal{K}_{ij}(u) g(y_{kl}(t-u)) \Delta u y_{ij}(t) \right) \Bigg] \Delta t \\ &+ \sum_{C_{kl} \in N_{r_3}(i,j)} D_{ij}^{kl}(t) \left(\delta_{ij}(t, x_{ij}(t)) - \delta_{ij}(t, y_{ij}(t)) \right) \Delta w_{ij}(t), \quad t \in \mathbb{T}_\omega.\end{aligned} \tag{6.5.7}$$

Consequently, we have

$$
\begin{aligned}
z_{ij}(t) &= e_{-a_{ij}}(t,0)z_{ij}(0) + \int_0^t e_{-a_{ij}}(t,\sigma(s))\Bigg(- \sum_{C_{kl}\in N_{r_1}(i,j)} C_{ij}^{kl}(s)\left(f(x_{kl}(s-\eta_{kl}(s)))x_{ij}(t)\right.\\
&\quad \left. -f(y_{kl}(t-\eta_{kl}(t)))y_{ij}(t)\right) - \sum_{C_{kl}\in N_{r_2}(i,j)} B_{ij}^{kl}(t)\left(\int_0^\infty \mathcal{K}_{ij}(u)g(x_{kl}(s-u))\Delta u x_{ij}(s)\right.\\
&\quad \left. - \int_0^\infty \mathcal{K}_{ij}(u)g(y_{kl}(s-u))\Delta u y_{ij}(s)\right)\Bigg)\Delta s - \int_0^t e_{-a_{ij}}(t,\sigma(s)) \sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)\Big(\delta_{ij}(s,x_{ij}(s))\\
&\quad - \delta_{ij}(s,y_{ij}(s))\Big)\Delta w_{ij}(s). \qquad (6.5.8)
\end{aligned}
$$

For $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, let us define a function

$$f_{ij}^p(x) := \left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right) - x - \mathcal{M}_p\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)(1+\bar{\mu}x).$$

It is clear that $f_{ij}^p(x)$ is continuous on $[0,\infty]$ and $\lim_{x\to\infty} f_{ij}^p(x) = -\infty$. In addition to this, if we consider $\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right) > 0$, then by assumption (**A6**) we have $f_{ij}^p(0) > 0$ for all $p \geq 2$. Thus, by the intermediate value theorem, there exists $\alpha_{ij} > 0$ such that $f_{ij}^p(\alpha_{ij}) = 0$ and $f_{ij}^p(x) > 0$ for $x \in (0,\alpha_{ij})$. Consequently, we can choose a constant $0 < \lambda_p < \min_{i,j}\left\{\alpha_{ij}, \left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)\right\}$ such that $f_{ij}^p(\lambda_p) > 0$. This implies that

$$M := \max_{i,j} \frac{\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right) - \lambda_p}{\mathcal{M}_p\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)(1+\bar{\mu}\lambda_p)} > 1. \qquad (6.5.9)$$

From (6.5.9) we obtain the following important inequality, which will be useful in the subsequent analysis.

$$\frac{1}{M} - \min_{i,j} \frac{\mathcal{M}_p\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right)(1+\bar{\mu}\lambda_p)}{4^{p-1}\left(\underline{a}_{ij} + \frac{ln|c|}{\omega}\right) - \lambda_p} \leq 0. \qquad (6.5.10)$$

Now, for each i,j and $t \in [-\theta,0]_{\mathbb{T}_\omega}$, we clearly have

$$
\begin{aligned}
\mathbb{E}\|c^{-t/\omega}(x_{ij}(t) - y_{ij}(t))\|^p &= \mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t) - \psi_{ij}(t))\|^p \\
&\leq Me_{\ominus\lambda_p}(t,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}} \mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t) - \psi_{ij}(t))\|^p\right).
\end{aligned}
$$

Thus, for any $\varepsilon > 0$ and $t \in [-\theta,0]_{\mathbb{T}_\omega}$, we can write

$$\mathbb{E}\|c^{-t/\omega}(x_{ij}(t) - y_{ij}(t))\|^p < Me_{\ominus\lambda_p}(t,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}} \mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t) - \psi_{ij}(t))\|^p + \varepsilon\right) \qquad (6.5.11)$$

We claim that the above inequality is also satisfied for every $t \in \mathbb{T}_\omega^+$. If not, then there exists some $t_1 > 0$ such that

$$\mathbb{E}\|c^{-t_1/\omega}(x_{ij}(t_1) - y_{ij}(t_1))\|^p = Me_{\ominus\lambda_p}(t_1,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}} \mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t) - \psi_{ij}(t))\|^p + \varepsilon\right) \quad \text{and}$$

$$\mathbb{E}\|c^{-t/\omega}(x_{ij}(t)-y_{ij}(t))\|^p < Me_{\ominus\lambda_p}(t,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right)\forall\ t\in(0,t_1)_{\mathbb{T}_\omega}. \tag{6.5.12}$$

Now, proceeding similar to the calculation of $\mathbb{E}[\mathscr{H}_{ij}{}^{(1)}]$ in Theorem 6.5.4 and employing the inequality given in (6.5.12), we can have

$$\begin{aligned}
&\mathbb{E}[\mathscr{L}_{ij}{}^{(1)}]\\
&:=\mathbb{E}\left\|c^{-t_1/\omega}\int_0^{t_1}e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_1}(i,j)}C_{ij}^{kl}(s)\left(f(x_{kl}(s-\eta_{kl}(s)))x_{ij}(s)-f(y_{kl}(s-\eta_{kl}(s)))y_{ij}(s)\right)\Delta s\right\|^p\\
&\le\frac{2^{p-1}\left(\int_0^t e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t,s)\Delta s\right)^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p}\Bigg[\hat{f}^p\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\sum_{i,j}\mathbb{E}\|c^{-s/\omega}\left(x_{ij}(s)-y_{ij}(s)\right)\|^p\Delta s\\
&\quad+\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\left(\sum_{C_{kl}\in N_{r_1}(i,j)}\mathbb{E}\|\left(f(x_{kl}(s-\eta_{ij}(s))-f(y_{kl}(s-\eta_{ij}(s)))\right)c^{-s/\omega}y_{ij}(s)\|^p\right)\Delta s\Bigg]\\
&\le\Bigg[\hat{f}^p\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)Me_{\ominus\lambda_p}(s,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right)\Delta s\\
&\quad+\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\left(\mathrm{L}_f^p\kappa^p\max_{i,j}\left(\sup_{t\in\mathbb{T}_0^+}\mathbb{E}\|c^{-t/\omega}\left(x_{ij}(t)-y_{ij}(t)\right)\|^p\right)\right)\Delta s\Bigg]\frac{mn2^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p-1}}\\
&\qquad\qquad\qquad\qquad\text{(by (6.5.12) and Hölder's inequality followed by Assumption }(\mathbf{A2}))\\
&\le\frac{Mmn2^{p-1}\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p-1}}\left[\hat{f}^p\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)e_{\ominus\lambda_p}(s,0)\Delta s+\mathrm{L}_f^p\kappa^p\int_0^{t_1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right]\\
&\quad\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right)\quad\text{(by (6.5.12))}\\
&\le\frac{Me_{\ominus\lambda_p}(t_1,0)mn2^{p-1}\left(\hat{f}^p+\mathrm{L}_f^p\kappa^p\right)\mathfrak{S}_C}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p-1}}\left(\int_0^{t_1}e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right)\\
&\quad\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right).
\end{aligned}$$

Similarly, proceeding as to the calculation of $\mathbb{E}[\mathscr{H}_{ij}{}^{(2)}]$ and $\mathbb{E}[\mathscr{H}_{ij}{}^{(3)}]$ in Theorem 6.5.4 and employing the inequality given in (6.5.12), we can obtain the following bounds:

$$\begin{aligned}
\mathbb{E}[\mathscr{L}_{ij}{}^{(2)}]\quad:=\quad&\mathbb{E}\Bigg\|c^{-t/\omega}\int_0^t e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_2}(i,j)}B_{ij}^{kl}(s)\Bigg(\int_0^\infty\mathcal{K}_{ij}(u)g(x_{kl}(s-u))\Delta u x_{ij}(s)\\
&\qquad-\int_0^\infty\mathcal{K}_{ij}(u)g(y_{kl}(s-u))\Delta u y_{ij}(s)\Bigg)\Delta s\Bigg\|^p
\end{aligned}$$

$$\leq \frac{Me_{\ominus\lambda_p}(t_1,0)mn2^{p-1}\bar{\mathcal{K}}_{ij}^p\left(\hat{g}^p + \mathrm{L}_g^p\kappa^p\right)\mathfrak{S}_B}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{p-1}}\left(\int_0^{t_1} e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right)$$
$$\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right),$$

and

$$\mathbb{E}[\mathscr{L}_{ij}{}^{(3)}] := \mathbb{E}\left\|c^{-t/\omega}\int_0^t e_{-a_{ij}}(t,\sigma(s))\sum_{C_{kl}\in N_{r_3}(i,j)} D_{ij}^{kl}(s)\left(\sigma(s,x_{ij}(s)-\sigma(s,y_{ij}(s))\right)\Delta w_{ij}(s)\right\|^p$$
$$\leq \begin{cases} \frac{Me_{\ominus\lambda_p}(t_1,0)C_p mnl_{ij}^p\mathfrak{S}_D}{(1-\bar{\mu}\underline{a}_{ij})^p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)^{\left(\frac{p}{2}-1\right)}}\left(\int_0^{t_1} e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right) & \\ \qquad \max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right) & \text{when } p>2, \\ \frac{Me_{\ominus\lambda_p}(t_1,0)mnl_{ij}^2}{(1-\bar{\mu}\underline{a}_{ij})^2}\sum_{C_{kl}\in N_{r_3}(i,j)}(\bar{D}_{ij}^{kl})^2\left(\int_0^{t_1} e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right) & \\ \qquad \max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right) & \text{when } p=2. \end{cases}$$

Based on Eq. (6.5.8), we can obtain

$$\begin{aligned}
&\mathbb{E}\|c^{-t/\omega}(x_{ij}(t_1)-y_{ij}(t_1))\|^p \\
&\leq 4^{p-1}\left[\mathbb{E}\|c^{-t/\omega}e_{-a_{ij}}(t_1,0)z_{ij}(0)\|^p+\mathbb{E}[\mathscr{L}_{ij}{}^{(1)}]+\mathbb{E}[\mathscr{L}_{ij}{}^{(2)}]+\mathbb{E}[\mathscr{L}_{ij}{}^{(3)}]\right] \\
&\leq 4^{p-1}e_{\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,0)\mathbb{E}\|x_{ij}(0)-y_{ij}(0)\|^p + M\mathcal{M}_p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)e_{\ominus\lambda_p}(t_1,0) \\
&\quad \left(\int_0^{t_1} e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,s)\Delta s\right)\max_{i,j}\left(\sup_t \mathbb{E}\|c^{-t/\omega}\left(x_{ij}(t)-y_{ij}(t)\right)\|^p\right) \\
&\leq Me_{\ominus\lambda_p}(t_1,0)\left[4^{p-1}\left(\frac{1}{M}-\frac{\mathcal{M}_p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)(1+\bar{\mu}\lambda_p)}{4^{p-1}\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}-\lambda_p\right)}\right)e_{\lambda_p\ominus\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)}(t_1,0)\right. \\
&\quad \left.+\frac{\mathcal{M}_p\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}\right)(1+\bar{\mu}\lambda_p)}{\left(\underline{a}_{ij}+\frac{ln|c|}{\omega}-\lambda_p\right)}\right]\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right) \\
&< Me_{\ominus\lambda_p}(t_1,0)\max_{i,j}\left(\sup_{t\in[-\theta,0]_{\mathbb{T}_\omega}}\mathbb{E}\|c^{-t/\omega}(\phi_{ij}(t)-\psi_{ij}(t))\|^p+\varepsilon\right),
\end{aligned}$$

which contradicts (6.5.12). Thus, (6.5.11) holds for all $t\in\mathbb{T}_\omega^+$. Next, letting $\varepsilon\to 0^+$, (6.5.11) finally yield (6.5.6). This completes the proof. □

Remark 6.5.9. *The graininess function $\mu(t)$ determines the local spacing of points in the time scale and enters the generalized exponential $e_\alpha(t,s)$. It directly modulates the effective decay in $e_{\ominus\lambda_p}(t,s)$ and, together with the scaling c (via $\ln|c|/\omega$) and λ_p, determines the stability bound M in the p^{th}-mean exponential* *estimate (6.5.6). Larger graininess, corresponding to more left-scattered points, reduces the effective decay per step, potentially increasing $\mathcal{M}$ and making stability harder to achieve. Thus, the time scale structure, as encoded by $\mu(t)$, and the scaling c critically influence the resulting stability estimates.*

Remark 6.5.10. *The stability of stochastic delayed SICNNs on time scales with p^{th}-mean S-asymptotically (ω, c)-periodic solutions is influenced differently by delays and stochastic perturbations. While the conditions ensuring existence, uniqueness, and exponential stability are independent of the delay lengths, they are sensitive to the diffusion intensities. Applying the Burkholder–Davis–Gundy inequality on time scales bounds the stochastic integrals in the p^{th}-mean, contributing to the estimation of κ in Assumption (**A5**), which in turn governs the contraction constant $\mathcal{M}_p$ and the stability constant M in (6.5.6). The p^{th}-mean exponential stability is preserved as long as the intrinsic decay rates a_{ij} dominate the combined effects of nonlinearities and stochastic perturbations. If the diffusion intensities increase sufficiently, these bounds may be violated, potentially destroying both stability and the S-asymptotic (ω, c)-periodicity of the solution. Hence, we can conclude that delays have a limited impact, whereas stochastic perturbations play a decisive role in dictating the stability of the system.*

6.6 Examples and numerical simulations

The purpose of this section is to validate the results obtained in this paper using numerical examples and their simulations. We present three examples: the first on a continuous time scale, the second on a discrete time scale, and the third on a mixed time scale. We also illustrate the effect of c by choosing $c > 1$ in the first two examples and $c < 1$ in the last example.

Example 6.6.1. *Assuming $\mathbb{T} = \mathbb{H}_{\mathbb{F}} = \mathbb{F} = \mathbb{R}$, consider the system* (6.3.1) *with $m = n = 2$, $r_1 = r_2 = 1$, $r_3 = 0$ and choose*

$$\mathcal{K}_{ij}(t) = 0.75e^{-t}, \quad f(x(t)) = 0.05\tanh(c^{-t}x(t)), \quad g(x(t)) = 0.05\arctan(c^{-t}x(t)),$$

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.5 + 0.1\sin 2\pi t & 0.4 + 0.2|\cos \pi t| \\ 2\exp(\cos 2\pi t) & 1 + 0.5\cos(\sin 2\pi t) \end{bmatrix},$$

$$\begin{bmatrix} \eta_{11}(t) & \eta_{12}(t) \\ \eta_{21}(t) & \eta_{22}(t) \end{bmatrix} = \begin{bmatrix} 1 + 0.5|\sin \pi t| & 1 + \exp(\cos 2\pi t) \\ 2 + 0.5\cos 2\pi t & 1.5 + |\sin \pi t| \end{bmatrix},$$

$$\begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.09 + 0.03\sin 2\pi t & 0.05 + 0.02\cos 2\pi t \\ 0.07 + 0.05\cos 2\pi t & 0.10 + 0.07\sin 2\pi t \end{bmatrix},$$

$$\begin{bmatrix} \delta_{11}(t,x) & \delta_{12}(t,x) \\ \delta_{21}(t,x) & \delta_{22}(t,x) \end{bmatrix} = 0.02\begin{bmatrix} (x + \sin(|x|))/2 + c^t\cos\sqrt{|t|} & x + \arctan x \\ x + \tanh x & (x + \sin x)/2 + c^t\cos(\ln(1 + |t|)) \end{bmatrix}$$

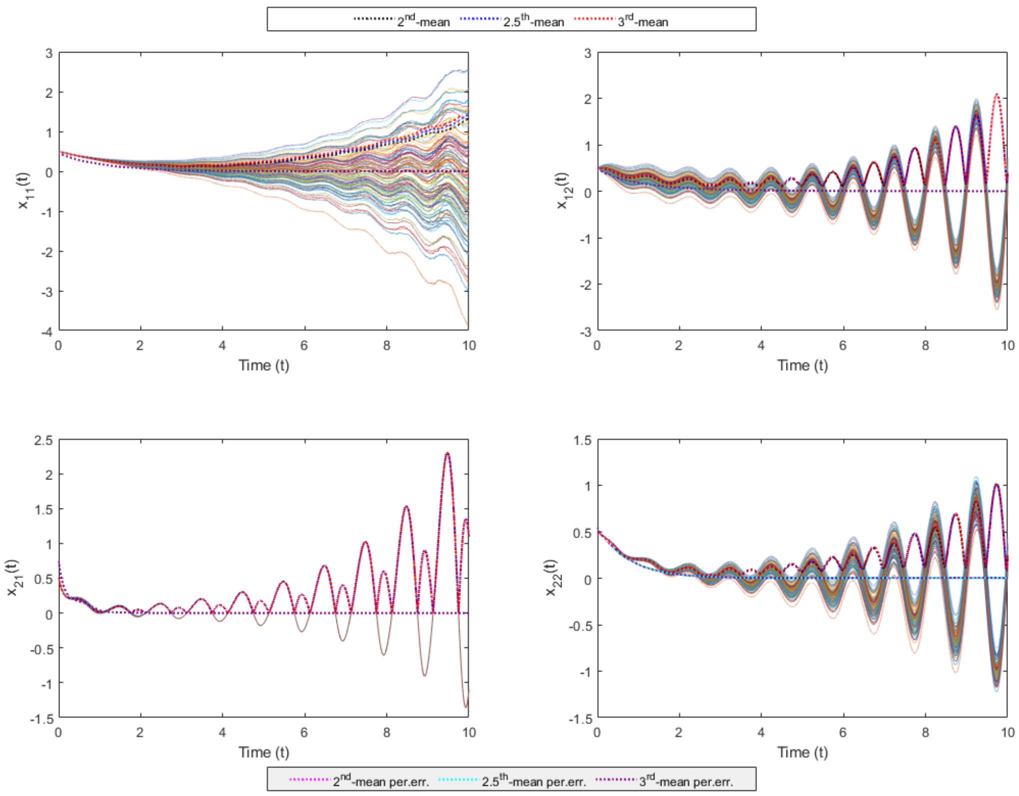

Figure 6.1: Cell activities (x_{ij}) of the SICNN (6.3.1), with parameters specified in Example 6.6.1 and initial function $\phi_{ij} = 0.5$, for 100 random sample paths. For $p = 2, 2.5, 3$, the p-th mean of the 100 simulated solutions are shown in the figures. The corresponding p-th mean (ω, c)-periodic errors, defined in (6.4.3), are also plotted to analyze the exact behavior of the solutions.

$$\begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix} = 0.25 \begin{bmatrix} c^t \left(0.09\sin(2\pi t) + 0.1\sin\left(\ln\left(1+|t|\right)\right)X\right) & c^t \left(\cos(2\pi t) + (1/(|t|+1)^2)X\right) \\ c^t \left(\sin(2\pi t) + 0.1\exp(-t^2)X\right) & c^t \left(0.5\cos(2\pi t) + 0.05(\sin\sqrt{|t|})X\right) \end{bmatrix},$$

*where $c = 1.5$, $X \in N(0,1)$, i.e., $X : \Omega \to \mathbb{R}$ is a measurable function such that its probability distribution is the standard normal distribution with mean 0 and variance 1. Note that $\mathbb{E}|c^{-t}L_{ij}(t)|^p \le \frac{1+\mathbb{E}|X|^p}{2^{p+1}}$. For $\omega = 1$, it is easy to verify that all the conditions of assumptions (**A1**)–(**A4**) hold for the above parameters with $p = 2, 2.5$ and 3. Furthermore, for $\kappa \le 2$, computing the bounds, we get*

$$\min_{i,j} \underline{a}_{ij} = 0.4, \quad \hat{f} = \hat{g} = \max_{i,j} \hat{\delta}_{ij} = \mathtt{L}_f = \mathtt{L}_g = l_{ij} = 0.05, \quad \max_{i,j} \hat{L}_{ij} \le 0.5, \quad \max_{i,j} \bar{\mathcal{K}}_{ij} \le 1,$$

$$\sum_{C_{kl}\in N_1(1,1)} \bar{C}_{11}^{kl} = \sum_{C_{kl}\in N_1(1,2)} \bar{C}_{12}^{kl} = \sum_{C_{kl}\in N_1(2,1)} \bar{C}_{21}^{kl} = \sum_{C_{kl}\in N_1(2,2)} \bar{C}_{22}^{kl} = 0.48,$$

$$\sum_{C_{kl}\in N_1(1,1)} \bar{B}_{11}^{kl} = \sum_{C_{kl}\in N_1(1,2)} \bar{B}_{12}^{kl} = \sum_{C_{kl}\in N_1(2,1)} \bar{B}_{21}^{kl} = \sum_{C_{kl}\in N_1(2,2)} \bar{B}_{22}^{kl} = 0.48,$$

$$\sum_{C_{kl}\in N_0(1,1)} \bar{D}_{11}^{kl} = 0.12, \quad \sum_{C_{kl}\in N_0(1,2)} \bar{D}_{12}^{kl} = 0.07, \quad \sum_{C_{kl}\in N_0(2,1)} \bar{D}_{21}^{kl} = 0.12, \quad \sum_{C_{kl}\in N_0(2,2)} \bar{D}_{22}^{kl} = 0.17.$$

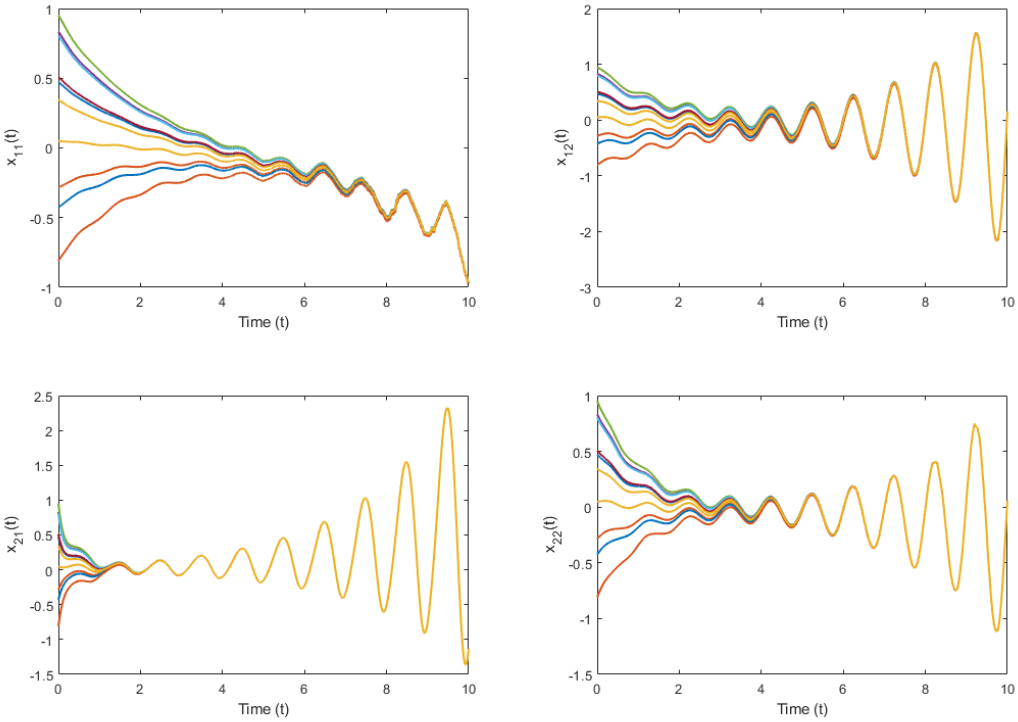

Figure 6.2: Stability of the states x_{ij}, $(1 \le i, j \le 2)$, of (6.3.1) with parameters given in Example 6.6.1, tested with the initial functions randomly chosen in the range $[-1, 1]$.

Then, for $p = 2, 2.5, 3$ *we can, respectively, find* $\kappa \in [1.2605, 4.07], [1.434, 4.14], [1.567, 4.17]$*, so that assumptions (**A5**) and (**A6**) hold simultaneously. In particular, for* $\kappa = 2$*, we have*

$$\mathcal{N}_2 > 0.6294 > 0.25, \quad \mathcal{M}_2 < 0.2856 < 1; \quad \mathcal{N}_{2.5} > 0.4064 > 0.1768, \quad \mathcal{M}_{2.5} < 0.1878 < 1;$$
$$\text{and} \quad \mathcal{N}_3 > 0.25994 \ge 0.125, \quad \mathcal{M}_3 < 0.1237 < 1.$$

Thus, in view of Theorems 6.5.7 and 6.5.8, we can conclude that, for $p = 2,\ 2.5,\ 3$*, the system* (6.3.1) *has a unique p-th mean S-asymptotically* $(1, 1.5)$*-periodic solution, which is stochastically* $(1, 1.5)$*-bounded by* $\kappa = 2$ *and globally exponentially stable in the sense of* (6.5.6)*. This is also verified by the numerical simulations in Figures 6.1 and 6.2. We have plotted the cell activities of each cell (i.e., the solutions) with a fixed initial function and 100 random sample paths in Figure 6.1. Since the p-th mean* (ω, c)*-periodic errors (defined by* (6.4.3)*) of the solutions corresponding to these sample paths appear to converge to zero over time, and consequently with time the p-th mean exhibits behavior close to* $(1, 1.5)$*-periodicity, it follows that the solution is p-th mean S-asymptotically* $(1, 1.5)$*-periodic. In Figure 6.2, we have plotted the solution for a fixed sample path with 10 initial values randomly chosen from the interval* $[-1, 1]$*. Over time, it seems to converge to a unique solution, confirming its stability.*

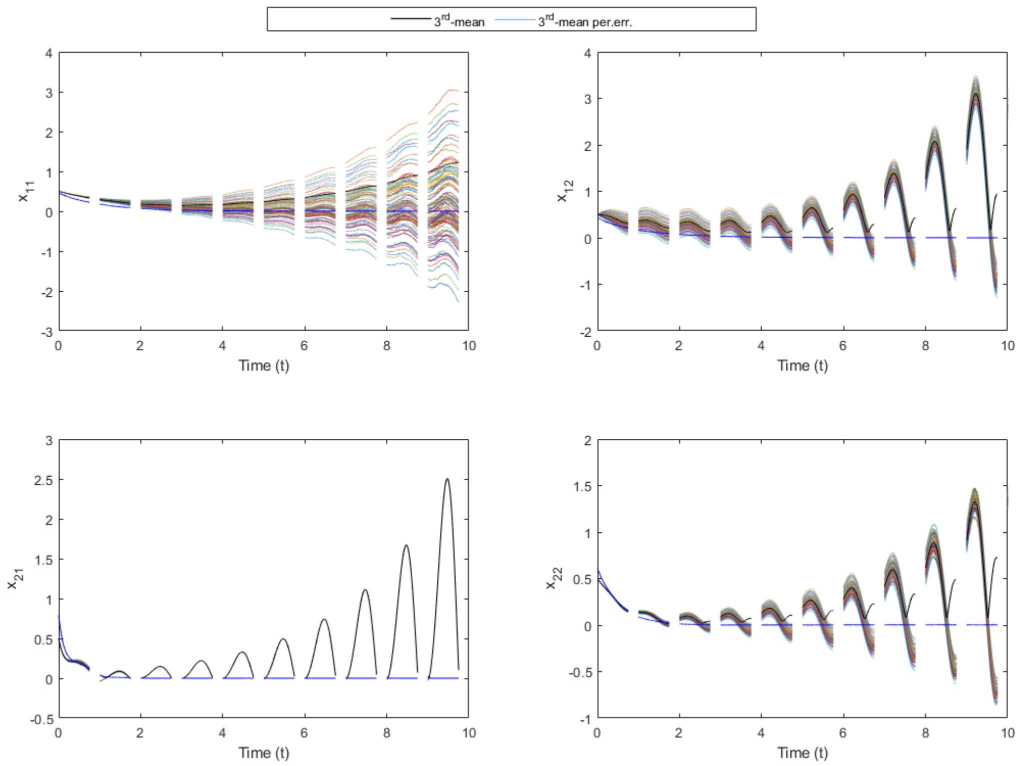

Figure 6.3: Solutions of (6.3.1), with parameters specified in Example 6.6.2 and initial function $\phi_{ij} = 0.5$ for 100 random sample paths.

Example 6.6.2. *Let* $\mathbb{T} = \bigcup\limits_{n\in\mathbb{Z}} [n, n+0.75]$, *and consider the system* (6.3.1) *with the same model parameters as in Example* 6.6.1, *except that we set* $\eta_{ij} = 0$ *whenever* $\eta_{ij} \notin \mathbb{T}$. *Then, with the same bounds as in Example* 6.6.1, *we can easily calculate*

$$\mathcal{N}_3 > 0.1324 > 0.125, \quad \mathcal{M}_3 < 0.2416 < 1.$$

Thus, for $p = 3$, *the system* (6.3.1) *has a unique* p*-th mean* S*-asymptotically* $(1, 1.5)$*-periodic solution in* $\bigcup\limits_{n\in\mathbb{Z}} [n, n+0.75]$, *which is stochastically* $(1, 1.5)$*-bounded by* 2 *and globally exponentially stable in the sense of* (6.5.6). *This is also verified by the numerical simulations in Figures* 6.3 *and* 6.4.

Example 6.6.3. *Let* $\mathbb{T} = 0.5\mathbb{Z}$ *and* $\mathbb{H}_{\mathbb{F}} = \mathbb{F} = \mathbb{R}$. *Consider the system* (6.3.1) *with* $m = n = 2$, $r_1 = r_2, r_3 = 1$ *and take*

$$\mathcal{K}_{ij}(t) = 0.75e^{-t}, \quad f(x(t)) = 0.02\tanh(c^{-t/2}x(t)), \quad g(x(t)) = 0.02\arctan(c^{-t/2}x(t)),$$

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \begin{bmatrix} 1 + 0.2\cos\pi t & 0.8 + 0.2|\cos(\pi t/2)| \\ 0.5 + \exp(-\sin\pi t) & 0.5 + \cos(\sin\pi t) \end{bmatrix}, \quad \begin{bmatrix} \eta_{11}(t) & \eta_{12}(t) \\ \eta_{21}(t) & \eta_{22}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

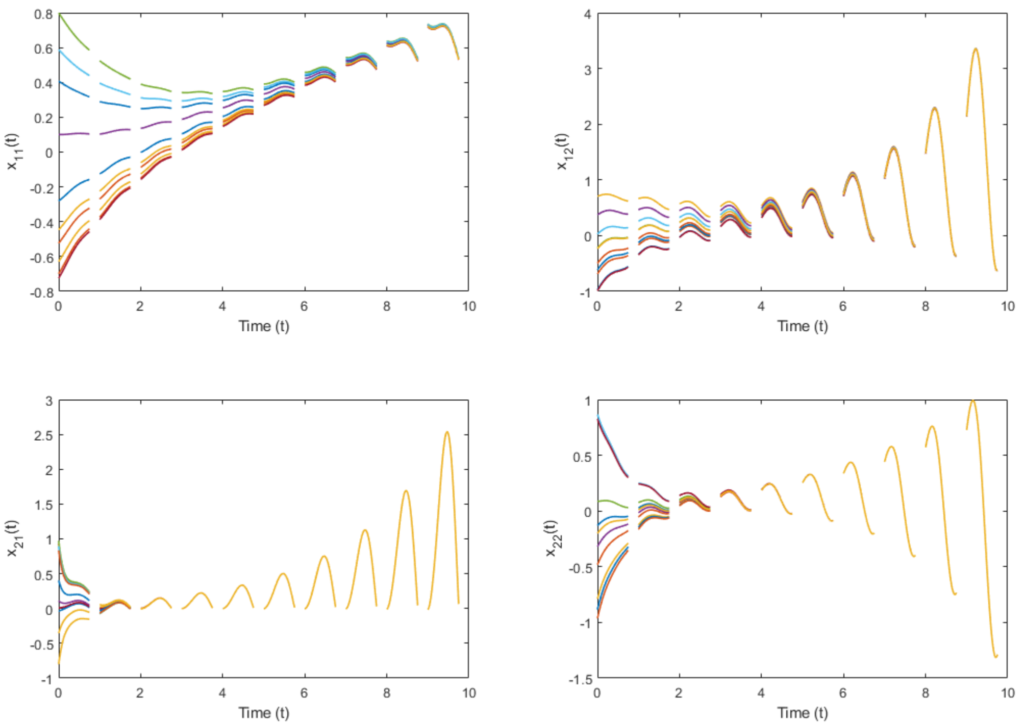

Figure 6.4: Stability of states x_{ij} $(1 \le i,j \le 2)$ of (6.3.1) with parameters given in Example 6.6.2, tested with the initial functions randomly chosen in the range $[-1,1]$.

$$\begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.09+0.03\sin\pi t & 0.05+0.02\cos\pi t \\ 0.07+0.05\cos\pi t & 0.10+0.07\sin\pi t \end{bmatrix},$$

$$\begin{bmatrix} \delta_{11}(t,x) & \delta_{12}(t,x) \\ \delta_{21}(t,x) & \delta_{22}(t,x) \end{bmatrix} = 0.01 \begin{bmatrix} (x+\sin(|x|))/2+(c/2)^{t/2} & x+\arctan x \\ x+\tanh x & x+\big(\sin(|x|)+c^{t/2}\sin(\ln(1+|t|))\big)/2 \end{bmatrix}$$

$$\begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix} = 0.02 \begin{bmatrix} c^{t/2}\left(\sin(\pi t)+0.05\sin(\ln(1+|t|))X\right) & c^{t/2}\left(\cos(\pi t)+(1/(|t|+1)^2)X\right) \\ c^{t/2}\left(\sin(\pi t)+exp(-t^2)X\right) & c^{t/2}\left(\cos(\pi t)+0.07(\cos\sqrt{|t|})X\right) \end{bmatrix},$$

where $c=0.8$, $X \in N(0,1)$. *Computing the bounds, for* $\kappa \le 1$, *we get*

$$\mu \equiv 0.5, \quad \omega = 2, \quad \min_{i,j} \underline{a}_{ij} = 0.8, \quad \hat{f} = \hat{g} = \max_{i,j} \hat{\delta}_{ij} = \mathtt{L}_f = \mathtt{L}_g = l_{ij} = 0.02, \quad \max_{i,j} \hat{L}_{ij} < 0.05, \quad \max_{i,j} \bar{\mathcal{K}}_{ij} \le 1,$$

$$\max_{i,j} \sum_{C_{kl}\in N_1(i,j)} \bar{C}_{ij}^{kl} = \max_{i,j} \sum_{C_{kl}\in N_1(i,j)} \bar{B}_{ij}^{kl} = \max_{i,j} \sum_{C_{kl}\in N_1(i,j)} \bar{D}_{ij}^{kl} = 0.48.$$

Then, for $p=3$ *we have* $\kappa = 1$ *so that*

$$\mathcal{N}_3 > 34.659436 > \frac{1}{\kappa^3}, \quad \textit{and} \quad \mathcal{M}_3 < 0.0277 < 1.$$

Thus, for $p = 3$, the system (6.3.1) *has a unique p-th mean S-asymptotically $(2, 0.8)$-periodic solution in $\mathbb{Z}$, which is stochastically $(2, 0.8)$-bounded by 1 and globally exponentially stable in the sense of* (6.5.6)*. The result is also verified by the numerical simulations in Figures 6.5 and 6.6.*

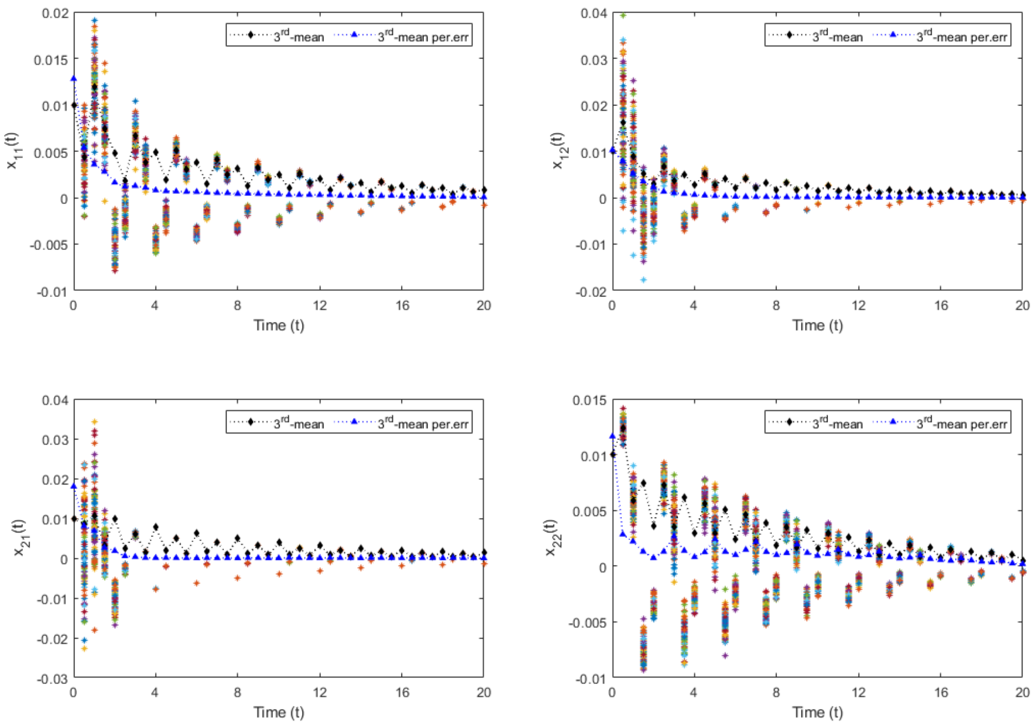

Figure 6.5: Solutions of (6.3.1), with parameters specified in Example 6.6.3 and initial function $\phi_{ij} = 0.01$ for 100 random sample paths.

6.7 Conclusion

This chapter introduced the notion of S-asymptotically (ω, c)-periodic processes on time scales in the p-th mean sense, with $p \geq 2$. This framework extends existing concepts of generalized periodicity and, as a deterministic restriction, yields the corresponding notion of S-asymptotically (ω, c)-periodic functions on time scales, thereby generalizing asymptotically (ω, c)-periodic functions within the unified time-scale setting. It is worth noting that, due to the monotonicity of L^p-norms, if a stochastic process is S-asymptotically (ω, c)-periodic in the p-th mean sense for some $p > 1$, then it is also S-asymptotically (ω, c)-periodic in the first mean sense. Thus, the results obtained for $p \geq 2$ automatically imply the corresponding conclusions in the stochastic mean for $p = 1$. We further investigated a class of stochastic shunting inhibitory

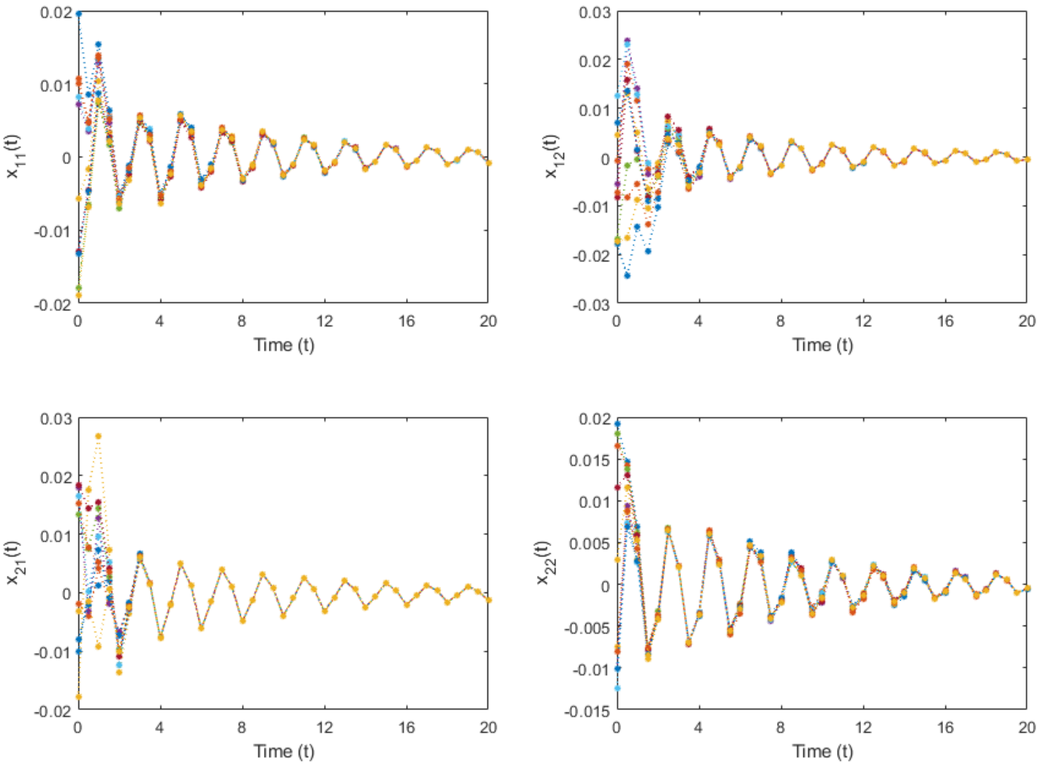

Figure 6.6: Stability of states x_{ij} $(1 \leq i, j \leq 2)$ of (6.3.1) with parameters given in Example 6.6.3, tested with the initial functions randomly chosen in the range $[-0.02, 0.02]$.

delayed cellular neural networks driven by a Wiener process and established sufficient conditions for the existence and stability of S-asymptotically (ω, c)-periodic solutions. The theoretical results are verified through several numerical examples and their graphical simulations on different time scales. For graphical illustration, the associated stochastic dynamic equations were solved numerically to capture the system's temporal evolution under both deterministic and stochastic influences. The simulations were performed using discretization schemes compatible with the underlying time scale, with the Euler–Maruyama method employed to efficiently approximate the stochastic dynamics.

CHAPTER 7

Conclusion and Future Scopes

This chapter concludes the thesis by summarizing the key findings and novel contributions of the research and by identifying possible directions for future work that may be pursued based on the outcomes of this study.

7.1 Conclusion

This thesis has established a unified and comprehensive study of (ω, c)-periodic and generalized (ω, c)-periodic phenomena on ω-periodic time scales in both deterministic and stochastic settings. The foundation of this study is built upon the concept of (ω, c)-periodic functions on time scales, based on which the thesis has further developed the notions of (ω, c)-periodic functions, (ω, c)-asymptotically periodic functions, (doubly) weighted pseudo (ω, c)-periodic functions, S-asymptotically (ω, c)-periodic functions, and S-asymptotically (ω, c)-periodic stochastic processes in the p-th mean sense within the framework of time scales, together with a detailed analysis of the structural properties of the associated function spaces. By integrating time scale calculus with these generalized periodicity concepts, the study has significantly extended existing theories, bridged continuous and discrete dynamics, and established new analytical frameworks for investigating the long-term behavior of nonlinear dynamical systems on time scales under deterministic and stochastic influences. In particular, the thesis has established rigorous results on qualitative properties, including the existence, uniqueness, stability, and synchronization of solutions for various classes of dynamic equations exhibiting structured oscillatory behavior in the presence of time-varying delays, impulsive effects, stochastic perturbations, and abstract dynamical system formulations. From a methodological perspective, the thesis has employed a combination of fixed-point theory, semigroup theory, evolution family techniques, control strategies, Lyapunov-based stability and synchronization analysis, and stochastic analysis within the unified framework of time scale calculus. The applicability of the developed theory has been demonstrated through its implementation in several mathematical models, including population dynamics models (Nicholson's blowflies, Lasota–Ważewska-type systems, and the Mackey–Glass system with immigration or harvesting effects), deterministic and stochastic neural network systems (Hopfield neural networks, cellular neural networks, and shunting inhibitory cellular neural networks), economic dynamics (Keynesian–Cross model), and impulsive evolution systems, where (ω, c)-periodic dynamics can be naturally observed with or without external interventions. The theoretical findings have been further supported and validated through numerical examples and simulations, complemented by graphical representations illustrating system behavior.

The theoretical frameworks developed in this thesis offer a strong foundation for further mathematical developments and for the modeling and qualitative analysis of real-world systems exhibiting structured yet non-ideal oscillatory behavior.

7.2 Future research directions

The results presented in this thesis provide a solid foundation for further investigations in the theory of generalized (ω, c)-periodic phenomena on time scales. Several potential directions for future research naturally emerge from the present work, both at the theoretical and applied levels.

- Relaxation of analytical assumptions: In the present study, the existence and uniqueness results were primarily established using the Banach fixed point theorem, which requires relatively strong contraction-type assumptions. Future research may investigate alternative analytical tools, such as Schauder's fixed point theorem, Krasnoselskii's fixed point theorem, or topological degree theory, in order to relax these assumptions and extend the applicability of the results to broader classes of nonlinear systems. Similarly, the stability and synchronization analyses in this thesis were carried out using specific analytical techniques under certain structural assumptions. Exploring alternative approaches, such as comparison principles, invariant set methods, or other qualitative techniques, may lead to different sets of sufficient conditions and provide further insight into the qualitative behavior of generalized (ω, c)-periodic solutions.

- Analysis of broader classes of models: The proposed framework is sufficiently general to be applicable to a wide range of dynamical systems beyond the models considered in this thesis. Future research may focus on applying the developed theory to other classes of models arising in biology, economics, engineering, and control theory, including reaction–diffusion systems, epidemiological models, networked systems, and hybrid control systems on time scales, in order to investigate the existence and other qualitative behavior of generalized (ω, c)-periodic solutions.

- Further generalizations of (ω, c)-periodicity: The notions of (ω, c)-periodicity, asymptotic periodicity, weighted pseudo periodicity, S-asymptotic periodicity, and stochastic generalizations introduced in this work may themselves be further extended. Possible directions include the study of deformed or variable-factor (ω, c)-periodic functions, multi-frequency or quasi-(ω, c)-periodic structures, and recurrent or almost automorphic (ω, c)-type behaviors on time scales, which may offer a more refined description of complex oscillatory dynamics.
- Extension to nonperiodic time scales: Throughout this thesis, the analysis was primarily conducted on periodic time scales. A challenging and promising direction for future research is the extension of the developed concepts and results to nonperiodic time scales. Such extensions would further enhance the flexibility of the framework and enable the study of generalized periodic-type behavior in systems with nonuniform temporal structures.

These directions highlight the potential for further theoretical development and broader applicability of the framework introduced in this thesis, and provide a promising avenue for future research on dynamical systems and generalized periodic phenomena on time scales.

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List of Publications

Thesis Publications

1. **Puja Bharti**, Soniya Dhama, Martin Bohner. Dynamics beyond bounds: (ω, c)-periodic functions and their impact on delayed population models on time scales, *Discrete and Continuous Dynamical Systems - Series S*, 18 (9) (2025), 2586–2621. https://doi.org/10.3934/dcdss.2025049

2. **Puja Bharti**, Soniya Dhama. (ω, c)-asymptotically periodic oscillation of cellular neural networks on time scales with leakage delays and mixed time-varying delays. *Neural Networks*, 185 (2025), 107174. https://doi.org/10.1016/j.neunet.2025.107174

3. **Puja Bharti**, Soniya Dhama, Amar Debbouche. Exponential stability of (ω, c)-periodic solutions for nonautonomous abstract dynamic equations on time scales. (*Under review*)

4. **Puja Bharti**, Soniya Dhama. Exponential synchronization of weighted pseudo-(ω, c)-periodic solution for neutral-type impulsive evolution system on time scales, *Communication in Nonlinear Science and Numerical Simulation*, 161 (2026) 110177. https://www.sciencedirect.com/science/article/pii/S1007570426005344

5. **Puja Bharti**, Soniya Dhama. S-asymptotically (ω, c)-periodic behavior of hybrid-time shunting inhibitory cellular neural networks with delays and stochastic perturbations, *Applied Mathematics and Computation*, 526 (2026), 130071. https://doi.org/10.1016/j.amc.2026.130071

Other Publications

1. **Puja Bharti**, Soniya Dhama. Stellar model for charged and isotropic rotating neutron stars in gravity. Iranian Journal of Science 47 (2023), 601–615. https://doi.org/10.1007/s40995-023-01429-3

2. **Puja Bharti**, Soniya Dhama. (ω, c)-asymptotically periodic solutions to population models governed by dynamic equations on time scales. (*Accepted for publication in a book titled Advances in Applied Analysis.*)